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SEMICOMMUTATIVE PROPERTY ON NILPOTENT PRODUCTS

Nam Kyun Kim, Tai Keun Kwak, and Yang Lee

Abstract. The semicommutative property of rings was introduced initially by Bell, and has done important roles in noncommutative ring theory. This concept was generalized to one of *nil-semicommutative* by Chen. We first study some basic properties of nil-semicommutative rings. We next investigate the structure of Ore extensions when upper nilradicals are σ-rigid δ-ideals, examining the nil-semicommutative ring property of Ore extensions and skew power series rings, where σ is a ring endomorphism and δ is a σ-derivation.

1. Introduction

Throughout this note every ring is an associative ring with identity unless otherwise stated. Given a ring R, $N^*(R)$ and $N(R)$ denote the upper nilradical (i.e., sum of nil ideals) and the set of all nilpotent elements in R, respectively. Note $N^*(R) \subseteq N(R)$. The polynomial ring with an indeterminate x over a ring R is denoted by $R[x]$. Let $C_{f(x)}$ denote the set of all coefficients of given a polynomial $f(x)$. $\mathbb{Z}$ and $\mathbb{Z}_n$ denote the ring of integers and the ring of integers modulo n. Denote the n by n ($n \geq 2$) full (resp., upper triangular) matrix ring over R by $\mathrm{Mat}_n(R)$ (resp., $U_n(R)$). Use e_{ij} for the matrix with (i,j)-entry 1 and elsewhere 0.

Due to Bell [6], a ring R is called to satisfy the *Insertion-of-Factors-Property* if $ab = 0$ implies $aRb = 0$ for $a, b \in R$. Narbonne [20] and Shin [23] used the terms *semicommutative* and *SI* for the IFP, respectively. (In this paper, we choose "a semicommutative ring" in the above names, so as to cohere with other related references.) Commutative rings clearly are semicommutative, and any reduced ring (i.e., a ring without nonzero nilpotent elements) is semicommutative by a simple computation. There exist many non-reduced commutative rings (e.g., $\mathbb{Z}_{n^l}$ for $n, l \geq 2$), and many noncommutative reduced rings (e.g.,

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direct products of noncommutative domains). A ring is called *Abelian* if every idempotent is central. Semicommutative rings are Abelian by a simple computation.

According to Marks [18], R is called *NI* if $N^*(R) = N(R)$. Note that R is NI if and only if $N(R)$ forms an ideal if and only if $R/N^*(R)$ is reduced. It is well-known that semicommutative rings are NI, but not conversely. Following [7, Definition 2.1], a ring R is called *nil-semicommutative* if whenever $ab \in N(R)$ for $a, b \in R$, then $arb \in N(R)$ for any $r \in R$. It is shown that every NI ring is nil-semicommutative and that the converse is true if Köthe's conjecture holds in [7]. Notice that the class of nil-semicommutative rings is clearly closed under subrings, and that semicommutative rings are clearly nil-semicommutative.

On the other hand, a ring R is called *weak symmetric* [21, Definition 1] if $abc \in N(R)$ implies $acb \in N(R)$ for all $a, b, c \in R$.

Proposition 1.1. *A ring R is nil-semicommutative if and only if R is weak symmetric.*

Proof. Assume that R is nil-semicommutative and let $abc \in N(R)$ for $a, b, c \in R$. Then $(acb)^2 = a \cdot c \cdot b \cdot a \cdot c \cdot b \cdot 1 \in N(R)$ by assumption, and so $acb \in N(R)$. Thus R is weak symmetric.

Conversely, assume that R is weak symmetric and let $ab \in N(R)$ for $a, b \in R$, say $(ab)^n = 0$ for a positive integer n. Let $r \in R$. Then $(ab)^n = 0$ implies

$$\begin{aligned}(ab)^n r^n = 0 \Rightarrow ab(ab)^{n-1} r^{n-1} r = 0 &\Rightarrow (arb)(ab)^{n-1} r^{n-1} \in N(R)\\ &\Rightarrow ((arb)a)\left(b(ab)^{n-2} r^{n-2}\right) r \in N(R)\\ &\Rightarrow (arb)^2 (ab)^{n-2} r^{n-2} \in N(R)\\ &\cdots\cdots\\ &\Rightarrow (arb)^n \in N(R)\\ &\Rightarrow arb \in N(R),\end{aligned}$$

by assumption. This shows that R is nil-semicommutative. □

Note that the weak symmetric ring property is left-right symmetric by the similar computation to the proof of Proposition 1.1. Hence, we will use this fact and Proposition 1.1 without reference.

