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THE CANONICAL FORM OF INVOLUTARY FUZZY MATRICES

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Abstract. We study special types of matrices. The involutary fuzzy matrices are important in various applications and have many interesting properties. Using the graphical method, we have the zero patterns of involutary fuzzy matrix, that is, involutary Boolean matrices. And we give the construction of all involutary fuzzy matrices for some dimensions and suggest the canonical form of involutary fuzzy matrix.

1. Introduction

The theory of a fuzzy matrix is very useful in the discussion of fuzzy relations. We can represent basic propositions of the theory of fuzzy relations in terms of matrix operations. Futhermore we can deal with the fuzzy relations in the matrix form. In the study of the theory of a fuzzy matrix, a canonical form of some fuzzy matrices has received increasing attention in [5]. For example, Kim and Roush [1, 2] studies the canonical form of some special matrix in 1980 and 1981. Hashimoto [3] studied the canonical form of a transitive matrix in 1983. And Fuzhen Zhang [6] introduce the result that is:

A square matrix A is said to be involutary if $A^2 = I$ where I is identity. Let A be an $n \times n$ complex matrix. Then A is involutary if and if A similar to a diagonal matrix of the form $diag(1, \cdots, 1, -1, \cdots, -1)$. The above result give the canonical form of involutary matrices that have simple structures under similarity. Clearly, the study of this kind of canonical forms is important to develop the theory of a fuzzy matrix.

We define some operations for fuzzy matrices whose elements are in the unit interval $[0, 1]$. First for $x, y \in [0, 1]$ we define $x \vee y$, and $x \wedge y$

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as follows:

$$x \vee y = max(x, y)$$
$$x \wedge y = min(x, y).$$

Next we define the following matrix operations for $n \times n$ fuzzy matrices $R = [r_{ij}]$ and $S = [s_{ij}]$:

$\cdot\ R \vee S = [r_{ij} \vee s_{ij}]$;

$\cdot\ R \wedge S = [r_{ij} \wedge s_{ij}]$;

$\cdot\ R \times S = [(r_{i1} \wedge s_{1j}) \vee (r_{i2} \wedge s_{2j}) \vee \cdots \vee (r_{in} \wedge s_{nj})]$;

$\cdot\ R^T = [r_{ji}]$ (the transpose of R);

$\cdot\ R \leq S$ if and only if $r_{ij} \leq s_{ij}$ for all i, j.

We define some special kinds of fuzzy matrices where I is an identity and O is a zero matrix. A fuzzy matrix R is said to be:

$\cdot$ idempotent if $R^2 = R$;

$\cdot$ nilpotent if $R^2 = O$;

$\cdot$ involutary if $R^2 = I$

Accordingly, any nilpotent fuzzy matrix and idempotent fuzzy matrix except identity is not involution.

2. The properties of involutary fuzzy matrices

First we examine some basic properties of involutary fuzzy matrices. They are useful in the following discussion. We know that an 1×1 fuzzy matrix whose entry is 1 is only involutary. Hence, in this paper, we deal only with square fuzzy matrix that dimension n, $n \geq 2$. Let F_I be the set of all involutary fuzzy matrices.

LEMMA 2.1. *The set of all lnvolutary fuzzy matrices, F_I, is closed under the following operations:*

(i) *permutation similarity; and*
(ii) *transposition*

DEFINITION 2.2. *Let A be a fuzzy matrix. For some nonzero entry a of A, $Z(a) = [z_{ij}]$ is defined by*

$$z_{ij} = \begin{cases} 1 \ \ if \ \ 0 < a_{ij} \leq a \\ 0 \ \ if \ \ otherwise. \end{cases}$$

$Z(a) = [z_{ij}]$ is called a zero pattern of A.

EXAMPLE 2.3. *If a fuzzy matrix*

$$A = \begin{bmatrix} 0.3 & 0.8 & 1 \\ 0.8 & 0 & 0.3 \\ 0.5 & 1 & 0 \end{bmatrix},$$

then all its zero patterns are

$$Z(0.3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad Z(0.5) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},$$

$$Z(0.8) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \text{ and } Z(1) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

LEMMA 2.4. [1] *A fuzzy matrix is idempotent if and only if all its zero patterns are idempotent.*

LEMMA 2.5. [6] *For a matrix A, $\frac{1}{2}(I + A)$ is idempotent if and only if A is an involution.*

By the above Lemma 2.4, 2.5, we examine the properties of $(0,1)$-fuzzy matrices and obtain a theorem and canonical form of the $(0,1)$-fuzzy matrices. Thus we will be able to characterize the structure of the set of all involutary fuzzy matrices, F_I. Let B_I be the set of all involutary $(0,1)$-fuzzy matrices. In fact, B_I is the set of Boolean involutary matrices. In this paper, the set of numbers $\{1, 2, 3, \cdots n-1, n\}$ will be denoted by N.

