J. Korean Math. Soc. **51** (2014), No. 6, pp. 1291–1304
http://dx.doi.org/10.4134/JKMS.2014.51.6.1291

A NOTE ON KADIRI'S EXPLICIT ZERO FREE REGION FOR RIEMANN ZETA FUNCTION

Woo-Jin Jang and Soun-Hi Kwon

Abstract. In 2005 Kadiri proved that the Riemann zeta function $\zeta(s)$ does not vanish in the region

$$\mathrm{Re}(s) \geq 1 - \frac{1}{R_0 \log|\mathrm{Im}(s)|}, \ |\mathrm{Im}(s)| \geq 2$$

with $R_0 = 5.69693$. In this paper we will show that R_0 can be taken $R_0 = 5.68371$ using Kadiri's method together with Platt's numerical verification of Riemann Hypothesis.

1. Introduction

It is well known that the distribution of prime numbers is deeply related with the zeros of the Riemann zeta function $\zeta(s)$. Let $\pi(x)$ be the number of primes up to x. Hadamard and de la Vallée Poussin proved the prime number theorem which states that $\pi(x)$ is asymptotic to $Li(x) = \int_2^x \frac{dt}{\log t}$. In 1899, de la Vallée Poussin proved that $\zeta(s)$ does not vanish in the region

$$\mathrm{Re}(s) \geq 1 - \frac{1}{R_0 \log|\mathrm{Im}(s)|}, \ |\mathrm{Im}(s)| \geq 2$$

with $R_0 = 34.82$. From this zero-free region for $\zeta(s)$ the error term $\pi(x) - Li(x)$ can be estimated:

$$\pi(x) - Li(x) = O\left(x \ \exp\left(-\sqrt{\frac{\log x}{R_0}}\right)\right) \text{ when } x \to \infty.$$

See [18] and Section 1 of [7]. The constant R_0 has been improved by many authors, particularly by Rosser [13, 14, 15], Schoenfeld [14, 15], Stechkin [16], Ford [1, 2], and Kadiri [7]. Recently Kadiri improved significantly on R_0.

Received March 14, 2014.
2010 *Mathematics Subject Classification.* Primary 11M06, 11M26; Secondary 11Y35.
Key words and phrases. Riemann zeta function, zero-free regions, Riemann Hypothesis.
This research was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education (NRF-2010-0009540).

Theorem 1 (Kadiri [7]). *$\zeta(s)$ does not vanish in the region*

$$\mathrm{Re}(s) \geq 1 - \frac{1}{R_0 \log|\mathrm{Im}(s)|}, \ |\mathrm{Im}(s)| \geq 2$$

with $R_0 = 5.69693$.

The aim of this paper is to show that R_0 can be slightly reduced to 5.68371. In order to enlarge zero-free regions for $\zeta(s)$ Kadiri used Weil's explicit formula together with a generalization of Stechkin's work. In [6] Heath-Brown used Weil's explicit formula to get better zero-free regions for Dirichlet L-functions. Kadiri adapted Heath-Brown's method for $\zeta(s)$. Moreover, Heath-Brown suggested several test functions with which one can use Weil's explicit formula. Kadiri used one of those functions and Wedeniwski's numerical verification of the Riemann Hypothesis to prove Theorem 1. In this paper we will use Kadiri's method with different test functions together with Platt's numerical verification of the Riemann Hypothesis and obtain $R_0 = 5.68371$. For the numerical verification of the Riemann Hypothesis, see van de Lune, te Riele, and Winter [10], Wedeniwski [19], Gourdon [3], Platt [11, 12], and Kadiri [7, 8].

The structure of this paper is as follows. In Section 2 we review briefly some of well-known patterns for studying the zeros of $\zeta(s)$. In Section 3 we present Kadiri's method. In Section 4 we give our numerical results using various test functions and trigonometric inequalities.

2. The logarithmic derivative of $\zeta(s)$ and the zeros of $\zeta(s)$

To begin with we consider the logarithmic derivative of $\zeta(s)$

$$-\frac{\zeta'}{\zeta}(s) = \sum_{n\geq 1} \frac{\Lambda(n)}{n^s},$$

where $\Lambda(n)$ is the von Mangoldt function. Using the functional equation for $\zeta(s)$ and the Hadamard product formula we get

$$-\mathrm{Re}\frac{\zeta'}{\zeta}(s) = \frac{-\log\pi}{2} + \frac{1}{2}\mathrm{Re}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}+1\right) + \mathrm{Re}\left(\frac{1}{s-1}\right) - \sum_{\rho\in Z(\zeta)} \mathrm{Re}\left(\frac{1}{s-\rho}\right),$$

where $Z(\zeta)$ denotes the set of non-trivial zeros of $\zeta(s)$. Let $\rho_0 = \beta_0 + i\gamma_0$ be some particular zero with $|\gamma_0| \geq 2$. Note that if $\mathrm{Re}(s) > 1$, then $\mathrm{Re}\frac{1}{s-\rho} > 0$. Discarding $-\sum_{\substack{\rho\in Z(\zeta)\\ \rho\neq\rho_0}} \mathrm{Re}\frac{1}{s-\rho} < 0$ we obtain

$$-\mathrm{Re}\frac{\zeta'}{\zeta}(s) \leq \frac{-\log\pi}{2} + \frac{1}{2}\mathrm{Re}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}+1\right) + \mathrm{Re}\left(\frac{1}{s-1}\right) - \mathrm{Re}\left(\frac{1}{s-\rho_0}\right)$$

for s with $\mathrm{Re}(s) > 1$. Let $P(v) = \sum_{k=0}^{K} a_k \cos(kv) \geq 0$ with $a_k \geq 0$ and let $s = \sigma + it$. Then

$$\sum_{k=0}^{K} a_k\left(-\mathrm{Re}\frac{\zeta'}{\zeta}(\sigma+ikt)\right) = \mathrm{Re}\sum_{k=0}^{K} a_k \sum_{n\geq 1}\frac{\Lambda(n)}{n^{\sigma+ikt}}$$

$$= \sum_{k=0}^{K} a_k \sum_{n\geq 1} \frac{\Lambda(n)}{n^\sigma} \cos(kt\log n) \geq 0.$$

