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SOME REMARKS ON TYPES OF NOETHERIAN LOCAL RINGS

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Abstract. We study some results which concern the types of Noetherian local rings, and improve slightly the previous result: For a complete unmixed (or quasi-unmixed) Noetherian local ring A, we prove that if either $A_\mathfrak{p}$ is Cohen-Macaulay, or $r(A_\mathfrak{p}) \leq \operatorname{depth} A_\mathfrak{p} + 1$ for every prime ideal $\mathfrak{p}$ in A, then A is Cohen-Macaulay. Also, some analogous results for modules are considered.

1. Backgrounds on types

Throughout this paper, we assume that $(A, \mathfrak{m})$ is a commutative Noetherian local ring of dimension d, and M is a finitely generated A-module. We also assume that all modules are unitary.

In this section, we introduce some previously known results about types of rings. Although most of the backgrounds on types quoted here are already present in [7,8], we have included them for the readers' convenience.

We first define the type of a ring. The i-th Bass number of M at a prime $\mathfrak{p}$, denoted $\mu_i(\mathfrak{p}, M)$, is defined to be $\dim_{k(\mathfrak{p})} \operatorname{Ext}^i_{A_\mathfrak{p}}(k(\mathfrak{p}), M_\mathfrak{p})$, where $k(\mathfrak{p}) = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$: we set $\mu_i(A)=\mu_i(\mathfrak{m}, A)$ for brevity. The *type* of A, denoted by $r(A)$, is defined to be $\mu_d(A)$. Many mathematicians have been interested on the conditions related on types, which make a ring A Cohen-Macaulay ([1,4,5,6,9], etc).

Gorenstein rings were characterized as Cohen-Macaulay rings A with $r(A) = 1$ by Bass ([2]), and then it was conjectured by Vasconcelos that the condition $r(A) = 1$ is sufficient for A to be Gorenstein ([14]). In [5], Foxby proved this conjecture for essentially equicharacteristic rings using a version of the Intersection Theorem. The conjecture was proven

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in general by Roberts ([13]): he showed that a Noetherian local ring of type one is Cohen-Macaulay, (and hence Gorenstein).

THEOREM 1.1. [13] *Let A be a commutative Noetherian local ring, let n be the Krull dimension of A, and suppose that $\mu_n(A) = 1$. Then A is Gorenstein.*

In [9], the above theorem was slightly developed using the behavior of maps in minimal injective resolution of a module as follows:

THEOREM 1.2. [9] *Let $(A, \mathfrak{m}, k)$ be a Noetherian local ring of dimension n and M a finitely generated A-module of dimension d. If $\mu_t(\mathfrak{q}, M) = 1$ for some $\mathfrak{q} \in \operatorname{Supp}(M)$ and some $t \leq \dim M_{\mathfrak{q}}$, then $\mu_j(\mathfrak{q}, M) = 0$ for all $j < t$.*

Costa, Huneke and Miller showed the following:

THEOREM 1.3. [4] *Let A be a local ring whose completion is a domain. If $r(A) = 2$, then A is Cohen-Macaulay.*

After they gave two examples: a complete equidimensional local ring of type two that is not Cohen-Macaulay, and a complete reduced local ring of type two that is not Cohen-Macaulay, they posed a question: *Does there exist a complete, equidimensional, reduced local ring A with $r(A) = 2$ that is not Cohen-Macaulay?* However, Marley answered this question in negative by proving the following theorem:

THEOREM 1.4. [10] *Let A be an unmixed local ring of type two. Then A is Cohen-Macaulay.*

Marley also asked that if a complete local ring of type n satisfies Serre's condition (S_{n-1}), then it is Cohen-Macaulay: we say that a finite A-module M satisfies a Serre's condition (S_t) if depth $M_{\mathfrak{p}} \geq \min\{t, \dim M_{\mathfrak{p}}\}$ for every prime $\mathfrak{p}$ in $Supp(M)$. Kawasaki answered this question in the affirmative when rings contain a field and $n \geq 3$ ([6]), and later Aoyama gave a general proof.