We obtain basic equivalences for nil-semicommutative rings as follows.

Theorem 1.2. *Given a ring R, the following conditions are equivalent:*

(1) *R is nil-semicommutative.*
(2) *If $a_1 a_2 \cdots a_n \in N(R)$ for $a_1, \ldots, a_n \in R$, then $a_{\theta(1)} a_{\theta(2)} \cdots a_{\theta(n)} \in N(R)$ for any permutation θ of the set $\{1, 2, \ldots, n\}$, where n is any positive integer.*
(3) *$abc \in N(R)$ implies $bac \in N(R)$ for $a, b, c \in R$.*
(4) *If $a_1 \cdots a_n \in N(R)$ for $a_1, \ldots, a_n \in R$, then $r_1 a_1 r_2 a_2 \cdots r_n a_n r_{n+1} \in N(R)$ for all $r_1, \ldots, r_{n+1} \in R$.*

Proof. (1)⇒(2): Let R be a nil-semicommutative ring and suppose that $a_1 \cdots a_i \cdots a_j \cdots a_n \in N(R)$ for $a_1, \ldots, a_i, \cdots, a_j, \ldots, a_n \in R$ $(i < j)$. Then we get the following directions:

$$\begin{aligned} &(a_1 \cdots a_{i-1})(a_i \cdots a_{j-1})(a_j \ldots a_n) \in N(R) \\ \Rightarrow\ &(a_1 \cdots a_{i-1})(a_j \ldots a_n)(a_i \cdots a_{j-1}) \in N(R); \end{aligned}$$

$$\begin{aligned} &(a_1 \cdots a_{i-1}a_j)(a_{j+1} \ldots a_n a_i)(a_{i+1} \cdots a_{j-1}) \in N(R) \\ \Rightarrow\ &(a_1 \cdots a_{i-1}a_j)(a_{i+1} \cdots a_{j-1})(a_{j+1} \ldots a_n a_i) \in N(R); \end{aligned}$$

and

$$\begin{aligned} &(a_1 \cdots a_{i-1}a_j a_{i+1} \cdots a_{j-1})(a_{j+1} \ldots a_n)a_i \in N(R) \\ \Rightarrow\ &(a_1 \cdots a_{i-1})a_j(a_{i+1} \cdots a_{j-1})a_i(a_{j+1} \ldots a_n) \\ =\ &(a_1 \cdots a_{i-1}a_j a_{i+1} \cdots a_{j-1})a_i(a_{j+1} \ldots a_n) \in N(R), \end{aligned}$$

using Proposition 1.1. Since any permutation is a product of finite number of transpositions, we have $a_{\theta(1)}a_{\theta(2)} \cdots a_{\theta(n)} \in N(R)$ for any permutation θ of the set $\{1, 2, \ldots, n\}$.

(2)⇒(3) is clear, and (3)⇒(1) is obtained by Proposition 1.1.

(1)⇔(4) is similar to one of Proposition 1.1, applying $n+1$ times of the nil-semicommutativity of R for $1 \cdot a_1 \cdot a_2 \cdots a_n \cdot 1 \in N(R)$ where $a_1, \ldots, a_n \in R$. □

Proposition 1.3. *Let R be a nil-semicommutative ring. Then we have the following result.*

(1) *If $a \in N(R)$, then both Ra and aR are nil.*

(2) *$N(R)$ is multiplicatively closed, and $N(R) = \bigcup_{a \in N(R)} Ra = \bigcup_{b \in N(R)} bR$.*

(3) *If $N(R)$ is additively closed, then RaR is nil for any $a \in N(R)$.*

Proof. (1) Assume that $a^n = 0 \in N(R)$ for $n \geq 1$. Letting $r_1 = \cdots = r_n$ and $r_{n+1} = 1$ (resp., $r_2 = \cdots = r_{n+1}$ and $r_1 = 1$) in (1), we get that Ra (resp., aR) is nil.

(2) This comes from (1).

