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CURVES OF MAXIMAL GENUS ON SURFACE SCROLLS

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Abstract. We investigate a minimal set of generators of a homogeneous ideal of a curve of degree d on a linearly normal smooth surface scroll W in $\mathbf{P}^r$ whose arithmetic genus is maximal among curves of degree d on W.

1. Introduction

For given two integers d, r with $d \geq r \geq 3$, Castelnuovo [2] proved that there is an upper bound $\pi(d,r)$ for the arithmetic genus of irreducible, nondegenerate curves of degree d in $\mathbf{P}^r$. Here $\pi(d,r)$ is given by

$$\pi(d,r) = \binom{m}{2}(r-1) + m\epsilon$$

where $d = m(r-1) + \epsilon + l$, $\epsilon = 0, \ldots, r-2$. He also classified curves for which the bound is attained.

In this paper we will investigate a curve lying on a linearly normal smooth surface scroll W in $\mathbf{P}^r$ whose arithmetic genera are maximal among curves of degree d on W; Theorem 2.3. Especially we are interested in a minimal set of generators of homogeneous ideals of such curves; Proposition 2.5.

Every linearly normal surface scroll in $\mathbf{P}^r$ is the image of the unique ruled surface by an embedding defined by a unisecant linear series on the ruled surface. So a degree of a curve on W is given by a fixed ample divisor H on the ruled surface which gives the embedding of the ruled surface into $\mathbf{P}^r$. So the paper deals with a curve on a ruled surface with a fixed ample divisor on the ruled surface.

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2. Generators of homogeneous ideals of maximal curves

Let Y be a smooth curve of genus $g_y \geq 1$. Let $\pi : \mathbf{P}(\mathcal{E}) \to Y$ be a ruled surface over Y with the e-invariant e and let S_0 be a *minimal degree section* of $\mathbf{P}(\mathcal{E})$, that is, S_0 is the section of $\mathbf{P}(\mathcal{E})$ with $S_0^2 = -e$. Assume that $\mathcal{E}_0 = \mathcal{E} \otimes \mathcal{O}_Y(-N)$ is normalized. Setting $n = \deg N$ and $\mathcal{O}_Y(B) = \det \mathcal{E}$ with $b = \deg B$, we have

$$e = -\deg \mathcal{E}_0 = 2n - b. \tag{2.1}$$

Throughout this paper, fix a divisor $Z \in \mathrm{Div}(Y)$ with

$$z := \deg Z \geq \max\{2g_y + 1, 2g_y + 1 + e\}.$$

and set

$$H = S_0 + \pi^* Z \text{ and } r = \dim |H|.$$

The *H-degree* of a curve $X \subset \mathbf{P}(\mathcal{E})$ is defined by the intersection number $X.H$.

Lemma 2.1. *The linear series $|H|$ on $\mathbf{P}(\mathcal{E})$ is very ample and*

$$r = \dim |H| = -e + 2z - 2g_y + 1. \tag{2.2}$$

Proof. By [3, V, Ex. 2.11], the linear series $|H|$ is very ample. We now count $h^0(\mathbf{P}(\mathcal{E}), H)$. Since $\pi_*(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(S_0)) \cong \mathcal{E}_0$ by [3, V, 2.4], it follows by the projection formula that

$$\begin{aligned} h^0(\mathbf{P}(\mathcal{E}), H) &= h^0(Y, \mathcal{E}_0 \otimes \mathcal{O}_Y(Z)) \\ &= \deg(\mathcal{E}_0 \otimes \mathcal{O}_Y(Z)) - 2(g_y - 1) + h^1(Y, \mathcal{E}_0 \otimes \mathcal{O}_Y(Z)) \\ &= -e + 2z - 2g_y + 2 + h^0(Y, \mathcal{E}_0^\vee \otimes \mathcal{O}_Y(K_Y - Z)). \end{aligned}$$

It is enough to prove that $h^0(Y, \mathcal{E}_0^\vee \otimes \mathcal{O}_Y(K_Y - Z)) = 0$. Note that we have

$$\mathcal{E}_0^\vee \cong \mathcal{E}_0 \otimes \det \mathcal{E}_0{}^{-1} \cong \mathcal{E}_0 \otimes (\det \mathcal{E}^{-1} \otimes \mathcal{O}_Y(2N)) = \mathcal{E}_0 \otimes \mathcal{O}_Y(-B + 2N);$$

hence, it follows that

$$h^0(Y, \mathcal{E}_0^\vee \otimes \mathcal{O}_Y(K_Y - Z)) = h^0(Y, \mathcal{E}_0 \otimes \mathcal{O}_Y(K_Y - Z - B + 2N)).$$