THEOREM 1.5. [1] *Let $n \geq 3$ be an integer. If $r(A) \leq n$ and $\hat{A}$ is (S_{n-1}), then A is Cohen-Macaulay.*

Kawasaki conjectured the following which is still open:

CONJECTURE 1.6. [6] *Let A be a complete unmixed local ring of type n. If $A_{\mathfrak{p}}$ is Cohen-Macaulay for all $\mathfrak{p}$ in $Spec(A)$ such that $ht(\mathfrak{p}) < n$, then A is Cohen-Macaulay.*

We note that the condition "$A_\mathfrak{p}$ is Cohen-Macaulay for all $\mathfrak{p}$ in $Spec(A)$ such that $ht(\mathfrak{p}) < n$" is weaker than (S_{n-1}), and that if A is complete and (S_2), then A is unmixed; hence Conjecture 1 implies Theorem 4. If $r(A) > \dim A$ in the above conjecture, then the conjecture trivially holds. The case of $r(A) \leq \dim A$ seems not easy to be proven. The author conjectured the following for the rings with $r(A) \leq \operatorname{depth} A + 1$:

Conjecture 1.7. [7] *Let $(A, \mathfrak{m})$ be a complete unmixed Noetherian local ring. If $r(A) \leq \operatorname{depth} A + 1$, where $r(A)$ is the type of A, then a ring A is Cohen-Macaulay.*

The above conjectures were answered in the affirmative under some additional conditions as follows:

Theorem 1.8. [7] *Let $(A, \mathfrak{m})$ be a quasi-unmixed Noetherian local ring of dimension d such that $r(A) \leq \operatorname{depth} A + 1$. Suppose that $\hat{A}_\mathfrak{p}$ is Cohen-Macaulay for a prime $\mathfrak{p} \neq \hat{\mathfrak{m}}$ in $Spec(\hat{A})$. Then A is Cohen-Macaulay.*

Theorem 1.9. [8] *Let $(A, \mathfrak{m})$ be a complete unmixed Noetherian local ring, and $\mathfrak{q}$ an prime ideal of A. Suppose that $r(A_\mathfrak{p}) \leq \operatorname{depth} A_\mathfrak{p} + 1$ for each prime ideal $\mathfrak{p} \subseteq \mathfrak{q}$. Then $A_\mathfrak{q}$ is Cohen-Macaulay. In particular, if $r(A_\mathfrak{p}) \leq \operatorname{depth} A_\mathfrak{p} + 1$ for each prime ideal $\mathfrak{p}$ of A, then a ring A is Cohen-Macaulay.*

Theorem 1.10. [8] *Let A be a complete unmixed Noetherian local ring of type n with $\dim A \leq \operatorname{depth} A + 1$. Suppose that $A_\mathfrak{p}$ is Cohen-Macaulay for all $\mathfrak{p}$ in $Spec(A)$ with $ht(\mathfrak{p}) < n$. Then A is Cohen-Macaulay.*

2. Main theorems

In this section, we slightly improve Theorem 7 in Section 1. We also prove some theorems for the case of modules.

Let's recall that A is *equidimensional* if $\dim A/\mathfrak{p} = \dim A$ for every minimal prime $\mathfrak{p}$ of A. A is said to be *quasi-unmixed* (or formally equidimensional) if its completion $\hat{A}$ is equidimensional, and to be *unmixed* if $\dim \hat{A}/\mathfrak{p} = \dim \hat{A}$ for each associated prime $\mathfrak{p} \in Ass(\hat{A})$.

Now, we prove one of the main theorems in this article.

Theorem 2.1. *Let A be a complete unmixed (or quasi-unmixed) Noetherian local ring. Suppose that for every prime ideal $\mathfrak{p}$ in A, either* *(i) $A_{\mathfrak{p}}$ is Cohen-Macaulay, or (ii) $r(A_{\mathfrak{p}}) \leq \operatorname{depth} A_{\mathfrak{p}} + 1$. Then A is Cohen-Macaulay.*

Proof. If A is not Cohen-Macaulay, then we have $r(A) \leq \operatorname{depth} A + 1$ by assumption (ii). Note that A is complete and quasi-unmixed, and so according to Theorem 1.6, it is enough to show that $A_{\mathfrak{p}}$ is Cohen-Macaulay for every non-maximal prime ideal $\mathfrak{p}$ in A.

Now, suppose that $A_{\mathfrak{p}}$ is Cohen-Macaulay for every prime $\mathfrak{p}$ of A such that $ht(\mathfrak{p}) < t$ for some integer t, and $r(A_{\mathfrak{q}}) \leq \operatorname{depth} A_{\mathfrak{q}} + 1$ for some prime $\mathfrak{q}$ with $ht(\mathfrak{q}) = t$. We may assume $t \geq 1$ since $A_{\mathfrak{p}_0}$ is Cohen-Macaulay for every minimal prime $\mathfrak{p}_0$, i.e., $ht(\mathfrak{p}_0) = 0$.