(3) First note that $ras \in N(R)$ for any $a \in N(R)$ and $r, s \in R$ by (2). So if $N(R)$ is additively closed, then RaR is nil. □

Proposition 1.3(3) leads the following result.

Corollary 1.4. *Let R be a nil-semicommutative ring. Then $N(R)$ is additively closed if and only if R is NI.*

Proposition 1.5. (1) *If there exists a nil-semicommutative ring R but not NI, then for some $a, b \in N(R)$, $(a+b)\mathbb{Z}_n[a+b] \cong x\mathbb{Z}_n[x]$ or $(a+b)\mathbb{Z}_n[a+b]$ contains a nonzero idempotent, where $n = 0$ or $n \geq 2$.*

(2) *If R is a nil-semicommutative ring and $N(R)[x] \subseteq N(R[x])$, then R is NI.*

Proof. (1) Let R be a nil-semicommutative ring but not NI. Then there exist $0 \neq a, b \in N(R)$ with $a+b \notin N(R)$ by Corollary 1.4. Let S be the subring of R generated by $a+b$. Then $S = (a+b)\mathbb{Z}_n[a+b]$, where $n = 0$ or $n \geq 2$. Consider the subset $T = \{(a+b)^t \mid t \geq 1\}$ of S. Assume that $(a+b)^m = (a+b)^n$ for some $m \neq n$ (otherwise, $(a+b)\mathbb{Z}_n[a+b] \cong x\mathbb{Z}_n[x]$). Then $(a+b)^s$ is an idempotent for some $s \geq 1$ by the proof of [12, Proposition 16]. But $(a+b)^s$ is nonzero.

(2) Suppose that R is a nil-semicommutative ring and $N(R)[x] \subseteq N(R[x])$. Let $a, b \in N(R)$ for $a, b \in R$. Then $a + bx \in N(R)[x] \subseteq N(R[x])$, and so $(a+bx)^n = 0$ for some $n \geq 1$. Hence $(a+b)^n = 0$ and thus $a+b \in N(R)$, showing that $N(R)$ is additively closed. Therefore R is NI by Corollary 1.4. □

Proposition 1.6. (1) *The class of nil-semicommutative rings is closed under direct sums.*

(2) *For any family $\{R_\gamma \mid \gamma \in \Gamma\}$ of rings, suppose that the direct product $R = \prod_{\gamma\in\Gamma} R_\gamma$ is of bounded index of nilpotency. Then R_γ is a nil-semicommutative ring for all $\gamma \in \Gamma$ if and only if R is.*

(3) *The classes of nil-semicommutative rings are closed under direct limits.*

(4) *Let $e \in R$ be a central idempotent. Then R is nil-semicommutative if and only if eR and $(1-e)R$ are nil-semicommutative rings.*

Proof. (1) Let R_u be nil-semicommutative for all $u \in U$ and $A = \oplus_{u\in U} R_u$, the direct sum of R_u's. It can be easily checked that $N(A) = \oplus_{u\in U} N(R_u)$. Thus this entails that A is nil-semicommutative.

(2) Let k be the bounded index of R. Then R_γ is also of bounded index $\leq k$ for each $\gamma \in \Gamma$. By the same computation as in the proof of (1), the proof is completed.

(3) Let $D = \{R_i, \alpha_{ij}\}$ be a direct system of nil-semicommutative rings R_i for $i \in I$ and ring homomorphisms $\alpha_{ij} : R_i \to R_j$ for each $i \leq j$ satisfying $\alpha_{ij}(1) = 1$, where I is a directed partially ordered set. Let $R = \varinjlim R_i$ be the direct limit of D with $\iota_i : R_i \to R$ and $\iota_j\alpha_{ij} = \iota_i$. If we take $a, b \in R$, then $a = \iota_i(a_i)$, $b = \iota_j(b_j)$ for some $i, j \in I$ and there is $k \in I$ such that $i \leq k$, $j \leq k$. Define

$$a + b = \iota_k(\alpha_{ik}(a_i) + \alpha_{jk}(b_j)) \text{ and } ab = \iota_k(\alpha_{ik}(a_i)\alpha_{jk}(b_j)),$$

where $\alpha_{ik}(a_i)$ and $\alpha_{jk}(b_j)$ are in R_k. Then R forms a ring with $0 = \iota_i(0)$ and $1 = \iota_i(1)$. Let $abc \in N(R)$. There is $k \in I$ such that $a = \iota_i(a_i), b = \iota_j(b_j), c = \iota_l(c_l)$ and $i, j, l \leq k$. Then $abc = \iota_k(\alpha_{ik}(a_i)\alpha_{jk}(b_j)\alpha_{lk}(c_l)) \in N(R_k)$. Since R_k is nil-semicommutative, $acb \in N(R_k)$ and this implies that R is nil-semicommutative by Proposition 1.1.