Let us consider for example the classical inequality

$$P(v) = 3 + 4\cos v + \cos(2v) = 2(1+\cos v)^2 \geq 0.$$

Suppose that $\sigma > 1$. We have

$$-\frac{\zeta'}{\zeta}(\sigma) \leq \frac{-\log\pi}{2} + \frac{1}{2}\mathrm{Re}\frac{\Gamma'}{\Gamma}\left(\frac{\sigma}{2}+1\right) + \frac{1}{\sigma-1} - \mathrm{Re}\left(\frac{1}{\sigma-\rho_0}\right),$$

$$-\mathrm{Re}\frac{\zeta'}{\zeta}(\sigma+i\gamma_0) \leq \frac{-\log\pi}{2} + \frac{1}{2}\mathrm{Re}\frac{\Gamma'}{\Gamma}\left(\frac{\sigma+i\gamma_0}{2}+1\right) + \mathrm{Re}\left(\frac{1}{\sigma+i\gamma_0-1}\right) - \frac{1}{\sigma-\beta_0},$$

and

$$-\mathrm{Re}\frac{\zeta'}{\zeta}(\sigma+i2\gamma_0) \leq \frac{-\log\pi}{2} + \frac{1}{2}\mathrm{Re}\frac{\Gamma'}{\Gamma}\left(\frac{\sigma+i2\gamma_0}{2}+1\right) + \mathrm{Re}\left(\frac{1}{\sigma+i2\gamma_0-1}\right) - \mathrm{Re}\left(\frac{1}{\sigma-\beta_0+i\gamma_0}\right).$$

From

$$\mathrm{Re}\left(-3\frac{\zeta'}{\zeta}(\sigma) - 4\frac{\zeta'}{\zeta}(\sigma+i\gamma_0) - \frac{\zeta'}{\zeta}(\sigma+i2\gamma_0)\right) \geq 0$$

we get

$$\frac{4}{\sigma-\beta_0} \leq \frac{3}{\sigma-1} + c_1\log|\gamma_0|,$$

where c_1 is some positive constant that can be estimated explicitly. If we choose σ optimally, then we obtain an upper bound for β_0. See Chapter III of [17], Section 1 of [15], Section 4 of [6], and § 8 of [9].

In [16] Stechkin proved the following.

Stechkin's Lemma. *Let $\beta \in [1/2, 1]$, $y > 0$, $\sigma > 1$ and let $\tau = (1+\sqrt{1+4\sigma^2})/2$. Then*

$$\mathrm{Re}\left[\frac{1}{\sigma-\beta+iy} + \frac{1}{\sigma-1+\beta+iy} - \frac{1}{\sqrt{5}}\left(\frac{1}{\tau-\beta+iy} + \frac{1}{\tau-1+\beta+iy}\right)\right] \geq 0.$$

If ρ is a zero of $\zeta(s)$, then $\bar\rho$ and $1-\bar\rho$ are also zeros of $\zeta(s)$. In view of Stechkin's lemma we have

$$\mathrm{Re}\left[\frac{1}{\sigma+it-\rho} + \frac{1}{\sigma+it-1+\bar\rho} - \frac{1}{\sqrt{5}}\left(\frac{1}{\sigma+\delta+it-\rho} + \frac{1}{\sigma+\delta+it-(1-\bar\rho)}\right)\right] \geq 0,$$

where $\delta = \tau - \sigma$. We consider

$$\begin{aligned} &-\mathrm{Re}\frac{\zeta'}{\zeta}(s) + \frac{1}{\sqrt{5}}\mathrm{Re}\frac{\zeta'}{\zeta}(s+\delta) \\ &= \frac{-\log\pi}{2}\left(1-\frac{1}{\sqrt{5}}\right) + \frac{1}{2}\left[\mathrm{Re}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}+1\right) - \frac{1}{\sqrt{5}}\mathrm{Re}\frac{\Gamma'}{\Gamma}\left(\frac{s+\delta}{2}+1\right)\right] \end{aligned}$$

$$+\,\mathrm{Re}\frac{1}{s-1}-\frac{1}{\sqrt{5}}\mathrm{Re}\frac{1}{s+\delta-1}-\frac{1}{2}\sum_{\rho\in Z(\zeta)}\mathrm{Re}\bigg[\frac{1}{s-\rho}+\frac{1}{s-(1-\bar{\rho})}$$
$$-\frac{1}{\sqrt{5}}\bigg(\frac{1}{s+\delta-\rho}+\frac{1}{s+\delta-1+\bar{\rho}}\bigg)\bigg].$$