3. The involutary $(0,1)$-fuzzy matrices

To simplify notation, in the remainder of this paper we let the index set $\{1, 2, \cdots, n\}$ be represented by N. A matrix A of order $n \geq 2$ is said to be *reducible* if there exists a permutation matrix P such that

$$PAP^T = \begin{bmatrix} B & C \\ O & D \end{bmatrix}$$

where B and D are square matrices of order at least one. Otherwise A is called *irreducible*.

LEMMA 3.1. *Let $A = [a_{ij}]$ be an $n \times n$ $(2 \leq n)$ reducible involutary fuzzy matrix. If for some permutation matrix P,*

$$PAP^T = \begin{bmatrix} B & C \\ O & D \end{bmatrix}$$

where B is irreducible with maximal $k \times k$ dimension, then the matrix block C is zero.

Proof. Suppose that the matrix block C of A is not zero. Let the entry a_{1k+1} of A be non-zero. Then the product block matrices $C \times D$ is zero because A is in B_I. Therefore the entries a_{k+1q} are all zero where q is from $k+1$ to n. It is an immediate consequence of the fuzzy matrix product that the $k+1$'s row of A is zero. It is impossible. □

In the special quality of permutation matrix, each $n \times n$ $(2 \leq n)$ permutation fuzzy matrix $P = [p_{ij}]$ has the transpose matrix as the inverse matrix, That is $PP^T = I$.

LEMMA 3.2. *If an $n \times n$ permutation fuzzy matrix P is symmetric, then P is involution.*

Proof. Let P be a symmetric fuzzy matrix. Then $P^T = P$ and $P^2 = PP = PP^T = I$. It is an immediate consequence of the fuzzy matrix product. □

LEMMA 3.3. *If $A = [a_{ij}]$ is an $n \times n$ matrix in B_I and $a_{ii} = 0$, then for each i, A has nonzero pair a_{ik} and a_{ki} where $k \in N$.*

Proof. Let $A^2 = [a'_{ij}] = I$ and $a_{ii} = 0$. Since $a_{ii} \wedge a_{ii} = 0$ and $a'_{ii} = 1 = \bigvee_{k=1}^{n}(a_{ik} \wedge a_{ki})$, there exists nonzero pair a_{ik} and a_{ki} where $k \in N$. It is an immediate consequence of the fuzzy matrix product. □

LEMMA 3.4. *If $A = [a_{ij}]$ is an $n \times n$ involutary $(0,1)$-fuzzy matrix and $a_{ii} = 1$, then for all $k, k \neq i$, a_{ik} and a_{ki} are all zero.*

Proof. Suppose that $a_{ik} = 1$ for some $k, k \neq i$ and $A^2 = [a'_{ij}]$. Then $a'_{ik} = 1 = \bigvee_{k=1}^{n}(a_{ik} \wedge a_{kk})$ by the nonzero term $a_{ii} \wedge a_{ik}$. It is an immediate consequence of the fuzzy matrix product. □

LEMMA 3.5. *If $A = [a_{ij}]$ is a $(0,1)$-fuzzy matrix and for two distinct p, q, $a_{ip} = a_{iq} = 1$, then the fuzzy matrix A is not involution.*

Proof. Suppose that $a_{ip} = a_{iq} = 1$ and $A \in B_I$. Then by Lemma 3.3, 3.4, $a_{pi} = a_{qi} = 1$ of A and $a_{ii} = 0$. Now we consider the entry a'_{pq} of $A^2 = [a'_{ij}]$. Then $a'_{pq} = 1 = \bigvee_{k=1}^{n}(a_{pk} \wedge a_{kq})$ by the nonzero term $a_{pi} \wedge a_{iq}$. Because that $a'_{pq} = 1$, A is not in B_I. □