Since A is quasi-unmixed, so is $A_{\mathfrak{q}}$. To show that $A_{\mathfrak{q}}$ is Cohen-Macaulay, we will prove that $(\widehat{A_{\mathfrak{q}}})_{\tilde{\mathfrak{p}}}$ is Cohen-Macaulay for every prime ideal $\tilde{\mathfrak{p}}$ (of $\widehat{A_{\mathfrak{q}}}$), which is properly contained in $\widehat{\mathfrak{q}A_q}$: if then, Theorem 1.6 implies that $A_{\mathfrak{q}}$ is Cohen-Macaulay since $r(A_{\mathfrak{q}}) \leq \operatorname{depth} A_{\mathfrak{q}} + 1$.

Since A is complete, there is a Gorenstein (and so Cohen-Macaulay) local ring S such that $S \to A \to 0$ is surjective, which implies that $S_{\mathfrak{q}} \to A_{\mathfrak{q}} \to 0$ is also surjective. Thus A and $A_{\mathfrak{q}}$ are homomorphic images of Cohen-Macaulay rings. Thus by Lemma 2.8 in [7] (or see Lemma 2.4 below), showing that $(\widehat{A_{\mathfrak{q}}})_{\tilde{\mathfrak{p}}}$ is Cohen-Macaulay for every prime ideal $\tilde{\mathfrak{p}}$ (of $\widehat{A_{\mathfrak{q}}}$), which is properly contained in $\widehat{\mathfrak{q}A_q}$ is equivalent to showing that $A_{\mathfrak{p}}(= (A_{\mathfrak{q}})_{\mathfrak{p}})$ is Cohen-Macaulay for every prime ideal $\mathfrak{p}$, which is properly contained in $\mathfrak{q}$. We know that the latter part is true by assumption that $A_{\mathfrak{p}}$ is Cohen-Macaulay for every prime $\mathfrak{p}$ of A such that $ht(\mathfrak{p}) < t = ht(\mathfrak{q})$. Therefore, by Theorem 1.6, $A_{\mathfrak{q}}$ is Cohen-Macaulay, and hence we may assume that $A_{\mathfrak{q}}$ is Cohen-Macaulay for every prime $\mathfrak{q}$ of A with $ht(\mathfrak{q}) = t$.

Using induction on t, we can show that $A_{\mathfrak{p}}$ is Cohen-Macaulay for every non-maximal prime $\mathfrak{p}$ of A. Again by Theorem 1.6, we conclude that A is Cohen-Macaulay. □

We now turn to Kawasaki's conjecture, and answer it affirmatively with some extra condition.

Corollary 2.2. *Let A be a complete unmixed Noetherian local ring of type n. Suppose that $A_{\mathfrak{p}}$ is Cohen-Macaulay for all $\mathfrak{p}$ in $Spec(A)$ with $ht(\mathfrak{p}) < n$. Also we assume that and $r(A_{\mathfrak{p}}) \leq \operatorname{depth} A_{\mathfrak{p}} + 1$ for all $\mathfrak{p}$ in $Spec(A)$ with $ht(\mathfrak{p}) \geq n$. Then A is Cohen-Macaulay.*

Proof. It is clear by Theorem 2.1. □

Also, it is easy to obtain the following corollary from Theorem 2.1:

COROLLARY 2.3. [8, Theorem 2.4] *Let* $(A, \mathfrak{m})$ *be a complete unmixed Noetherian local ring. If* $r(A_{\mathfrak{p}}) \leq \operatorname{depth} A_{\mathfrak{p}} + 1$ *for each prime ideal* $\mathfrak{p}$ *of* A*, then a ring* A *is Cohen-Macaulay.*

For the rest of this section, we will investigate the cases of modules. Although the basic strategy of proofs used in the case of rings would be used again for modules, we include detailed proofs for completeness since there is no complete proofs of them in the literature.

We start with recalling some definitions and fact used in the sequel: An A- module M is said to be *equidimensional* if $\dim A/\mathfrak{p} = \dim M$ for every minimal prime $\mathfrak{p}$ in $\operatorname{Supp}(M)$. M is said to be *quasi-unmixed* if its completion $\hat{M}$ is equidimensional, and to be *unmixed* if $\dim \hat{A}/\mathfrak{p} = \dim \hat{M}$ for all $\mathfrak{p} \in Ass(\hat{M})$.