One may discard the sum

$$\sum_{\substack{\rho\in Z(\zeta)\\ \rho\neq\rho_0,\ 1-\bar{\rho_0}}}\mathrm{Re}\bigg[\frac{1}{s-\rho}+\frac{1}{s-(1-\bar{\rho})}-\frac{1}{\sqrt{5}}\bigg(\frac{1}{s+\delta-\rho}+\frac{1}{s+\delta-1+\bar{\rho}}\bigg)\bigg]\geq 0$$

and get

$$-\,\mathrm{Re}\bigg(\frac{\zeta'}{\zeta}(s)-\frac{1}{\sqrt{5}}\mathrm{Re}\frac{\zeta'}{\zeta}(s+\delta)\bigg)$$
$$\leq\frac{-\log\pi}{2}\bigg(1-\frac{1}{\sqrt{5}}\bigg)+\frac{1}{2}\mathrm{Re}\bigg[\frac{\Gamma'}{\Gamma}\bigg(\frac{s}{2}+1\bigg)-\frac{1}{\sqrt{5}}\frac{\Gamma'}{\Gamma}\bigg(\frac{s+\delta}{2}+1\bigg)\bigg]$$
$$+\,\mathrm{Re}\bigg(\frac{1}{s-1}-\frac{1}{\sqrt{5}}\frac{1}{s+\delta-1}\bigg)-\mathrm{Re}\bigg[\frac{1}{s-\rho_0}+\frac{1}{s-(1-\bar{\rho_0})}$$
$$-\frac{1}{\sqrt{5}}\bigg(\frac{1}{s+\delta-\rho_0}+\frac{1}{s+\delta-(1-\bar{\rho_0})}\bigg)\bigg].$$

Combining this bound and

$$\sum_{k=0}^{K}a_k\mathrm{Re}\bigg(\frac{1}{\sqrt{5}}\frac{\zeta'}{\zeta}(\sigma+\delta+ikt)-\frac{\zeta'}{\zeta}(\sigma+ikt)\bigg)$$
$$=\sum_{n\geq 1}\frac{\Lambda(n)}{n^{\sigma}}(1-\frac{1/\sqrt{5}}{n^{\delta}})P(t\log n)\geq 0$$

one can obtain a zero free region for $\zeta(s)$. Note that using Stechkin's lemma one can reduce the constant $\frac{1}{2}$ to $\frac{1}{2}\big(1-\frac{1}{\sqrt{5}}\big)$. This allows to get wider zero-free region for $\zeta(s)$: in particular, Rosser and Schoenfeld obtained $R_0=9.645908801$ in [15]. For more detail see Stechkin [16], Rosser and Schoenfeld [15], and Graham [4].

In [6] Heath-Brown established the explicit formula that relating the sum

$$\sum_{n}\Lambda(n)\frac{\chi(n)}{n^s}f\bigg(\frac{\log n}{\log q}\bigg)$$

to a sum over zeros of Dirichlet L-function $L(s,\chi)$ associated with χ, where χ is a primitive Dirichlet character modulo $q>1$, and $f(t)$ is a positive smooth function satisfying certain conditions given in Section 5 of [6]. Note that Heath-Brown gave several functions for $f(t)$. Using this explicit formula Heath-Brown improved zero-free regions for Dirichlet L-functions (see [6]). In 2005 Kadiri generalized Stechkin's work and used it together with Weil's explicit formula for $\zeta(s)$. We will describe Kadiri's method in the next section.

3. Kadiri's method

In this section we will describe Kadiri's method in brief. We will use the same notations as [7].

Let f be a positive function with compact support that is in $C^2([0,d])$. Assume that

$$f(d) = f'(0) = f'(d) = f''(d) = 0. \tag{1}$$

Let

$$F(z) = \int_0^d e^{-zt} f(t)dt$$

be the Laplace transform of f. In Proposition 2.1 of [7] Kadiri proved the following.

Proposition 1 (Weil's explicit formula)**.** *Let f be a function as above and $s = \sigma + it$ a complex number. Then we have*

$$\begin{aligned}
&\mathrm{Re}\bigg(\sum_{n\geq 1} \frac{\Lambda(n)}{n^s} f(\log n)\bigg) \\
&= f(0)\bigg(-\frac{1}{2}\log\pi + \mathrm{Re}\frac{1}{2}\frac{\Gamma'}{\Gamma}\Big(\frac{s}{2}+1\Big)\bigg) \\
&\quad + \mathrm{Re}F(s-1) - \sum_{\rho\in Z(\zeta)} \mathrm{Re}F(s-\rho) \\
&\quad + \mathrm{Re}\bigg(\frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} \mathrm{Re}\frac{\Gamma'}{\Gamma}\Big(\frac{z}{2}\Big)\frac{F_2(s-z)}{(s-z)^2}dz + \frac{F_2(s)}{s^2}\bigg),
\end{aligned}$$

where F_2 is the Laplace transform of f'' and the sum is over the non-trivial zeros ρ of $\zeta(s)$.

Indeed Kadiri established the explicit formula in more general setting. See Theorem 1 in [7]. For more on explicit formula, see Guinand [5] and Weil [20].

Kadiri considered

$$\mathrm{Re}\bigg[\sum_{n\geq 1} \frac{\Lambda(n)}{n^s} f(\log n)\Big(1 - \frac{\kappa}{n^\delta}\Big)\bigg],$$

where $0 < \kappa < 1$ and $0 < \delta < 1$ are the constants that will be fixed later. In view of Proposition 1 we get

$$\begin{aligned}
&\mathrm{Re}\sum_{n\geq 1} \frac{\Lambda(n)}{n^s} f(\log n)\Big(1 - \frac{\kappa}{n^\delta}\Big) \\
&= f(0)\Delta_1(s) + D(s-1) - \sum_{\rho\in Z(\zeta)} D(s-\rho) + \Delta_2(s),
\end{aligned} \tag{2}$$

where

$$\Delta_1(s) = T_1(s) - \kappa T_1(s+\delta),\ \ \Delta_2(s) = T_2(s) - \kappa T_2(s+\delta),$$

$$D(s) = \mathrm{Re}F(s) - \kappa \mathrm{Re}F(s+\delta),\ T_1(s) = -\frac{1}{2}\log\pi + \frac{1}{2}\mathrm{Re}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}+1\right),$$

$$\begin{aligned} T_2(s) &= \mathrm{Re}\left(\frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} \mathrm{Re}\frac{\Gamma'}{\Gamma}\left(\frac{z}{2}\right)\frac{F_2(s-z)}{(s-z)^2}dz + \frac{F_2(s)}{s^2}\right) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{Re}\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+i\frac{t}{2}\right)\mathrm{Re}\frac{F_2(s-1/2-it)}{(s-1/2-it)^2}dt + \mathrm{Re}\frac{F_2(s)}{s^2}. \end{aligned}$$

