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VECTOR GENERATORS OF THE REAL CLIFFORD ALGEBRA $C\ell_{0,n}$

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Abstract. In this paper, we present new vector generators of a matrix subalgebra $L_{0,n}$, which is isomorphic to the Clifford algebra $C\ell_{0,n}$, and we obtain the matrix form of inverse of a vector in $L_{0,n}$. Moreover, we consider the solution of a linear equation $xg_2 = g_2x$, where g_2 is a vector generator of $L_{0,n}$.

1. Introduction

Let $\mathbb{R}^{p,q}$ be the standard n-dimensional pseudo-Euclidean space endowed with the quadratic form $Q(v) = \sum_{i=1}^{p} v_i^2 - \sum_{i=p+1}^{p+q} v_i^2$ of signature (p, q) with $p + q = n$. Also, let $C\ell_{p,q}$ be the corresponding real Clifford algebra of $\mathbb{R}^{p,q}$.

The Clifford algebras are isomorphic to some matrix algebras. In particular, we constructed the subalgebra $L_{0,n}(\mathbb{R})$ of the $2^n \times 2^n$ real matrix algebra $M_{2^n}(\mathbb{R})$ for every $n \in \mathbb{N}$ which is isomorphic to the real Clifford algebra $C\ell_{0,n}$ and called it the "OE-construction" [2]. Also, we showed $g_2, g_3, g_7, \ldots, g_{2^n-1}$ are vector generators of $L_{0,n}(\mathbb{R})$ and proved some interesting properties.

In section 2, we will show that $g_2, g_4, g_8, \ldots, g_{2^n}$ are another vector generators of $L_{0,n}(\mathbb{R})$.

In section 3, we will prove some interesting properties of the vector generators $g_2, g_4, g_8, \ldots, g_{2^n}$ comparing with those of vector generators $g_2, g_3, g_7, \ldots, g_{2^n-1}$. More concretely, we will calculate the determinant

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of a linear combination of generators in $L_{0,n}(\mathbb{R})$ which are different from those in [2] and we obtain the matrix form of inverse of a vector in $L_{0,n}$.

In section 4, we will consider the existence of solutions for some simple linear equation $xa = ax$ in $L_{0,n}(\mathbb{R})$. In fact, by using the construction of matrix representation in [2], the solution set can be obtained easily in some sense. Furthermore the solution set of the equation can be considered in the Clifford algebra $C\ell_{0,n}$, since $L_{0,n}(\mathbb{R})$ is isomorphic to the Clifford algebra $C\ell_{0,n}$.

2. Generators of the algebra $L_{0,n}(\mathbb{R})$

In [2], we constructed vector generators $g_2, g_3, g_7, \ldots, g_{2^n-1}$ of the subalgebra $L_{0,n}(\mathbb{R})$ of the $2^n \times 2^n$ real matrix algebra $M_{2^n}(\mathbb{R})$ for every $n \in \mathbb{N}$ and proved some interesting properties. In this section, we will show $g_2, g_4, g_8, \ldots, g_{2^n}$ are another vector generators of $L_{0,n}(\mathbb{R})$ and prove some interesting properties comparing with those of vector generators $g_2, g_3, g_7, \ldots, g_{2^n-1}$. First of all, recall some notations given in [2].

NOTATION. Let

$$E = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Moreover, let $K_1 = J$ and

$$K_{m-2} = \begin{pmatrix} O_2 & \cdots & O_2 & J \\ O_2 & \cdots & J & O_2 \\ \vdots & \ddots & \vdots & \vdots \\ J & \cdots & O_2 & O_2 \end{pmatrix} \in M_{2^{m-2}}(\mathbb{R}) ,$$

for $4 \leq m \leq n$. Also, let

$$T_{m-1} = \begin{pmatrix} O_{2^{m-2}} & -K_{m-2} \\ K_{m-2} & O_{2^{m-2}} \end{pmatrix} \in M_{2^{m-1}}(\mathbb{R}) ,$$

for $3 \leq m \leq n$.