We can localize M at a prime ideal when we need to localize its completion $\hat{M}$ at a prime ideal according to the following lemma:

LEMMA 2.4. [8] *Let* $(A, \mathfrak{m})$ *be a Noetherian local ring, and* M *a finitely generated* A*-module. Suppose that* A *is a homomorphic image of a Cohen-Macaulay ring, and* M *is quasi-unmixed. Then* $M_{\mathfrak{p}}$ *is Cohen-Macaulay for every prime* $\mathfrak{p} \neq \mathfrak{m}$ *in* $\operatorname{Supp}(M)$ *if and only if* $\hat{M}_{\mathfrak{p}}$ *is Cohen-Macaulay for every* $\mathfrak{p} \neq \hat{\mathfrak{m}}$ *in* $\operatorname{Supp}(\hat{M})$.

For the proofs of our theorems, the following theorem is used:

THEOREM 2.5. [8] *Let* $(A, \mathfrak{m})$ *be a Noetherian local ring, and* M *a finitely generated quasi-unmixed* A*-module of dimension* d *such that* $r(M) \leq \operatorname{depth} M + 1$. *Suppose that* $\hat{M}_{\mathfrak{p}}$ *is Cohen-Macaulay for every prime* $\mathfrak{p} \neq \hat{\mathfrak{m}}$ *in* $Supp(\hat{M})$. *Then* M *is Cohen-Macaulay.*

Now, we have the following result for modules, which is analogous to Theorem 2.1.

THEOREM 2.6. *Let* $(A, \mathfrak{m})$ *be a complete Noetherian local ring, and* M *a finitely generated unmixed* A*-module. Suppose that for every prime* $\mathfrak{p}$ *in* $\operatorname{Supp}(M)$*, either (i)* $M_{\mathfrak{p}}$ *is Cohen-Macaulay, or (ii)* $r(M_{\mathfrak{p}}) \leq \operatorname{depth} M_{\mathfrak{p}} + 1$. *Then* M *is Cohen-Macaulay.*

Proof. We will use Theorem 2.5 to complete the proof. If M is not Cohen-Macaulay, then we know $r(M) \leq \operatorname{depth} M + 1$ by assumption (ii). Thus to use Theorem 2.5, we may assume that $r(M) \leq \operatorname{depth} M + 1$.

Since M is quasi-unmixed and $r(M) \leq \operatorname{depth} M + 1$, it is enough to show that $\hat{M}_{\mathfrak{p}}$ is Cohen-Macaulay for every prime $\mathfrak{p} \neq \hat{\mathfrak{m}}$ in $Supp(\hat{M})$ by Theorem 2.5. Since A is complete, there is a Gorenstein (and so

Cohen-Macaulay) local ring S such that $S \to A \to 0$ is surjective. Thus A is a homomorphic image of a Cohen-Macaulay ring. By Lemma 2.4, it also suffices to show that $M_{\mathfrak{p}}$ is Cohen-Macaulay for every prime $\mathfrak{p} \neq \mathfrak{m}$ in $Supp(M)$.

Now, suppose that $M_{\mathfrak{p}}$ is Cohen-Macaulay for every prime $\mathfrak{p}$ in $\mathrm{Supp}(M)$ such that $\dim M_{\mathfrak{p}} < t$ for some integer t, and $r(M_{\mathfrak{q}}) \leq \operatorname{depth} M_{\mathfrak{q}} + 1$ for some prime $\mathfrak{q}$ with $\dim M_{\mathfrak{q}} = t$. We may assume $t \geq 1$ since $M_{\mathfrak{p}_0}$ is Cohen-Macaulay for every prime $\mathfrak{p}_0$ with $\dim M_{\mathfrak{p}_0} = 0$.

Note that M is quasi-unmixed if and only if $A/ann(M)$ is quasi-unmixed. By Theorem 31.6 in [11], $A_{\mathfrak{q}}/ann(M)A_{\mathfrak{q}}$ is quasi-unmixed, and so $M_{\mathfrak{q}}$ is quasi-unmixed. Also, we note that $r(M_{\mathfrak{q}}) \leq \operatorname{depth} M_{\mathfrak{q}} + 1$. Thus to show that M_q is Cohen-Macaulay, we will prove that $(\widehat{M_{\mathfrak{q}}})_{\tilde{\mathfrak{p}}}$ is Cohen-Macaulay for every prime ideal $\tilde{\mathfrak{p}}$ (in $\mathrm{Supp}\widehat{M_{\mathfrak{q}}}$), which is properly contained in $\widehat{\mathfrak{q}A_q}$: if then, Theorem 2.5 implies that $M_{\mathfrak{q}}$ is Cohen-Macaulay.