(4) This directly follows from (1) and the fact that the class of nil-semicommutative rings is closed under subrings, since $R \cong eR \oplus (1-e)R$. □

For $n \geq 2$, the n by n full matrix ring over any ring need not nil-semicommutative by the following example.

Example 1.7. Let R be any ring. For

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix},\ B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\ C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathrm{Mat}_2(R),$$

$ABC = 0 \in N(\mathrm{Mat}_2(R))$ but $ACB = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right) \notin N(\mathrm{Mat}_2(R))$. Thus $\mathrm{Mat}_2(R)$ is not nil-semicommutative and so $\mathrm{Mat}_n(R)$ for $n \geq 2$ is not nil-semicommutative.

Let R be the ring of quaternions with integer coefficients. Then R is a domain and thus nil-semicommutative. However, for any odd prime integer q, there exists a ring isomorphism $R/qR \cong \mathrm{Mat}_2(\mathbb{Z}_q)$ by the argument in [8, Exercise 2A]. But $\mathrm{Mat}_2(\mathbb{Z}_q)$ is not nil-semicommutative by Example 1.7, and thus R/qR cannot be nil-semicommutative. Therefore the class of nil-semicommutative rings is not closed under homomorphic images.

Alhevaz et al. [1] and Nasr-Isfahani [19] introduced a *skew triangular matrix ring* as a set of all upper triangular matrices with addition point-wise and new multiplication defined by

$$(a_{ij})(b_{ij}) = (c_{ij}),$$

where $c_{ij} = a_{ii}b_{ij} + a_{i(i+1)}\sigma(b_{(i+1)j}) + \cdots + a_{ij}\sigma^{j-i}(b_{jj})$ for each $i \leq j$ where $(a_{ij}), (b_{ij}) \in U_n(R)$ $(n \geq 2)$ over a ring R with an endomorphism σ, and denoted by $U_n(R, \sigma)$. Note that $U_n(R, 1_R) = U_n(R)$, where 1_R is the identity endomorphism of R.

Let $D_n(R)$ be the ring of all matrices in $U_n(R)$ whose diagonal entries are all equal, and $V_n(R)$ be the ring of all matrices (a_{ij}) in $D_n(R)$ such that $a_{st} = a_{(s+1)(t+1)}$ for $s = 1, \ldots, n-2$ and $t = 2, \ldots, n-1$. Then $D_n(R, \sigma)$ and $V_n(R, \sigma)$ are subrings of $U_n(R, \sigma)$, and $V_n(R, \sigma)$ is a subring of $D_n(R, \sigma)$. There is a ring isomorphism $\phi : R[x; \sigma]/(x^n) \to V_n(R, \sigma)$, given by $\phi(a_0 + a_1x + \cdots + a_{n-1}x^{n-1} + (x^n)) = (a_0, a_1, \ldots, a_{n-1})$, with $a_i \in R$, where $R[x; \sigma]$ denotes the skew polynomial ring with an indeterminate x over R, subject to $xr = \sigma(r)x$ for $r \in R$ and (x^n) is the ideal generated by (x^n). So $V_n(R, \sigma) \cong R[x; \sigma]/(x^n)$.

Theorem 1.8. *For a ring R with an endomorphism σ and $n \geq 2$, the following conditions are equivalent*:

(1) *R is nil-semicommutative.*
(2) *$U_n(R, \sigma)$ is nil-semicommutative.*
(3) *$D_n(R, \sigma)$ is nil-semicommutative.*
(4) *$V_n(R, \sigma)$ is nil-semicommutative.*

Proof. It is enough to show (1)⇒(2) since the class of nil-semicommutative rings is closed under subrings. Assume that R is a nil-semicommutative ring and $n \geq 2$. For a nilpotent ideal

$$I = \{A \in U_n(R, \sigma) \mid \text{ each diagonal entry of } A \text{ is zero}\}$$

of $U_n(R, \sigma)$, we have $\frac{U_n(R,\sigma)}{I} \cong \oplus_{i=1}^n R_i$, where $R_i = R$, is nil-semicommutative by Proposition 1.6(1). Hence $U_n(R, \sigma)$ is also a nil-semicommutative ring by [7, Corollary 2.4]. □

The following corollary which includes [7, Proposition 2.5] and [21, Proposition 2.3] directly comes from Theorem 1.8.