REMARK 2.1. By using the above notations, $g_{2^i} \in L_{0,n}(\mathbb{R})$ for $i = 1, 2, \ldots, n$ can be written as follows:

$$g_2 = \begin{pmatrix} E & O_2 & \cdots & O_2 \\ O_2 & E & \cdots & O_2 \\ \vdots & \vdots & \ddots & \vdots \\ O_2 & O_2 & \cdots & E \end{pmatrix} \in M_{2^n}(\mathbb{R}),$$

$$g_{2^{m-1}} = \begin{pmatrix} T_{m-1} & O_{2^{m-1}} & \cdots & O_{2^{m-1}} \\ O_{2^{m-1}} & T_{m-1} & \cdots & O_{2^{m-1}} \\ \vdots & \vdots & \ddots & \vdots \\ O_{2^{m-1}} & O_{2^{m-1}} & \cdots & T_{m-1} \end{pmatrix} \in M_{2^n}(\mathbb{R}),$$

for $3 \leq m \leq n$, and

$$g_{2^n} = \begin{pmatrix} O_2 & \cdots & O_2 & O_2 & \cdots & -J \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ O_2 & \cdots & O_2 & -J & \cdots & O_2 \\ O_2 & \cdots & J & O_2 & \cdots & O_2 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ J & \cdots & O_2 & O_2 & \cdots & O_2 \end{pmatrix} \in M_{2^n}(\mathbb{R}) .$$

Let $\Gamma = \{g_2, g_{2^2}, \ldots, g_{2^n}\}$. Then, $g_{2^i} \in \Gamma$ has the following properties:

LEMMA 2.2. *Let $g_{2^i} \in \Gamma$. Then,*

(1) *g_{2^i} is antisymmetric for all $i = 1, 2, \ldots, n$.*
(2) *$g_{2^i}^2 = -I_{2^n}$ for all $i = 1, 2, \ldots, n$.*

Proof. (1) It is obvious from the definition of g_{2^i}.
(2) Since $E^2 = -I_2$ and $J^2 = I_2$, we obtain $g_{2^i}^2 = -I_{2^n}$ by straightforward computations. □

Moreover, any two elements in Γ are anticommutative as follows:

PROPOSITION 2.3. *For all $i \geq 2$, we have $g_2\, g_{2^i} = -g_{2^i}\, g_2$.*

Proof. For $i = 2$, $g_2\, g_{2^i} = -g_{2^i}\, g_2$, since $EJ = -JE$.
Now, assume that $g_2\, g_{2^k} = -g_{2^k}\, g_2$. Note that

$$g_2 = \begin{pmatrix} E & O_2 & \cdots & O_2 \\ O_2 & E & \cdots & O_2 \\ \vdots & \vdots & \ddots & \vdots \\ O_2 & O_2 & \cdots & E \end{pmatrix}, \quad g_{2^k} = \begin{pmatrix} T_k & O_{2^k} & \cdots & O_{2^k} \\ O_{2^k} & T_k & \cdots & O_{2^k} \\ \vdots & \vdots & \ddots & \vdots \\ O_{2^k} & O_{2^k} & \cdots & T_k \end{pmatrix}.$$

Set

$$g_2^{(k)} = \begin{pmatrix} E & O_2 & \cdots & O_2 \\ O_2 & E & \cdots & O_2 \\ \vdots & \vdots & \ddots & \vdots \\ O_2 & O_2 & \cdots & E \end{pmatrix} \in M_{2^k}(\mathbb{R}).$$

Then, from the equation $g_2\, g_{2^k} = -g_{2^k}\, g_2$, we have

$$g_2^{(k)} T_k = -T_k g_2^{(k)}.$$

Thus,

$$\begin{pmatrix} g_2^{(k-1)} & O_{2^{k-1}} \\ O_{2^{k-1}} & g_2^{(k-1)} \end{pmatrix} \begin{pmatrix} O_{2^{k-1}} & -K_{k-1} \\ K_{k-1} & O_{2^{k-1}} \end{pmatrix} = - \begin{pmatrix} O_{2^{k-1}} & -K_{k-1} \\ K_{k-1} & O_{2^{k-1}} \end{pmatrix} \begin{pmatrix} g_2^{(k-1)} & O_{2^{k-1}} \\ O_{2^{k-1}} & g_2^{(k-1)} \end{pmatrix},$$

which implies that $g_2^{(k-1)} K_{k-1} = -K_{k-1} g_2^{(k-1)}$.