Since A is complete, A is a homomorphic image of a Cohen-Macaulay ring, and so is $A_{\mathfrak{q}}$. Thus by Lemma 2.4, it suffices to show that $(M_{\mathfrak{q}})_{\mathfrak{p}}(= M_{\mathfrak{p}})$ is Cohen-Macaulay for every prime ideal $\mathfrak{p}A_q$ (in $\mathrm{Supp}M_{\mathfrak{q}}$), which is properly contained in $\mathfrak{q}A_q$, i.e., $\dim M_{\mathfrak{p}} < t = \dim M_{\mathfrak{q}}$. We know that this is true by assumption. Therefore, $M_{\mathfrak{q}}$ is Cohen-Macaulay by Theorem 2.5, and hence we may assume that $M_{\mathfrak{q}}$ is Cohen-Macaulay for every prime $\mathfrak{q}$ $\mathrm{Supp}(M)$ with $\dim M_{\mathfrak{q}} = t$.

Using induction on t, we can show that $M_{\mathfrak{p}}$ is Cohen-Macaulay for every prime $\mathfrak{p} \neq \mathfrak{m}$ in $Supp(M)$. Again by Theorem 2.5, we conclude that M is Cohen-Macaulay. □

The following corollary was stated in [8, Theorem 3.4] with the extra condition, "$\hat{M}_{\mathfrak{p}}$ is Cohen-Macaulay for every $\mathfrak{p} \neq \hat{\mathfrak{m}}$ in $\mathrm{Supp}(\hat{M})$", but Theorem 2.5 enables us to remove this extra condition as follows:

Corollary 2.7. *Let $(A, \mathfrak{m})$ be a complete Noetherian local ring, and M a finitely generated unmixed A-module. Suppose $r(M_{\mathfrak{p}}) \leq \operatorname{depth} M_{\mathfrak{p}} + 1$ for each prime ideal $\mathfrak{p}$ in $\mathrm{Supp}(M)$. Then M is Cohen-Macaulay.*

Proof. It is trivial by Theorem 2.6. □

We close this section by proving our last theorem. The following theorem shows that the conjecture 1 in Section 1 holds when $\dim M \leq \operatorname{depth} M + 1$, which was stated in [8] without proof. Here we include a complete proof for the readers' convenience.

THEOREM 2.8. *Let A be a complete Noetherian local ring, and M be a finitely generated unmixed A-module of type n with* $\dim M \leq \operatorname{depth} M + 1$. *If $M_{\mathfrak{p}}$ is Cohen-Macaulay for all $\mathfrak{p}$ in* $\operatorname{Supp}(M)$ *such that* $\dim M_{\mathfrak{p}} < n$, *then M is Cohen-Macaulay.*

Proof. We may assume that $r(M) \leq \dim M$ since if $r(M) > \dim M$, then M is Cohen-Macaulay by assumption. Then $r(M) \leq \dim M \leq \operatorname{depth} M + 1$. Thus by Theorem 2.5, it is enough to show that $\hat{M}_{\mathfrak{p}}$ is Cohen-Macaulay for every prime $\mathfrak{p} \neq \hat{\mathfrak{m}}$ in $Supp(\hat{M})$ (which implies that M is Cohen-Macaulay).

The completeness of A gives us the fact that A is a homomorphic image of a Cohen-Macaulay ring. Since M is quasi-unmixed, we note (by Lemma 2.4) that $\hat{M}_{\mathfrak{p}}$ is Cohen-Macaulay for every $\mathfrak{p} \neq \hat{\mathfrak{m}}$ in $\operatorname{Supp}(\hat{M})$ is if and only if $M_{\mathfrak{p}}$ is Cohen-Macaulay for every prime $\mathfrak{p} \neq \mathfrak{m}$ in $\operatorname{Supp}(M)$. Therefore, we will show that $M_{\mathfrak{p}}$ is Cohen-Macaulay for every prime $\mathfrak{p} \neq \mathfrak{m}$ in $\operatorname{Supp}(M)$.