Corollary 1.9. *For a ring R and $n \geq 2$, the following conditions are equivalent*:
(1) *R is nil-semicommutative.*
(2) *$U_n(R)$ is nil-semicommutative.*
(3) *$D_n(R)$ is nil-semicommutative.*
(4) *$V_n(R)$ is nil-semicommutative.*

Armendariz [5, Lemma 1] proved that $ab = 0$ for all $a \in C_{f(x)}, b \in C_{g(x)}$ whenever $f(x)g(x) = 0$ for $f(x), g(x) \in R[x]$, where R is a reduced ring. Based on this result, Rege-Chhawchharia [22] called a ring (not necessarily reduced) *Armendariz* if it satisfies Armendariz's result. So reduced rings are clearly Armendariz. Typical examples of non-reduced Armendariz rings are $D_n(A)$ for $n = 2, 3$ over a reduced ring A, by [15, Proposition 2]. Armendariz rings are Abelian by the proof of [3, Theorem 6].

Observe that the class of Armendariz rings and the class of nil-semicommutative rings do not imply each other by the next example.

Example 1.10. (1) We apply [4, Example 4.8]. Let K be a field and $A = K\langle a, b\rangle$ be the free algebra generated by the noncommuting indeterminates a, b. Let I be the ideal of A generated by a^2 and $R = A/I$. Then R is Armendariz by [4, Example 4.8] and so R is Abelian. Identify a and b with their images in R for simplicity. Then $(ba)ab \in N(R)$ but $(ba)ba \notin N(R)$ since $ba \notin N(R)$. Thus R is not nil-semicommutative.

(2) The ring $U_n(R)$ $(n \geq 2)$ over a nil-semicommutative ring R is nil-semicommutative by Theorem 1.8, but it can be checked that $U_2(A)$ over a division ring A is not Abelian, and hence it is not Armendariz.

Antoine called a ring R *nil-Armendariz* [4, Definition 2.3] if $ab \in N(R)$ for all $a \in C_{f(x)}$ and $b \in C_{g(x)}$ whenever two polynomials $f(x), g(x) \in R[x]$ satisfy $f(x)g(x) \in N(R)[x]$. Nil-Armendariz rings strictly contain both the class of NI rings and the class of Armendariz rings by [4, Proposition 2.1 and Proposition 2.7]. Nil-Armendariz rings need not be nil-semicommutative by Example 1.10(1). Note that for a nil-semicommutative ring, the concept of a nil-Armendariz ring coincides with the concept of an NI ring by [4, Lemma 3.2(d)] and Corollary 1.4. Consequently, a ring is nil-Armendariz and nil-semicommutative if and only if it is NI.

Moreover, we have the following result.

A ring R is called (*von Neumann*) *regular* if for each $a \in R$ there exists $b \in R$ such that $a = aba$.

Proposition 1.11. *For a regular ring R, R is nil-semicommutative if and only if R is Armendariz if and only if R is nil-Armendariz if and only if R is NI.*

Proof. Recall that a regular ring R is Armendariz if and only if R is nil-Armendariz if and only if R is NI if and only if R is reduced by [17, Theorem 20]. Let R be a regular and nil-semicommutative ring. Then we claim the R is reduced. Assume on the contrary that there exists $0 \neq a \in R$ with $a^2 = 0$. Since R is regular, there exists $b \in R$ such that $a = aba$. Since R is nil-semicommutative, $a^2b^2 = 0$ implies $abab \in N(R)$ and so $ab \in N(R)$. But $(ab)^n = ab \neq 0$ for any $n \geq 1$, a contradiction. Therefore R is reduced. □