To prove $g_2\, g_{2^{k+1}} = -g_{2^{k+1}}\, g_2$, it is enough to show that $g_2^{(k+1)} T_{k+1} = -T_{k+1} g_2^{(k+1)}$.

Since

$$g_2^{(k+1)} = \begin{pmatrix} g_2^{(k-1)} & O_{2^{k-1}} & O_{2^{k-1}} & O_{2^{k-1}} \\ O_{2^{k-1}} & g_2^{(k-1)} & O_{2^{k-1}} & O_{2^{k-1}} \\ O_{2^{k-1}} & O_{2^{k-1}} & g_2^{(k-1)} & O_{2^{k-1}} \\ O_{2^{k-1}} & O_{2^{k-1}} & O_{2^{k-1}} & g_2^{(k-1)} \end{pmatrix}$$

and

$$T_{k+1} = \begin{pmatrix} O_{2^{k-1}} & O_{2^{k-1}} & O_{2^{k-1}} & -K_{k-1} \\ O_{2^{k-1}} & O_{2^{k-1}} & -K_{k-1} & O_{2^{k-1}} \\ O_{2^{k-1}} & K_{k-1} & O_{2^{k-1}} & O_{2^{k-1}} \\ K_{k-1} & O_{2^{k-1}} & O_{2^{k-1}} & O_{2^{k-1}} \end{pmatrix},$$

we have

$$\begin{aligned} g_2^{(k+1)} T_{k+1} &= \begin{pmatrix} O_{2^{k-1}} & O_{2^{k-1}} & O_{2^{k-1}} & -g_2^{(k-1)} K_{k-1} \\ O_{2^{k-1}} & O_{2^{k-1}} & -g_2^{(k-1)} K_{k-1} & O_{2^{k-1}} \\ O_{2^{k-1}} & g_2^{(k-1)} K_{k-1} & O_{2^{k-1}} & O_{2^{k-1}} \\ g_2^{(k-1)} K_{k-1} & O_{2^{k-1}} & O_{2^{k-1}} & O_{2^{k-1}} \end{pmatrix} \\ &= \begin{pmatrix} O_{2^{k-1}} & O_{2^{k-1}} & O_{2^{k-1}} & K_{k-1} g_2^{(k-1)} \\ O_{2^{k-1}} & O_{2^{k-1}} & K_{k-1} g_2^{(k-1)} & O_{2^{k-1}} \\ O_{2^{k-1}} & -K_{k-1} g_2^{(k-1)} & O_{2^{k-1}} & O_{2^{k-1}} \\ -K_{k-1} g_2^{(k-1)} & O_{2^{k-1}} & O_{2^{k-1}} & O_{2^{k-1}} \end{pmatrix} \\ &= -T_{k+1} g_2^{(k+1)}. \end{aligned}$$

Thus, the equation $g_2^{(k+1)} T_k = -T_k g_2^{(k+1)}$ holds and so the proposition is proved. □

PROPOSITION 2.4. *For every $i, j \geq 2$ with $i \neq j$, we have $g_{2^i}\, g_{2^j} = -g_{2^j}\, g_{2^i}$.*

Proof. Note that

$$g_{2^i} = \begin{pmatrix} T_i & O_{2^i} & \cdots & O_{2^i} \\ O_{2^i} & T_i & \cdots & O_{2^i} \\ \vdots & \vdots & \ddots & \vdots \\ O_{2^i} & O_{2^i} & \cdots & T_i \end{pmatrix}, \quad g_{2^{i+1}} = \begin{pmatrix} T_{i+1} & O_{2^{i+1}} & \cdots & O_{2^{i+1}} \\ O_{2^{i+1}} & T_{i+1} & \cdots & O_{2^{i+1}} \\ \vdots & \vdots & \ddots & \vdots \\ O_{2^{i+1}} & O_{2^{i+1}} & \cdots & T_{i+1} \end{pmatrix}.$$