Then

$$\sum_{n\geq 1} f(\log n)\frac{\Lambda(n)}{n^\sigma}\left(1-\frac{\kappa}{n^\delta}\right)\sum_{k=0}^{K} a_k\cos(k\gamma_0\log n) \geq 0,$$

where $\sum_{k=0}^{K} a_k\cos(kv) \geq 0$ with $a_k \geq 0$. In view of (2) we get

$$\begin{aligned} &\sum_{k=0}^{K} a_k\Bigg[f(0)\Delta_1(\sigma+ik\gamma_o) + D(\sigma-1+ik\gamma_o) \\ &\quad - \sum_{\rho\in Z(\zeta)} D(\sigma+ik\gamma_o-\rho) + \Delta_2(\sigma+ik\gamma_o)\Bigg] \geq 0. \end{aligned} \tag{3}$$

Kadiri estimated upper bounds for $\sum_{k=0}^{K} a_k f(0)\Delta_1(\sigma+ik\gamma_0)$, $\sum_{k=0}^{K} a_k D(\sigma-1+ik\gamma_0)$, and $\sum_{k=0}^{K} a_k\Delta_2(\sigma+ik\gamma_0)$, and lower bound for $\sum_{k=0}^{K} a_k \sum_{\rho\in\mathbf{Z}(\zeta)} D(\sigma+ik\gamma_0-\rho)$. In [7] the following trigonometric inequality

$$P(v) = \sum_{k=0}^{4} a_k\cos(kv) = 8(0.91+\cos v)^2(0.265+\cos v)^2 \geq 0$$

is used. Kadiri introduced the variables η, r and ω as follows. Let $\eta = 1-\beta_0$ and write

$$\eta = \frac{1}{r\log\gamma_0},$$

where $5 \leq r \leq R$. Let $T_0 = 3330657430.697$ and assume that $\gamma_0 \geq T_0$. Note that in [7] Kadiri used Wedeniwski's numerical verification of the Riemann Hypothesis: if ρ is a non-trivial zero of $\zeta(s)$ and $0 < \mathrm{Im}(\rho) < 3330657430.697$, then $\mathrm{Re}(\rho) = \frac{1}{2}$, i.e., the Riemann Hypothesis is true for those zeros. See [19]. Let

$$\sigma = 1 - \frac{1}{R\log(4\gamma_0+t_0)}$$

and

$$\omega = \frac{1-\sigma}{\eta},$$

where t_0 is a real number with $t_0 > 1$. (Kadiri put $t_0 = 10$.) According to Theorem 1 of [15] one may take $R = 9.645908801$. Thus

$$\sigma \geq \sigma_0 = 1 - \frac{1}{R\log(4T_0+t_0)},$$

$$\eta \leq \eta_0 = \frac{1}{r \log T_0} \leq \frac{1}{5 \log T_0},$$

and

$$\omega = \frac{1-\sigma}{\eta} = \frac{r \log \gamma_0}{R \log(4\gamma_0 + t_0)} \geq \frac{r \log T_0}{R \log(4T_0 + t_0)} = \frac{1-\sigma_0}{\eta_0}.$$

We set

$$\omega_0 = \frac{1-\sigma_0}{\eta_0}.$$

So, $\omega_0 \leq \omega < 1$. Let

$$A = \sum_{k=1}^{4} a_k = 35.78532.$$

We now impose on f to satisfy

$$\mathrm{Re}(F(z)) \geq 0 \quad \text{for} \quad \mathrm{Re}(z) \geq 0. \tag{4}$$

Kadiri proved the followings.

Proposition 2. *There exists a function $C_3(\eta)$ such that*

$$\sum_{k=0}^{4} a_k \sum_{\rho \in Z(\zeta)} D(\sigma + ik\gamma_0 - \rho) \geq a_1 F(\sigma - \beta_0) - C_3(\eta),$$

where $C_3(\eta)$ is given in [7, Section 4.1].

Proof. See Section 4.1 of [7]. □

To get a lower bound for $\sum_{k=0}^{4} a_k \sum_{\rho \in Z(\zeta)} D(\sigma + ik\gamma_0 - \rho)$ Kadiri considered firstly the case where $k = 1$ and $\rho \in \{\rho_0, 1 - \bar{\rho}\}$:

$$D(\sigma - \beta_0) + D(\sigma - 1 + \beta_0) \geq F(\sigma - \beta_0) + E(\eta),$$

where $E(\eta)$ is given in (34) of [7]. In the case that $k = 0, 2, 3, 4$ or $k = 1$ with $\rho \notin \{\rho_0, 1 - \bar{\rho}_0\}$, Kadiri divided the set $Z(\zeta)^*$ into two parts $Z(\zeta)^* = S_1 \cup S_2$, where

$$Z(\zeta)^* = \begin{cases} Z(\zeta) \setminus \{\rho_0, 1 - \bar{\rho}_0\}, & \text{for } k = 1, \\ Z(\zeta), & \text{otherwise,} \end{cases}$$

$S_1 = \{\rho = \beta + i\gamma \in Z(\zeta)^* \mid \frac{1}{2} \leq \beta \leq \sigma\}$ and $S_2 = \{\rho = \beta + i\gamma \in Z(\zeta)^* \mid \sigma < \beta\}$. For the sum over $\rho \in S_1$ Kadiri generalized Stechkin's work to the Laplace form $F(z)$:

Proposition (A generalization of Lemma 2 of [16])**.** *For $\frac{1}{2} \leq \beta \leq \sigma$ and $y > 0$, we have*

$$D(\sigma - \beta + iy) + D(\sigma - 1 + \beta + iy) \geq 0,$$

where $\delta \geq 0.61963$ and $0 < \kappa$ verify the conditions given in [7, Proposition 4.2].