By Equation (2.1) and the assumption $z - e \geq 2g_y + 1$, we have

$$\deg(K_Y - Z - B + 2N) = 2g_y - 2 - (z - e) < 0;$$

however, $\mathcal{E}_0$ is normalized, hence

$$h^0(Y, \mathcal{E}_0 \otimes \mathcal{O}_Y(K_Y - Z - B + 2N)) = 0.$$

□

Remark 2.2. Let

$$\phi_H : \mathbf{P}(\mathcal{E}) \hookrightarrow \mathbf{P}^r$$

be an embedding defined by $|H|$ and set

$$W = \phi_H(\mathbf{P}(\mathcal{E})).$$

Since $H^2 = -e + 2z$, the surface scroll W is of degree $(r - 1 + 2g_y)$ in $\mathbf{P}^r$.

Theorem 2.3. *The maximal arithmetic genus of curves of degree d lying on $W = \phi_H(\mathbf{P}(\mathcal{E}))$ in $\mathbf{P}^r$ is equal to*

$$\binom{m}{2}(r - 1 + 2g_y) + m\varepsilon + g_y, \tag{2.3}$$

where

$$m = \left[\frac{d - 1 + g_y}{r - 1 + 2g_y}\right] \tag{2.4}$$

is the greatest integer not exceeding $(d - 1 + g_y)/(r - 1 + 2g_y)$ and

$$\varepsilon = (d - 1 + g_y) - (r - 1 + 2g_y)m.$$

Proof. Let X be a curve on $\mathbf{P}(\mathcal{E})$ of H-degree d. Set $X \sim iS_0 + \pi^* J$ for some $J \in \mathrm{Div}(Y)$. We have

$$d = X.H = -ie + iz + j,$$

where $j = \deg J$. By the adjunction formula, we get

$$p_a(X) = (i - 1)\left(d - 1 + g_y - \frac{i}{2}(r - 1 + 2g_y)\right) + g_y.$$

An elementary calculation shows that $p_a(X)$ is maximized for fixed d and r when

$$i = \left[\frac{d - 1 + g_y}{r - 1 + 2g_y}\right] + 1.$$

□

Remark 2.4. If $X \subset \mathbf{P}(\mathcal{E})$ is a curve of H-degree d with

$$p_a(X) = \binom{m}{2}(r - 1 + 2g_y) + m\varepsilon + g_y,$$

then we have

$$X \sim (m + 1)H + \pi^*(J - (m + 1)Z). \tag{2.5}$$

Note that

$$\deg(J - (m + 1)Z) = -r + \varepsilon + 2 - 3g_y. \tag{2.6}$$

Proposition 2.5. *Assume that* $z \geq 3+e$ *if* $g_y = 1$ *and* $z \geq \max\{3g_y + e, 3g_y + \frac{e}{2}\}$ *if* $g_y \geq 2$. *Let* $X \subset \mathbf{P}(\mathcal{E})$ *be a curve of* H*-degree* d *with the maximal arithmetic genus*

$$p_a(X) = \binom{m}{2}(r - 1 + 2g_y) + m\varepsilon.$$

Let $I \subset \mathbb{C}[X_0, \ldots, X_r]$ *be the homogeneous ideal defining* X *under the embedding by* $|H|$. *If* $|(m+1)Z - J|$ *is base-point-free, then a minimal set of generators for* I *consists of quadrics and polynomials of degree* $m+1$. *On the other hand, if* $|(m+1)Z - J|$ *has a base point, then a minimal set of generators for* I *consists of quadrics, polynomials of degree* $m+1$, *and, in addition, polynomials of degree* $m+2$.

Remark 2.6. By [4] and the assumption on the degree z, the very ample linear series $|H|$ satisfies N_1 property, that is, the embedding ϕ_H is a projectively normal embedding and the homogeneous ideal of W is generated by quadrics.

We divide the proof of Proposition 2.5 into the following three lemmas.

Lemma 2.7. *Every hypersurface of degree* $l \leq m$ *containing* X *contains* W.