2. Nil-semicommutative and NI properties of Ore extensions

Recall that for an endomorphism σ of a ring R, the additive map $\delta : R \to R$ is called a *σ-derivation* if

$$\delta(ab) = \delta(a)b + \sigma(a)\delta(b) \quad \text{for any } a, b \in R.$$

For a ring R with an endomorphism σ of R and a σ-derivation δ, the *Ore extension* $R[x; \sigma, \delta]$ of R is the ring obtained by giving the polynomial ring over R a new skew-multiplication

$$xr = \sigma(r)x + \delta(r)$$

for all $r \in R$. If $\delta = 0$, then we write $R[x; \sigma]$ for $R[x; \sigma, 0]$ and it is called an *Ore extension of endomorphism type*, and for an identity endomorphism 1_R of R, we write $R[x; \delta]$ for $R[x; 1_R, \delta]$ and it is called an *Ore extension of derivation type*. The ring $R[[x; \sigma]]$ is called a *skew power series ring*.

According to Krempa [16], an endomorphism σ of a ring R is called *rigid* if $a\sigma(a) = 0$ implies $a = 0$ for $a \in R$. Hong et al. [10] called R a *σ-rigid ring* if there exists a rigid endomorphism σ of R. Note that any rigid endomorphism of a ring is a monomorphism and σ-rigid rings are reduced rings by [10, Proposition 5]. Following [9], a ring R is called *σ-compatible* if for each $a, b \in R$, $ab = 0 \Leftrightarrow a\sigma(b) = 0$, and R is called *δ-compatible* if for each $a, b \in R$, $ab = 0 \Rightarrow a\delta(b) = 0$. If R is both σ-compatible and δ-compatible, then R is called (σ, δ)-compatible, in this case the endomorphism σ is clearly a monomorphism.

On the other hand, for a σ-ideal I (i.e., $\sigma(I) \subseteq I$) of a ring R, I is called a *σ-rigid ideal* of R [11] if $a\sigma(a) \in I$ for $a \in R$ implies $a \in I$. Obviously, R is a σ-rigid ring if and only if the zero ideal of R is a σ-rigid ideal. If R is a σ-rigid ring, then $N^*(R) = 0$ is clearly a σ-rigid ideal, but the converse does not hold by Example 2.4 to follow.

From now on, let σ be *a* non-zero non-identity endomorphism of each given ring and δ be a σ-derivation. For a σ-derivation δ, an ideal I of a ring R is called a *δ-ideal* of R if $\delta(I) \subseteq I$, and if I is both a σ-rigid ideal and a δ-ideal, then we call I a *σ-rigid δ-ideal.*

Lemma 2.1. *For a ring R, let $N^*(R)$ be a σ-rigid ideal of R. Then we get the following result.*

(1) *R is NI.*

(2) *If* $ab \in N(R)$ *for* $a, b \in R$, *then* $a\sigma^n(b), \sigma^n(a)b \in N(R)$ *for every positive integer* n. *Conversely, if* $a\sigma^k(b)$ *or* $\sigma^k(a)b \in N(R)$ *for some positive integer* k, *then* $ab \in N(R)$.

(3) *If* $a_1a_2\cdots a_n \in N(R)$ *for* $a_1, a_2, \ldots, a_n \in R$, *then*

$$\sigma^{l_1}(a_{\theta(1)})\sigma^{l_2}(a_{\theta(2)})\cdots\sigma^{l_n}(a_{\theta(n)}) \in N(R) \in N(R)$$

for any $l_i \geq 0$ *and any permutation* θ *of the set* $\{1, 2, \ldots, n\}$.

Conversely, if $\sigma^{k_1}(a_1)\sigma^{k_2}(a_2)\cdots\sigma^{k_n}(a_n) \in N(R)$ *where* $a_i \in R$ *and* $k_i \geq 0$ *for* $1 \leq i \leq n$, *then* $a_1a_2\cdots a_n \in N(R)$.

(4) *Suppose that* $N^*(R)$ *is a* δ*-ideal of* R. *Then*

(i) $ab \in N(R)$ *implies* $a\delta^n(b), \delta^n(a)b \in N(R)$ *for every positive integer* n;

(ii) $a_1a_2\cdots a_n \in N(R)$ *for* $a_1, a_2, \ldots, a_n \in R$ *implies*

$$\delta^{l_1}(a_{\theta(1)})\delta^{l_2}(a_{\theta(2)})\cdots\delta^{l_n}(a_{\theta(n)}) \in N(R)$$

for any $l_i \geq 0$ *and any permutation* θ *of the set* $\{1, 2, \ldots, n\}$.