Proof. See Proposition 4.2 of [7]. □

For these choices for δ and κ the sum $\sum_{k=0}^{4} a_k \sum_{\rho\in S_1} D(\sigma+ik\gamma_0-\rho)>0$ can be discarded. Note that $\sigma<\beta$ yields $|\gamma|>k\gamma_0+t_0$. For the sum over $\rho\in S_2$ Kadiri estimated the zero density of $\zeta(s)$.

Proposition 3. *There exists a function $C_1(\eta)$ such that*

$$f(0)\sum_{k=0}^{4} a_k\Delta_1(\sigma+ik\gamma_0)\le \frac{A}{2}(1-\kappa)f(0)\log\gamma_0+C_1(\eta),$$

where $C_1(\eta)$ is given in [7, Section 4.2].

Proof. See Section 4.2 of [7]. □

Proposition 4. *There exists a function $C_2(\eta)$ such that*

$$\sum_{k=0}^{4} a_k D(\sigma-1+ik\gamma_0)\le a_0\widetilde{F}(\sigma-1,0)+C_2(\eta),$$

where $C_2(\eta)$ is given in [7, Section 4.3].

Proof. See Section 4.3 of [7]. □

Proposition 5. *There exists a function $C_4(\eta)$ such that*

$$\sum_{k=0}^{4} a_k\Delta_2(\sigma+ik\gamma_0)\le C_4(\eta),$$

where $C_4(\eta)$ is given in [7, Section 4.4].

Proof. See Section 4.4 of [7]. □

In addition, for the proofs of Propositions 2-5 Kadiri estimated the upper bounds for $|\mathrm{Re}\frac{\Gamma'}{\Gamma}(z)|$ and observed in detail how behave the Laplace transform $F(s)$.

Gathering together (3) and Propositions 2-5 we obtained that

$$0\le \frac{A}{2}(1-\kappa)f(0)\log\gamma_0+a_0F(\sigma-1)-a_1F(\sigma-\beta_0)+C(\eta), \tag{5}$$

where

$$C(\eta)=C_1(\eta)+C_2(\eta)+C_3(\eta)+C_4(\eta).$$

Kadiri used for $f(t)$ the function given in Lemma 7.4 of [6] with $\lambda=1$. For $\theta\in]\pi/2,\pi[$ Kadiri set $f(t)=\eta h_\theta(\eta t)$, where

$$\begin{aligned}h_\theta(u)=(1+\tan^2\theta)\bigg\{&(1+\tan^2\theta)\left(\frac{-\theta}{\tan\theta}-\frac{u}{2}\right)\cos(u\tan\theta)+\frac{-2\theta}{\tan\theta}-u\\&-\frac{\sin(2\theta+u\tan\theta)}{\sin(2\theta)}+2\left(1+\frac{\sin(\theta+u\tan\theta)}{\sin\theta}\right)\bigg\}\end{aligned}$$

for $u \in [0, \frac{-2\theta}{\tan\theta}]$. Kadiri verified that $C(\eta) < 0$ for $\eta \in [0, \eta_0]$. From (5) it follows now that

$$\frac{A}{2}(1-\kappa)\eta h_\theta(0)\log\gamma_0 \geq a_1 F(\sigma-\beta_0) - a_0 F(\sigma-1).$$

Let

$$\begin{aligned} K(\omega,\theta) &:= a_1 F(\sigma-\beta_0) - a_0 F(\sigma-1) \\ &= \int_0^d \left(a_1 e^{(\beta_0-\sigma)t} - a_0 e^{(1-\sigma)t}\right) f(t)dt \\ &= \int_0^{d_1(\theta)} (a_1 e^{-t} - a_0) h_\theta(t) e^{\omega t} dt, \end{aligned}$$

where $d = \frac{d_1(\theta)}{\eta}$ and $d_1(\theta) = \frac{-2\theta}{\tan\theta}$. Ultimately we get

$$\eta \log\gamma_0 \geq \frac{K(\omega,\theta)}{\frac{A}{2}(1-\kappa)h_\theta(0)}. \tag{6}$$

Kadiri chose $r = 5.94292$, $\theta = 1.848$ for which $K(\omega,\theta)$ is increasing with $\omega \geq \omega_0 = 0.579471$. For the choices $\delta = 0.61963\cdots$, $\kappa = 0.44213\cdots$, we get

$$\eta\log\gamma_0 \geq \frac{K(\omega_0, 1.848)}{\frac{A}{2}(1-0.44213\cdots)h_{1.848}(0)} \geq \frac{1}{5.942924085}.$$

We let now $R = 5.942924086$ and repeat the same calculations. Kadiri iterated six times and obtained $R_0 = 5.69693$. This proves Theorem 1. Setting

$$\sigma = 1 - \frac{1}{R\log(4\gamma_0 + t_0)} < 1,\ f(t) = \eta h_\theta(\eta t),\ \omega = \frac{1-\sigma}{\eta},$$

and dealing skilfully with them are very original parts in Kadiri's method.