To prove the equality $g_{2^i}\, g_{2^{i+1}} = -g_{2^{i+1}}\, g_{2^i}$, it is enough to show that the following identity is satisfied:

$$\begin{pmatrix} T_i & O_{2^i} \\ O_{2^i} & T_i \end{pmatrix} T_{i+1} = -T_{i+1} \begin{pmatrix} T_i & O_{2^i} \\ O_{2^i} & T_i \end{pmatrix}.$$

Since

$$\begin{pmatrix} T_i & O_{2^i} \\ O_{2^i} & T_i \end{pmatrix} = \begin{pmatrix} O_{2^{i-1}} & -K_{i-1} & O_{2^{i-1}} & O_{2^{i-1}} \\ K_{i-1} & O_{2^{i-1}} & O_{2^{i-1}} & O_{2^{i-1}} \\ O_{2^{i-1}} & O_{2^{i-1}} & O_{2^{i-1}} & -K_{i-1} \\ O_{2^{i-1}} & O_{2^{i-1}} & K_{i-1} & O_{2^{i-1}} \end{pmatrix}$$

and

$$T_{i+1} = \begin{pmatrix} O_{2^{i-1}} & O_{2^{i-1}} & O_{2^{i-1}} & -K_{i-1} \\ O_{2^{i-1}} & O_{2^{i-1}} & -K_{i-1} & O_{2^{i-1}} \\ O_{2^{i-1}} & K_{i-1} & O_{2^{i-1}} & O_{2^{i-1}} \\ K_{i-1} & O_{2^{i-1}} & O_{2^{i-1}} & O_{2^{i-1}} \end{pmatrix},$$

the following equalities hold:

$$\begin{aligned} \begin{pmatrix} T_i & O_{2^i} \\ O_{2^i} & T_i \end{pmatrix} T_{i+1} &= \begin{pmatrix} O_{2^{i-1}} & O_{2^{i-1}} & K_{i-1}^2 & O_{2^{i-1}} \\ O_{2^{i-1}} & O_{2^{i-1}} & O_{2^{i-1}} & -K_{i-1}^2 \\ -K_{i-1}^2 & O_{2^{i-1}} & O_{2^{i-1}} & O_{2^{i-1}} \\ O_{2^{i-1}} & K_{i-1}^2 & O_{2^{i-1}} & O_{2^{i-1}} \end{pmatrix} \\ &= -T_{i+1} \begin{pmatrix} T_i & O_{2^i} \\ O_{2^i} & T_i \end{pmatrix}. \end{aligned}$$

Thus, the equality $g_{2^i}\, g_{2^{i+1}} = -g_{2^{i+1}}\, g_{2^i}$ is proved.

Now, we assume that the equality $g_{2^i}\, g_{2^{i+k}} = -g_{2^{i+k}}\, g_{2^i}$ is true for a natural number k. Then we show the equality $g_{2^i}\, g_{2^{i+k+1}} = -g_{2^{i+k+1}}\, g_{2^i}$.

Note that

$$g_{2^i} = \begin{pmatrix} T_i & O_{2^i} & \cdots & O_{2^i} \\ O_{2^i} & T_i & \cdots & O_{2^i} \\ \vdots & \vdots & \ddots & \vdots \\ O_{2^i} & O_{2^i} & \cdots & T_i \end{pmatrix}, \quad g_{2^{i+k}} = \begin{pmatrix} T_{i+k} & O_{2^{i+k}} & \cdots & O_{2^{i+k}} \\ O_{2^{i+k}} & T_{i+k} & \cdots & O_{2^{i+k}} \\ \vdots & \vdots & \ddots & \vdots \\ O_{2^{i+k}} & O_{2^{i+k}} & \cdots & T_{i+k} \end{pmatrix}.$$

Now, set

$$g_{2^i}^{(i+k)} = \begin{pmatrix} T_i & O_{2^i} & \cdots & O_{2^i} \\ O_{2^i} & T_i & \cdots & O_{2^i} \\ \vdots & \vdots & \ddots & \vdots \\ O_{2^i} & O_{2^i} & \cdots & T_i \end{pmatrix} \in M_{2^{i+k}}(\mathbb{R}).$$