We first claim that for every prime ideal $\mathfrak{p}$ in $Supp(M)$,

$$(i)\ \dim M_{\mathfrak{p}} - \operatorname{depth} M_{\mathfrak{p}} \leq \dim M - \operatorname{depth} M, \text{ and } (ii)\ r(M_{\mathfrak{p}}) \leq r(M)$$

For the claim (i), since $\mu_i(\mathfrak{p}, M) \leq \mu_{i+t}(\mathfrak{m}, M)$ for each i and $t = ht(\mathfrak{m}/\mathfrak{p})$ ([12]), if $\operatorname{depth} M_{\mathfrak{p}} = i$ and $ht(\mathfrak{m}/\mathfrak{p}) = t$, then

$$\operatorname{depth} M = \min\{s : \mu_s(\mathfrak{m}, M) \neq 0\} \leq i + t = \operatorname{depth} M_{\mathfrak{p}} + ht(\mathfrak{m}/\mathfrak{p}).$$

Thus the claim (i) is true by the following:

$$\begin{aligned} \dim M_{\mathfrak{p}} - \operatorname{depth} M_{\mathfrak{p}} &\leq (\dim M - ht(\mathfrak{m}/\mathfrak{p})) - (\operatorname{depth} M - ht(\mathfrak{m}/\mathfrak{p})) \\ &= \dim M - \operatorname{depth} M. \end{aligned}$$

For the claim (ii), we first prove that $\dim M_{\mathfrak{p}} + \dim A/\mathfrak{p} = \dim M$. Note that $\dim M = \dim A/ann(M)$, and if A is complete, then A/I is also complete for any ideal I of A. We know the fact that if a ring A is complete, then A is catenary, and for a catenary and local domain A, it is always true that $\dim A_{\mathfrak{p}} + \dim A/\mathfrak{p} = \dim A$ for any prime $\mathfrak{p}$ of A ([11]). Now let $\mathfrak{p} \in \operatorname{Supp}(M)(= V(ann(M)))$ and fix any $\mathfrak{p}_0 \in \operatorname{Min}(M)$ with $ht(\mathfrak{p}/\mathfrak{p}_0) = \dim M_{\mathfrak{p}}$. Then $\bar{A} = A/\mathfrak{p}_0$ is a catenary and local domain, and so we have $\dim \bar{A}_{\mathfrak{p}} + \dim \bar{A}/\mathfrak{p} = \dim \bar{A}$. Clearly, $\dim \bar{A}/\mathfrak{p} = \dim A/\mathfrak{p}$. Since M is unmixed and A is complete, we know that $\dim \bar{A} = \dim A/\mathfrak{p}_0 = \dim M$. Also, we know that $\dim \bar{A}_{\mathfrak{p}} = \dim A_{\mathfrak{p}}/\mathfrak{p}_0 A_{\mathfrak{p}} = ht(\mathfrak{p}/\mathfrak{p}_0) = \dim M_{\mathfrak{p}}$, and hence we finally have $\dim M_{\mathfrak{p}} + \dim A/\mathfrak{p} = \dim M$ for any prime $\mathfrak{p}$ in $\operatorname{Supp}(M)$. Again using the fact that if $ht(\mathfrak{q}/\mathfrak{p}) = t$ for primes $\mathfrak{p}$ and $\mathfrak{q}$, then $\mu_i(\mathfrak{p}, M) \leq$

$\mu_{i+t}(\mathfrak{q}, M)$, we can show (see [7]) that if $\dim M_{\mathfrak{p}} + \dim A/\mathfrak{p} = \dim M$, then

$$r(M_{\mathfrak{p}}) \leq r(M).$$

Now let $\mathfrak{p}$ be a non-maximal ideal in $\mathrm{Supp}(M)$. If $\dim M_{\mathfrak{p}} < n$, then $M_{\mathfrak{p}}$ is Cohen-Macaulay by assumption. Suppose $\dim M_{\mathfrak{p}} \geq n = r(M)$. Then by claims above, we have the following inequalities

$$r(M_{\mathfrak{p}}) \leq r(M) \leq \dim M_{\mathfrak{p}} \leq \operatorname{depth} M_{\mathfrak{p}} + (\dim M - \operatorname{depth} M) \leq \operatorname{depth} M_{\mathfrak{p}} + 1.$$

In all, we've proved that either $M_{\mathfrak{p}}$ is Cohen-Macaulay, or $r(M_{\mathfrak{p}}) \leq \operatorname{depth} M_{\mathfrak{p}} + 1$. By the above Theorem 2.6, we can conclude that M is Cohen-Macaulay. □

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