Proof. Consider the short exact sequence of ideal sheaves

$$0 \to \mathcal{I}_{W,\mathbf{P}^r}(l) \to \mathcal{I}_{X,\mathbf{P}^r}(l) \to \mathcal{I}_{X,W}(l) \to 0.$$

Since $l < m+1$, we have

$$\begin{aligned} H^0(W, \mathcal{I}_{X,W}(l)) &= H^0(\mathbf{P}(\mathcal{E}), -X + lH) \\ &= H^0(\mathbf{P}(\mathcal{E}), (l-m-1)S_0 + \pi^*(lZ - J)) = 0. \end{aligned}$$

Therefore

$$H^0(\mathbf{P}^r, \mathcal{I}_{X,\mathbf{P}^r}(l)) = H^0(\mathbf{P}^r, \mathcal{I}_{W,\mathbf{P}^r}(l))$$

for $l \leq m$. □

Lemma 2.8. *Under the hypothesis of Proposition 2.5, modulo the ideal of* W, *there are exactly* $h^0(Y, (m+1)Z - J)$ *linearly independent hypersurfaces of degree* $m+1$ *containing* X.

Proof. By Remark 2.6, we have

$$H^1(\mathbf{P}^r, \mathcal{I}_{W,\mathbf{P}^r}(m+1)) = 0.$$

From the short exact sequence of ideal sheaves, we have

$$0 \to H^0(\mathbf{P}^r, \mathcal{I}_{W,\mathbf{P}^r}(m+1)) \to H^0(\mathbf{P}^r, \mathcal{I}_{X,\mathbf{P}^r}(m+1)) \\ \to H^0(W, \mathcal{I}_{X,W}(m+1)) \to 0.$$

Since

$$H^0(W, \mathcal{I}_{X,W}(m+1)) \cong H^0(Y, (m+1)Z - J),$$

the proof is done. □

LEMMA 2.9. *Under the hypothesis of Proposition 2.5, the natural map*

$$H^0(W, \mathcal{I}_{X,W}(m+2)) \otimes H^0(W, \mathcal{O}_W(\alpha)) \to H^0(W, \mathcal{I}_{X,W}(m+2+\alpha))$$

is surjective for any $\alpha > 0$. Furthermore, if $|(m+1)Z - J|$ is base-point-free, then the natural map

$$H^0(W, \mathcal{I}_{X,W}(m+1)) \otimes H^0(W, \mathcal{O}_W(\alpha)) \to H^0(W, \mathcal{I}_{X,W}(m+1+\alpha))$$

is also surjective for any $\alpha > 0$.

Proof. Since

$$\mathcal{I}_{X,W}(m+2) \cong \mathcal{O}_{\mathbf{P}(\mathcal{E})}(S_0 + \pi^*((m+2)Z - J)),$$

we have to prove that the map

$$H^0(\mathbf{P}(\mathcal{E}), S_0 + \pi^*((m+2)Z - J)) \otimes H^0(\mathbf{P}(\mathcal{E}), \alpha S_0 + \pi^*(\alpha Z)) \\ \longrightarrow H^0(\mathbf{P}(\mathcal{E}), (\alpha+1)S_0 + \pi^*((m+2)Z - J + \alpha Z))$$

is surjective. Hence, it is equivalent to prove that the following map is surjective:

$$H^0(Y, \mathcal{E}_0 \otimes \mathcal{O}_Y((m+2)Z - J)) \otimes H^0(Y, \operatorname{sym}^\alpha \mathcal{E}_0 \otimes \mathcal{O}_Y(\alpha Z)) \\ \longrightarrow H^0(Y, \mathcal{E}_0 \otimes \operatorname{sym}^\alpha \mathcal{E}_0 \otimes \mathcal{O}_Y((m+2)Z - J + \alpha Z)).$$

For this, we have the following result:

LEMMA 2.10 ([1, Proposition 2.2]). *Let $\mathcal{F}$ and $\mathcal{G}$ be vector bundles over Y with $\mathcal{F}$ generated by global sections. Let $\mu^-(\mathcal{L}) := \min\{\mu(\mathcal{Q}) : \mathcal{L} \to \mathcal{Q} \to 0\}$, where $\mu(\mathcal{Q}) = \deg \mathcal{Q}/\operatorname{rank} \mathcal{Q}$. If*

1. *$\mu^-(\mathcal{G}) > 2g_y$ and*
2. *$\mu^-(\mathcal{G}) > 2g_y + \operatorname{rank}(\mathcal{F})(2g_y - \mu^-(\mathcal{F})) - 2h^1(Y, \mathcal{F})$,*

then the natural multiplication map

$$\tau : H^0(Y, \mathcal{F}) \otimes H^0(Y, \mathcal{G}) \to H^0(Y, \mathcal{F} \otimes \mathcal{G}),$$

is surjective.