In the next section we will apply Kadiri's method to various functions $f(t)$.

4. Various choices for $f(t)$ and trigonometric inequalities

In this Section we will apply Kadiri's methods to various functions and reduce R_0. Heath-Brown proposed several functions for $f(t)$ in Lemmas 7.1-7.4 of [6]. Kadiri used the function given in Lemma 7.4 of [6]. We will use the functions given in Lemmas 7.1-7.4 of [6] in Propositions 6-9 below, respectively. Moreover, Heath-Brown mentioned a suggestion for optimizing $f(t)$ that would lead a possible improvement in Section 16 of [6]. Following this idea Xylouris found a family of functions f in [21]. See also [22]. In Proposition 10 below we will use these functions.

In order to find optimal trigonometric inequalities we consider

$$2(a+\cos v)^2 \geq 0,$$

$$8(a+\cos v)^2(b+\cos v)^2 \geq 0,$$

$$32(a+\cos v)^2(b+\cos v)^2(c+\cos v)^2 \geq 0,$$

where $0 < a, b, c < 1$. We have observed numerically that the first and the third inequalities lead worse values for R_0. So, we take the trigonometric inequality of the form

$$\sum_{k=0}^{4} a_k \cos(kv) = 8(a + \cos v)^2(b + \cos v)^2$$

with $0 < a, b < 1$. According to [11] and [12] the Riemann Hypothesis is true for the non-trivial zeros ρ with $\mathrm{Im}(\rho) \le 30610046000$. From now on we let $T_0 = 30610046000$ and $R = 5.69693$. Let η, η_0, ω, ω_0, and A be as in Section 3.

Proposition 6. *Let* $0 < \theta < \frac{\pi}{2}$ *and let* $\lambda > 0$*. Set*

$$\begin{aligned} h_{1,\theta}(u) = \lambda(1 + \tan^2\theta)\Bigg\{ &\lambda(1 + \tan^2\theta)\left(\frac{\theta}{\lambda\tan\theta} - \frac{u}{2}\right)\cos(u\lambda\tan\theta) \\ &+ \frac{2\theta}{\tan\theta} - \lambda u + \frac{\sin(2\theta - u\lambda\tan\theta)}{\sin 2\theta} - 2\left(1 + \frac{\sin(\theta - u\lambda\tan\theta)}{\sin\theta}\right)\Bigg\} \end{aligned}$$

for $0 < u < \frac{2\theta}{\lambda\tan\theta}$*. Let* $f(t) = \eta h_{1,\theta}(\eta t)$*. If we let* $\lambda = 1.03669$*,* $\theta = 1.13537$*,* $a = 0.909198$*, and* $b = 0.261704$*, then* $R_0 = 5.68372$*.*

Proof. We verify numerically that the maximum

$$\max_{\substack{0<\lambda \\ 0<\theta<\frac{\pi}{2} \\ 0<a,b<1}} \frac{K(1,\theta)}{Ah_{1,\theta}(0)}$$

reaches approximately at $\lambda = 1.03669$, $\theta = 1.13537$, $a = 0.909198$, and $b = 0.261704$. So we choose these values for λ, θ, a, and b, respectively. For the choices $\lambda = 1.03669$ and $\theta = 1.13537$ we verify that $f(t)$ satisfies (1). Hence we can use Kadiri's method for $f(t)$. We proceed as [7] and perform four iterations. For each iteration, we determine δ, κ and the error term $C(\eta)$ as Section 4 of [7]. Our numerical results are as follows :

$$\delta = 0.620388, \quad \kappa = 0.439672, \qquad C(\eta) = -135.097\eta + 143.458\eta^2 + 117622\eta^3.$$

We verified that $C(\eta) < 0$ for $\eta \in [0, \eta_0]$ and $\frac{\partial}{\partial\omega}K(\omega, \theta) > 0$ for $\theta = 1.13537$. In view of (6) we get

$$\eta \log \gamma_0 \ge \frac{K(\omega_0, 1.13537)}{\frac{A}{2}(1 - 0.439672)h_{1,1.13537}(0)} \ge \frac{1}{5.68372}.$$

We conclude that $R_0 = 5.68372$. □

Proposition 7. *Let* $\lambda > 0$*. Set*

$$h_{2,\lambda}(u) = \frac{1}{30}\lambda(2 - \lambda u)^3(4 + 6\lambda u + \lambda^2 u^2)$$

for $0 < u < \frac{2}{\lambda}$*. Let* $f(t) = \eta h_{2,\lambda}(\eta t)$*. If* $\lambda = 1.98916$*,* $a = 0.909192$*, and* $b = 0.261661$*, then* $R_0 = 5.68483$*.*

Proof. Let

$$K(\omega,\lambda)=\int_0^{\frac{2}{\lambda}}(a_1e^{-t}-a_0)h_{2,\lambda}(t)e^{\omega t}dt.$$

We verify numerically that the maximum

$$\max_{\substack{0<\lambda\\0<a,b<1}}\frac{K(1,\lambda)}{Ah_{2,\lambda}(0)}$$

reaches approximately at $\lambda=1.98916$, $a=0.909192$, and $b=0.261661$. So we choose these values for λ, a, and b, respectively. For the choice $\lambda=1.98916$ we verify that $f(t)$ satisfies (1). Hence we can use Kadiri's method for $f(t)$. We proceed as [7] and perform four iterations. For each iteration, we determine δ, κ and the error term $C(\eta)$ as Section 4 of [7]. Our numerical results are as follows:

$$\delta=0.620396,\quad \kappa=0.439654,\quad C(\eta)=-48.0479\eta+52.8366\eta^2+43329.8\eta^3.$$

We verified that $C(\eta)<0$ for $\eta\in[0,\eta_0]$ and $\frac{\partial}{\partial\omega}K(\omega,\lambda)>0$ for $\lambda=1.98916$. In view of (6) we get

$$\eta\log\gamma_0\geq\frac{K(\omega_0,1.98916)}{\frac{A}{2}(1-0.439654)h_{2,1.98916}(0)}\geq\frac{1}{5.68483}.$$

We conclude that $R_0=5.68483$. □

As our proofs of Propositions 8 and 9 below are analogous to that of Proposition 6 we omit its proofs.