LEMMA 3.6. *If $A = [a_{ij}]$ is an $n \times n$ involutary $(0,1)$-fuzzy matrix and $a_{ij} = 1$, then the entry a_{ji} of A is nonzero.*

Proof. Suppose that $a_{ij} = 1$ and $a_{ii} = 0$. Let $A^2 = [a'_{ij}]$. Then $a'_{ii} = 1 = \bigvee_{k=1}^{n}(a_{ik} \wedge a_{ki})$ by the nonzero term $a_{ij} \wedge a_{ji}$. Therefore, $a_{ji} = 1$ by Lemma 3.5 . □

LEMMA 3.7. *If $A = [a_{ij}]$ is an $n \times n$ $(0, 1)$-fuzzy matrix and its graph has a simple cycle with length $k \geq 3$, then the fuzzy matrix A is not involution.*

Proof. Without loss of generality, suppose that $a_{12} = a_{23} = a_{31} = 1$ and $A \in B_I$. Then by Lemma 3.5, $a_{21} = a_{32} = a_{23} = 1$ of A and by the Lemma 3.6 $a'_{12} = a'_{23} = 1$ of $A^2 = [a'_{ij}]$. Therefore A is not in B_I. □

We obtain the following theorem by the above Lemmas.

THEOREM 3.8. *If $A = [a_{ij}]$ is an $n \times n$ involution $(0, 1)$-fuzzy matrix and irreducible, then its graph has only loops and simple 2-cycles.*

THEOREM 3.9. *A fuzzy matrix $A = [a_{ij}]$ is in F_I if and only if $A = [a_{ij}]$ is in B_I.*

Proof. Without loss of generality, Let A be in F_I and contains nonzero entry a_{ii} or a_{ij} except 1. It is an immediate consequence of the fuzzy matrix product. □

Let A an $n \times n$ matrix be given. Then either A is irreducible or there exists a permutation matrix P such that

$$P^T A P = \begin{bmatrix} A_1 & & * \\ & \ddots & \\ 0 & & A_k \end{bmatrix}$$

in which A_i is either irreducible or is 1×1 zero matrix, $i = 1, \cdots, k$. This is called the *Frobenious normal form* of A. If A is a reducible involution fuzzy matrix in Frobenious normal form, then it is clear that each irreducible diagonal block of A is involution fuzzy matrix. In the remainder of this paper, we use the results of above section, and assume that each nonzero irreducible diagonal block A_{ii} of A is $J_1 = [1]$ or

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Further, in terms of block multiplication where $P = A^2$,

$$P_{ij} = A_{ii}A_{ij} + A_{ii+1}A_{i+1j} + \cdots + A_{ij}A_{jj} = A_{ij}.$$

The key question here is: *What are the possible zero patterns of the off-diagonal blocks A_{ij} so that the above equation is satisfied?* By the Lemma

3.1, we have the answer for the above key question. Since a reducible involution fuzzy matrix is only zero off-diagonal block, we obtain the following:

THEOREM 3.10. *An $n \times n$ involutary fuzzy matrix $A = [a_{ij}]$ is reducible if and only if A is a direct sum of two matrices $J_1 = [1]$ or*

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Proof. Without loss of generality, if an $n \times n$ $(0,1)-$fuzzy matrix $A = [a_{ij}]$ has a $3-$cycle in its graph of A, then for some permutation fuzzy matrix P

$$PAP^T = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

The irreducible submatrix

$$PAP^T[1,2,3|1,2,3] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

is not involution. Therefore A is not involutary fuzzy matrix. □

EXAMPLE 3.11. *Let A is a reducible involution fuzzy matrix. Then for some permutation matrix P*

$$PAP^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

4. The canonical rorm of involutary $(0,1)$-fuzzy matrices

In this section, using the condition and properties in the above sections we give all involution fuzzy matrix from $n = 2$ to $n = 5$ by a special program that construct involution fuzzy zero patterns That is, an $n \times n$ $(0,1)$-fuzzy matrix as n^2-binary digit form.