Then, the equality $g_{2^i}\, g_{2^{i+k}} = -g_{2^{i+k}}\, g_{2^i}$ implies that

$$g_{2^i}^{(i+k)}\, T_{i+k} = -T_{i+k}\, g_{2^i}^{(i+k)},$$

and so

$$g_{2^i}^{(i+k-1)} K_{i+k-1} = -K_{i+k-1} g_{2^i}^{(i+k-1)},$$

since

$$T_{i+k} = \begin{pmatrix} O_{2^{i+k-1}} & -K_{i+k-1} \\ K_{i+k-1} & O_{2^{i+k-1}} \end{pmatrix}.$$

Note that

$$g_{2^i}\, g_{2^{i+k+1}} = \begin{pmatrix} T_i & O_{2^i} & \cdots & O_{2^i} \\ O_{2^i} & T_i & \cdots & O_{2^i} \\ \vdots & \vdots & \ddots & \vdots \\ O_{2^i} & O_{2^i} & \cdots & T_i \end{pmatrix} \begin{pmatrix} T_{i+k+1} & O_{2^{i+k+1}} & \cdots & O_{2^{i+k+1}} \\ O_{2^{i+k+1}} & T_{i+k+1} & \cdots & O_{2^{i+k+1}} \\ \vdots & \vdots & \ddots & \vdots \\ O_{2^{i+k+1}} & O_{2^{i+k+1}} & \cdots & T_{i+k+1} \end{pmatrix}$$

and

$$T_{i+k+1} = \begin{pmatrix} O_{2^{i+k-1}} & O_{2^{i+k-1}} & O_{2^{i+k-1}} & -K_{i+k-1} \\ O_{2^{i+k-1}} & O_{2^{i+k-1}} & -K_{i+k-1} & O_{2^{i+k-1}} \\ O_{2^{i+k-1}} & K_{i+k-1} & O_{2^{i+k-1}} & O_{2^{i+k-1}} \\ K_{i+k-1} & O_{2^{i+k-1}} & O_{2^{i+k-1}} & O_{2^{i+k-1}} \end{pmatrix}.$$

Thus, the equality

$$g_{2^i}\, g_{2^{i+k+1}} = -g_{2^{i+k+1}}\, g_{2^i}$$

holds, since $g_{2^i}^{(i+k-1)} K_{i+k-1} = -K_{i+k-1} g_{2^i}^{(i+k-1)}$. □

From Lemma 2.2 (2), Proposition 2.3 and Proposition 2.4, we obtain the main result as follows:

THEOREM 2.5. *$g_2, g_4, g_8, \ldots, g_{2^n}$ are vector generators of $L_{0,n}(\mathbb{R})$, which is isomorphic to the Clifford algebra $C\ell_{0,n}$.*

REMARK 2.6. Recall the Pauli spin matrices $\sigma_1, \sigma_2, \sigma_3$ defined by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and let $\sigma_4 = \sigma_1\sigma_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. From the process of "OE-construction" given in [2], we can express g_2, g_4, g_8 in $L_{0,3}(\mathbb{R})$ by means of tensor products. That is to say,

$$g_2 = I_2 \otimes I_2 \otimes \sigma_4 \ , \quad g_4 = I_2 \otimes \sigma_4 \otimes \sigma_1 \ , \quad g_8 = \sigma_4 \otimes \sigma_1 \otimes \sigma_1.$$

Since $\sigma_1^2 = I_2, \sigma_4^2 = -I_2$ and $\sigma_1\sigma_4 = -\sigma_4\sigma_1$, we obtain that $g_{2^i}^2 = -I_8$, and $g_{2^i}g_{2^j} = -g_{2^j}g_{2^i}$ for $i \neq j$.