Proposition 8. *Let $\theta>0$ and let $\lambda>0$. Set*

$$\begin{aligned}h_{3,\theta}(u)=\lambda(1-\tanh^2\theta)\Big\{&\lambda(1-\tanh^2\theta)\Big(\frac{\theta}{\lambda\tanh\theta}-\frac{u}{2}\Big)\cosh(u\lambda\tanh\theta)\\&+\frac{2\theta}{\tanh\theta}-\lambda u+\frac{\sinh(2\theta-u\lambda\tanh\theta)}{\sinh 2\theta}-2\Big(1+\frac{\sinh(\theta-u\lambda\tanh\theta)}{\sinh\theta}\Big)\Big\}\end{aligned}$$

for $0<u<\frac{2\theta}{\lambda\tanh\theta}$. Let $f(t)=\eta h_{3,\theta}(\eta t)$. If we let $\lambda=2.00106$, $\theta=0.135923$, $a=0.909215$, and $b=0.260906$, then $R_0=5.68484$.

Proposition 9. *Let $\frac{\pi}{2}<\theta<\pi$ and let $\lambda>0$. Set*

$$\begin{aligned}h_{4,\theta}(u)=\lambda(1+\tan^2\theta)\Big\{&\lambda(1+\tan^2\theta)\Big(\frac{-\theta}{\lambda\tan\theta}-\frac{u}{2}\Big)\cos(u\lambda\tan\theta)-\frac{2\theta}{\tan\theta}\\&-\lambda u-\frac{\sin(2\theta+u\lambda\tan\theta)}{\sin(2\theta)}+2\Big(1+\frac{\sin(\theta+u\lambda\tan\theta)}{\sin\theta}\Big)\Big\}\end{aligned}$$

for $0<u<\frac{-2\theta}{\lambda\tan\theta}$. Let $f(t)=\eta h_{4,\theta}(\eta t)$. If we let $\lambda=0.630225$, $\theta=1.75566$, $a=0.909282$, and $b=0.262108$, then $R_0=5.68486$.

Following the suggestions in Section 16 of [6] Xylouris found a family of functions for $f(t)$ in Chapter 5 of [21]. Let γ be a positive real number. Set

$$g(t) = c_1 \cos(x_1 t) + c_2 \cosh(x_2 t) - c_3$$

for $-\gamma \le t \le \gamma$ and $g(t) = 0$ for $|t| \ge \gamma$, where c_1, c_2, x_1, and x_2 are real numbers, and

$$c_3 = c_1 \cos(x_1 \gamma) + c_2 \cosh(x_2 \gamma).$$

Then g is continuous, non-negative, even function supported in an interval $(-\gamma, \gamma)$. Set

$$h_{5,\gamma}(u) = (g * g)(u) = \int_{u-\gamma}^{\gamma} g(x) g(u-x)\, dx,$$

for $0 < u < 2\gamma$. Then $h_{5,\gamma}$ satisfies the conditions (1) and (4). We will use those functions in Proposition 10 below.

Proposition 10. *Let γ, $g(t)$, and $h_{5,\gamma}(u)$ be as above. Let $f(t) = \eta h_{5,\gamma}(\eta t)$. If $a = 0.91$, $b = 0.265$, $\gamma = 0.509607$, $c_1 = 0.675805$, $x_1 = 0.463784$, $c_2 = 0.799229$, and $x_2 = 0.40991$, then $R_0 = 5.68371$.*

Proof. Let

$$K(\omega, \gamma) = \int_0^{2\gamma} (a_1 e^{-t} - a_0) h_{5,\gamma}(t) e^{\omega t} dt.$$

We verify numerically that the maximum

$$\max_{\substack{0<\gamma \\ 0<c_1,c_2,x_1,x_2<1}} \frac{K(1,\gamma)}{h_{5,\gamma}(0)}$$

reaches approximately at $\gamma = 0.509607$, $c_1 = 0.675805$, $x_1 = 0.463784$, $c_2 = 0.799229$, and $x_2 = 0.40991$. For those choices we proceed as [7] and perform four iterations. For each iteration, we determine δ, κ and the error term $C(\eta)$ as Section 4 of [7]. Our numerical results are as follows:

$$\begin{aligned}
&\delta = 0.620388, \quad \kappa = 0.439672, \\
&C_1(\eta) \le -0.0000159759\eta, \\
&C_2(\eta) \le -6.73037 \cdot 10^{-6}\eta + 1.12991 \cdot 10^{-25}\eta^2 + 0.000172321\eta^3, \\
&C_3(\eta) \le 2.90148 \cdot 10^{-6}\eta + 0.0000209955\eta^2 + 0.000455292\eta^3, \\
&C_4(\eta) \le 0.016588\eta^3,
\end{aligned}$$

and

$$C(\eta) = -0.0000198048\eta + 0.0000209955\eta^2 + 0.0172156\eta^3.$$

We verified that $C(\eta) < 0$ for $\eta \in [0, \eta_0]$ and $\frac{\partial}{\partial \omega} K(\omega, \gamma) > 0$ for $\gamma = 0.509607$. In view of (6) we get

$$\eta \log \gamma_0 \ge \frac{K(\omega_0, 0.509607)}{\frac{A}{2}(1 - 0.439672) h_{5,0.509607}(0)} \ge \frac{1}{5.68371}.$$

We conclude that $R_0 = 5.68371$. $\square$

In view of Propositions 6-10 we conclude therefore the following.