Remark 4.1. *There exists many zero patterns that is involutary fuzzy matrix for dimension n, we give the zero patterns that is the form $A(p)$ where p means the number of binary digit in the dimension n.*

$[Case,\ n = 2]$ There exist 2's involutary fuzzy matrices as follows;

$$A(6) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad A(9) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$[Case,\ n = 3]$ There exist 4's involutary fuzzy matrices as follows;

$$A(84) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A(161) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} A(266) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A(273) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$[Case,\ n = 4]$ There exist 10's involutary fuzzy matirces as follows;

$$A(4680) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} A(5160) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} A(8580) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$A(9345) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} A(18450) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} A(18465) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$A(33060) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A(33345) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A(33810) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A(33825) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$[Case,\ n=5]$ There exist 26's involutary fuzzy matrices as follows;

$$A(1118480)=\begin{bmatrix}0&0&0&0&1\\0&0&0&1&0\\0&0&1&0&0\\0&1&0&0&0\\1&0&0&0&0\end{bmatrix}A(1187920)=\begin{bmatrix}0&0&0&0&1\\0&1&0&0&0\\0&0&0&1&0\\0&0&1&0&0\\1&0&0&0&0\end{bmatrix}$$

$$A(1312912)=\begin{bmatrix}0&0&0&0&1\\0&0&1&0&0\\0&1&0&0&0\\0&0&0&1&0\\1&0&0&0&0\end{bmatrix}A(1314896)=\begin{bmatrix}0&0&0&0&1\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\1&0&0&0&0\end{bmatrix}$$

$$A(2134536)=\begin{bmatrix}0&0&0&1&0\\0&0&0&0&1\\0&0&1&0&0\\1&0&0&0&0\\0&1&0&0&0\end{bmatrix}A(2236929)=\begin{bmatrix}1&0&0&0&0\\0&0&0&0&1\\0&0&0&1&0\\0&0&1&0&0\\0&1&0&0&0\end{bmatrix}$$

$$A(2360836)=\begin{bmatrix}0&0&1&0&0\\0&0&0&0&1\\1&0&0&0&0\\0&0&0&1&0\\0&1&0&0&0\end{bmatrix}A(2363905)=\begin{bmatrix}1&0&0&0&0\\0&0&0&0&1\\0&0&1&0&0\\0&0&0&1&0\\0&1&0&0&0\end{bmatrix}$$

$$A(4243528)=\begin{bmatrix}0&0&0&1&0\\0&1&0&0&0\\0&0&0&0&1\\1&0&0&0&0\\0&0&1&0&0\end{bmatrix}A(4276481)=\begin{bmatrix}1&0&0&0&0\\0&0&0&1&0\\0&0&0&0&1\\0&1&0&0&0\\0&0&1&0&0\end{bmatrix}$$

$$A(4472866)=\begin{bmatrix}0&1&0&0&0\\1&0&0&0&0\\0&0&0&0&1\\0&0&0&1&0\\0&0&1&0&0\end{bmatrix}A(4472897)=\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&0&0&1\\0&0&0&1&0\\0&0&1&0&0\end{bmatrix}$$

$$A(8913988)=\begin{bmatrix}0&0&1&0&0\\0&1&0&0&0\\1&0&0&0&0\\0&0&0&0&1\\0&0&0&1&0\end{bmatrix}A(8915073)=\begin{bmatrix}1&0&0&0&0\\0&0&1&0&0\\0&1&0&0&0\\0&0&0&0&1\\0&0&0&1&0\end{bmatrix}$$

$$A(8917026) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} A(8917057) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$A(16812168) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} A(16814152) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A(16844036) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} A(16847105) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A(16916514) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} A(16916545) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A(17040452) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} A(17041537) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A(17043490) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} A(17043521) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

References

[1] K. H. Kim and F. W. Roush, *Generalized fuzzy matrices*, Fuzzy Sets and Systems **4** (1980), 293-315.

[2] K. H. Kim and F. W. Roush, *Idempotent fuzzy matrices*, Seminar Report, Alabama State Univ., Montgomery, AL (1981).

[3] H. Hashimoto, *Canonical form of a transitive fuzzy matrix*, Fuzzy Sets and Systems **11** (1983), 157-162.

[4] H. Schneider, *The influence of the marked reduced graph of a nonnegative matrix on the Jordan form and on related properties*, Linear Algebra Appl. **84** (1986), 161-189.
[5] H. Y. Lee and S. W. Park, *The Boolean idempotent matrices*, Journal of Applied Math. and computing. **15** (2004), 475-484.
[6] F. Zhang, *Matrix theory; Basic results and Techniques*, Springer-Verlag New York Inc. **15** (1999).

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