3. Inverse of vectors in $L_{0,n}(\mathbb{R})$

In this section, we will present the matrix representation of the inverse of a vector in $L_{0,n}(\mathbb{R})$. Set $\Delta = \{\sum_{i=1}^n a_i g_{2^i} : a_i \in \mathbb{R}, i = 1, 2, \ldots, n\}$, the set of all vectors of $L_{0,n}(\mathbb{R})$ in section 2. Then, the matrix $A \in \Delta$ satisfies some interesting properties as follows:

Proposition 3.1. *Let $A = \sum_{i=1}^n a_i g_{2^i} \neq O_{2^n} \in \Delta$. Then,*

(1) $\det(A) = \pm(\sum_{i=1}^n a_i^2)^{2^{n-1}}$.
(2) $A^{-1} = \frac{-1}{\sum_{i=i}^n a_i^2} A$.

Proof. (1) Since g_{2^i} is antisymmetric, $A^T = -\sum_{i=1}^n a_i g_{2^i}$. Also, by proposition 2.3, 2.4,

$$AA^T = (\sum_{i=1}^n a_i^2) I_{2^n}.$$

Thus,

$$\det(A)^2 = (\sum_{i=1}^n a_i^2)^{2^n}$$

and so

$$\det(A) = \pm(\sum_{i=1}^n a_i^2)^{2^{n-1}}.$$

(2) Since $AA^T = (\sum_{i=1}^n a_i^2) I_{2^n}$ and $A^T = -A$, the identity $A^{-1} = \frac{-1}{\sum_{i=1}^n a_i^2} A$ can be obtained. □

Example 3.2. *Let $n = 7$ and $A = g_2 - 3g_{16} + 2g_{64} + g_{128} \in L_{0,7}(\mathbb{R})$. Then*

$$\det(A) = (1^2 + (-3)^2 + 2^2 + 1^2)^{2^6} = 15^{64}.$$

Note that by using proposition 3.1, we can show that $A = \sum_{i=1}^n a_i g_{2^i} \neq O_{2^n} \in \Delta$ is an element in the Clifford group.

4. Existence of solutions for a linear equation $xa = ax$ in $L_{0,n}(\mathbb{R})$.

Now, we will consider the existence of solutions for a simple linear equation $xa = ax$ in $L_{0,n}(\mathbb{R})$. In fact, by using the matrix representation in [2], the solution set can be obtained easily in some sense. Furthermore, the solution set of the equation can be considered in the Clifford algebra $C\ell_{0,n}$, since $L_{0,n}(\mathbb{R})$ is isomorphic to the Clifford algebra $C\ell_{0,n}$.

THEOREM 4.1. *For $g_2 \in \Delta$, the equation $xg_2 = g_2x$ has solutions in $L_{0,n}(\mathbb{R})$ and the solution set of the equation in $L_{0,n}(\mathbb{R})$ is*

$$\left\{ \sum_{m=0}^{2^{n-2}-1} a_m g_{4m+1} + \sum_{m=0}^{2^{n-2}-1} b_m g_{4m+2} \mid a_m, b_m \in \mathbb{R} \right\}.$$

Proof. Let

$$x = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix} \in M_{2^n}(\mathbb{R}) \ ,$$

where $x_{ij} \in M_2(\mathbb{R})$ for all $1 \leq i, j \leq n$.

Then, the equation $xg_2 = g_2x$ is equivalent with $x_{ij}E = Ex_{ij}$ for all $1 \leq i, j \leq n$. Then we have

$$x_{ij} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix},$$

for some $a, b \in \mathbb{R}$, which implies that all the entries x_{i1} of the first column of x are of odd type. From the construction of $L_{0,n}$ in [2], we obtain $x_{2j1} = O_2$ for all $j = 1, 2, \ldots, n$. Thus, x is expressed by

$$x = \sum_{m=0}^{2^{n-2}-1} a_m g_{4m+1} + \sum_{m=0}^{2^{n-2}-1} b_m g_{4m+2},$$

for some $a_m, b_m \in \mathbb{R}$. □

EXAMPLE 4.2. *Let $n = 3$. Then, the equation $xg_2 = g_2x$ has solutions in $L_{0,3}(\mathbb{R})$ and the solution set of the equation in $L_{0,3}(\mathbb{R})$ is*

$$\{a_0g_1 + b_0g_2 + a_5g_5 + b_6g_6 \mid a_0, a_1, b_0, b_1 \in \mathbb{R}\}.$$

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