Proposition 11. *We can take* $R_0 = 5.68371$.

Remarks. (1) In the proof of Proposition 10 we obtained $R_0 = 5.68371$ with various values for a, b, γ, c_1, c_2, x_1, and x_2: e.g. $a = 0.909202$, $b = 0.261712$, $\gamma = 0.509747$, $c_1 = 0.824381$, $c_2 = 2.16732$, $x_1 = 2.26558$, and $x_2 = 0.0346198$.

(2) According to [21] one can take also

$$g(t) = c_1 \cosh(x_1 t)\cos(x_2 t) + c_2 \sinh(x_1 t)\sin(x_2 t) - c_3,$$

where $\gamma > 0$, c_1, c_2, x_1, and x_2 are real numbers, and

$$c_3 = c_1 \cosh(x_1\gamma)\cos(x_2\gamma) + c_2 \sinh(x_1\gamma)\sin(x_2\gamma).$$

Using those $g(t)$ we obtained rather worse values for R_0: e.g. for the choices $a = 0.91$, $b = 0.265$, $c_1 = 1$, $c_2 = 0.5$, $x_1 = 1$, $x_2 = 1$, and $\gamma = 0.508$, we obtained $R_0 = 5.68383$.

For the computations we have used Mathematica.

References

[1] K. Ford, *Vinogradov's integral and bounds for the Riemann zeta function*, Proc. London Math. Soc. (3) **85** (2002), no. 3, 565–633.

[2] ———, *Zero-free regions for the Riemann zeta function*, dans: Number Theory for the Millenium, II (Urbana, IL, 2000), 25–56, A K Peters, 2002.

[3] X. Gourdon, *The* 10^{13} *first zeros of the Riemann zeta function, and zeros computation at very large height*, http://numbers.computation.free.fr/Constants/Miscellaneous/zetazeros1e13-1e24.pdf, 2004.

[4] S. W. Graham, *On Linnik's constant*, Acta Arith. **39** (1981), no. 2, 163–179.

[5] A. P. Guinand, *A summation formula in the theory of prime numbers*, Proc. London Math. Soc. (2) **50** (1948), 107–119.

[6] D. R. Heath-Brown, *Zero-free regions for Dirichlet L-functions, and the least prime in an arithmetic progression*, Proc. London Math. Soc. **64** (1992), no. 2, 265–338.

[7] H. Kadiri, *Une région explicit sans zéro pour la fonction* ζ *de Riemann*, Acta Arith. **117** (2005), no. 4, 303–339.

[8] ———, *A zero density result for the Riemann zeta function*, Acta Arith. **160** (2013), no. 2, 185–200.

[9] J. C. Lagarias and A. M. Odlyzko, *Effective versions of the Chebotarev Density Theorem*, Algebraic Number fields, L-functions and Galois properties, (A. Fröhlich, Ed.) pp. 409–464, New York, London Academic Press, 1977.

[10] J. van de Lune, H. J. J. te Riele, and D. T. Winter, *On the zeros of the Riemann zeta function in the critical strip. IV*, Math. Comp. **46** (1986), no. 174, 667–681.

[11] D. Platt, *Computing degree 1 L-functions rigorously*, Ph. D. thesis, Univ. of Bristol, 2011.

[12] ———, *Computing zeta on the half-line*, preprint. arxiv:1305.3087 v1.

[13] J. B. Rosser, *Explicit bounds for some functions of prime numbers*, Amer. J. Math. **63** (1941), 211–232.

[14] J. B. Rosser and L. Schoenfeld, *Approximate formulas for some functions of prime numbers*, Illinois J. Math. **6** (1962), 64–94.

[15] ———, *Sharper bounds for the Chebyshev functions* $\theta(x)$ *and* $\psi(x)$, Math. Comp. **29** (1975), 243–269.

[16] S. B. Steckin, *The zeros of the Riemann zeta-function*, Math. Notes **8** (1970), 706–711.

[17] E. C. Titchmarsh, *The theory of the Riemann zeta-function*, second edition revised by D. R. Heath-Brown, Clarendon Press. Oxford, 1986.
[18] C. J. de la Vallée Poussin, *Sur la fonction $\zeta(s)$ de Riemann et le nombre des nombres premiers inférieurs à une limite donnée*, Mém. Couronnés et Autres Mém. Publ. Acad. Roy. Sci. Lett. Beaux-Arts Belg. **59** (1899-1900), 74 pp.
[19] S. Wedeniwski, *The first 10 billion zeros of the Riemann zeta function are calculated and satisfy the Riemann hypothesis*, http://www.zetagrid.net.
[20] A. Weil, *Sur les "formules explicites" de la théorie des nombres premiers*, Comm. Sém. Math. Univ. Lund (1952), 252–265.
[21] T. Xylouris, *On Linnik's constant*, http://arxiv.org/abs/0906.2749v1, 2009.
[22] ———, *On the least prime in an arithmetic progression and estimates for the zeros of Dirichlet L-functions*, Acta Arith. **150** (2011), no. 1, 65–91.

Woo-Jin Jang
Department of Mathematics
Korea University
Seoul 136-701, Korea
E-mail address: `woojinjang@korea.ac.kr`

Soun-Hi Kwon
Department of Mathematics Education
Korea University
Seoul 136-701, Korea
E-mail address: `sounhikwon@korea.ac.kr`