Set $\mathcal{F} = \mathcal{E}_0 \otimes \mathcal{O}_Y((m+2)Z - J)$ and $\mathcal{G} = \operatorname{sym}^\alpha \mathcal{E}_0 \otimes \mathcal{O}_Y(\alpha Z)$. First, we will show that $\mathcal{F}$ is generated by global sections. By [1, Lemma 1.12], a vector bundle $\mathcal{L}$ on Y is generated by global sections if $\mu^-(\mathcal{L}) \geq 2g_y$; hence it is enough to show that

$$\mu^-(\mathcal{F}) \geq 2g_y.$$

Note that $\mu^-(\mathcal{E}_0) = -e$ if $e \geq 0$ and $\mu^-(\mathcal{E}_0) = -\frac{e}{2}$ if $e < 0$; cf. [4, §2.4]. From [1, Lemma 2.5], we have $\mu^-(\mathcal{L} \otimes \mathcal{M}) = \mu^-(\mathcal{L}) + \mu^-(\mathcal{M})$ for vector bundles $\mathcal{L}$ and $\mathcal{M}$ on Y. Therefore we have

$$\begin{aligned}
\mu^-(\mathcal{F}) &= \mu^-(\mathcal{E}_0) + \mu^-(\mathcal{O}_Y((m+2)Z - J)) \\
&= \begin{cases} -e + r - \varepsilon - 2 + 3g_y + z & \text{if } e \geq 0, \\ -\dfrac{e}{2} + r - \varepsilon - 2 + 3g_y + z & \text{if } e < 0 \end{cases} && \text{by Equation (2.6)} \\
&\geq \begin{cases} -e + g_y + z & \text{if } e \geq 0, \\ -\dfrac{e}{2} + g_y + z & \text{if } e < 0 \end{cases} && \text{since } \varepsilon \leq r - 2 + 2g_y \\
&\geq 2g_y. && \text{by the assumption on } z
\end{aligned}$$

Second, we will prove that $\mu^-(\mathcal{G}) > 2g_y$. From [1, Lemma 2.5], we have $\mu^-(\operatorname{sym}^\alpha \mathcal{E}_0) = \alpha\mu^-(\mathcal{E}_0)$. Therefore we have

$$\begin{aligned}
\mu^-(\mathcal{G}) &= \alpha\mu^-(\mathcal{E}_0) + \mu^-(\mathcal{O}_Y(\alpha Z)) \\
&= \begin{cases} -\alpha e + \alpha z & \text{if } e \geq 0, \\ -\alpha \cdot \dfrac{e}{2} + \alpha z & \text{it } e \leq -1 \end{cases} \\
&> 2g_y.
\end{aligned}$$

Finally, since $2g_y - \mu^-(\mathcal{F}) \leq 0$, the condition (2) of Lemma 2.10 holds. So fat, we proved the first assertion of Lemma 2.9.

For the second assertion, we have

$$\mathcal{I}_{X,W}(m+1) \cong \mathcal{O}_{\mathbf{P}(\mathcal{E})}(\pi^*((m+1)Z - J));$$

hence it is equivalent to prove that the following map is surjective:

$$\begin{aligned}
&H^0(Y, (m+1)Z - J) \otimes H^0(Y, \operatorname{sym}^\alpha \mathcal{E}_0 \otimes \mathcal{O}_Y(\alpha Z)) \\
&\qquad \longrightarrow H^0(Y, \operatorname{sym}^\alpha \mathcal{E}_0 \otimes \mathcal{O}_Y((m+1)Z - J + \alpha Z)).
\end{aligned}$$

Set $\mathcal{F} = \mathcal{O}_Y((m+1)Z - J)$ and $\mathcal{G} = \operatorname{sym}^\alpha \mathcal{E}_0 \otimes \mathcal{O}_Y(\alpha Z)$. Since we assumed that $\mathcal{F}$ is generated by global sections, the second assertion also follows by Lemma 2.10. $\square$

Remark 2.11. It is clear that the natural map

$$H^0(W, \mathcal{I}_{X,W}(m+1)) \otimes H^0(W, \alpha H) \to H^0(W, \mathcal{I}_{X,W}(m+1+\alpha))$$

cannot be surjective if $|(m+1)Z - J|$ has a base point.

References

[1] D. C. Butler, *Normal generation of vector bundles over a curve*, J. Differential Geom. **39** (1994), no. 1, 1-34.

[2] G. Castelnuovo, *Ricerche di geometria sulle curve algebriche*, Ani R. Accad. Sci. Torino **24** (1889)

[3] R. Hartshorne, *Algebraic geometry*, Springer-Verlag, 1977, Graduate Texts in Mathematics, No. 52.

[4] E. Park, *On higher syzygies of ruled surfaces*, Trans. Amer. Math. Soc. **358** (2006), no. 8, 3733-3749.

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