Proof. (1) This is in [11, Corollary 2.3].

(2) and (4)-(i) follow from [11, Proposition 2.4] and (1).

(3) Since R is NI by (1), it is nil-semicommutative. Let $a_1a_2\cdots a_n \in N(R)$ for $a_1, a_2, \ldots, a_n \in R$. By Theorem 1.2(2), we obtain $a_{\theta(1)}a_{\theta(2)}\cdots a_{\theta(n)} \in N(R)$ for any permutation θ of the set $\{1, 2, \ldots, n\}$. Then

$$\begin{aligned}
&(a_{\theta(1)}a_{\theta(2)}\cdots a_{\theta(n-1)})a_{\theta(n)} \in N(R)\\
\Rightarrow\ &(a_{\theta(1)}a_{\theta(2)}\cdots a_{\theta(n-1)})\sigma^{l_n}(a_{\theta(n)}) \in N(R) \text{ for any } l_n \geq 0\\
\Rightarrow\ &(\sigma^{l_n}(a_{\theta(n)}))(a_{\theta(1)}a_{\theta(2)}\cdots a_{\theta(n-2)})a_{\theta(n-1)} \in N(R)\\
\Rightarrow\ &\sigma^{l_{n-1}}(a_{\theta(n-1)})\sigma^{l_n}(a_{\theta(n)})(a_{\theta(1)}a_{\theta(2)}\cdots a_{\theta(n-3)})a_{\theta(n-2)} \in N(R)\\
&\text{for any } l_n, l_{n-1} \geq 0\\
&\vdots\\
\Rightarrow\ &\sigma^{l_1}(a_{\theta(1)})\sigma^{l_2}(a_{\theta(2)})\cdots\sigma^{l_n}(a_{\theta(n)}) \in N(R) \quad \text{for any } \ l_i \geq 0
\end{aligned}$$

by (2).

Conversely, assume that $\sigma^{k_1}(a_1)\sigma^{k_2}(a_2)\cdots\sigma^{k_n}(a_n) \in N(R)$ where $a_i \in R$ and $k_i \geq 0$ for $1 \leq i \leq n$. By the inverse operation of the previous computation and (2), we have $a_1a_2\cdots a_n \in N(R)$.

(4)-(ii) This is obtained by the same argument as in the proof of (3) and (4)-(i), replacing σ with δ. □

Regarding Lemma 2.1(1), there exists an NI ring R but $N^*(R)$ is not a σ-rigid ideal of R by [11, Example 3.5]. For convenience, we will use the fact $N^*(R) = N(R)$ whenever $N^*(R)$ is a σ-rigid ideal of R without reference, in the procedure.

Lemma 2.2. *If R is a (σ,δ)-compatible ring, then*
(1) *$abc=0$ implies $a\sigma(b)c=0$ and so $a\sigma(b)\delta(c)=0$ for $a,b,c\in R$.*
(2) *$a^n=0$ for $a\in R$ and $n\geq 2$ implies $\sigma(a)\delta(a^{n-1})=0$ and $\delta(a)a^{n-1}=0$.*

Proof. (1) For $a,b,c\in R$,

$$abc=0 \ \Rightarrow\ a\sigma(bc)=0 \ \Rightarrow\ a\sigma(b)\sigma(c)=0 \ \Rightarrow\ a\sigma(b)c=0 \ \Rightarrow\ a\sigma(b)\delta(c)=0$$

by the (σ,δ)-compatibility of R.

(2) Let $a^n=0$ for $a\in R$ and $n\geq 2$. Then we directly get $\sigma(a)\delta(a^{n-1})=0$ by (1). Next,

$$a^n=0 \ \Rightarrow\ \delta(a^n)=0 \ \Rightarrow\ \delta(a)a^{n-1}+\sigma(a)\delta(a^{n-1})=0 \ \Rightarrow\ \delta(a)a^{n-1}=0$$

since $\sigma(a)\delta(a^{n-1})=0$. □

Proposition 2.3. *If R is a (σ,δ)-compatible NI ring, then $N^*(R)$ is a σ-rigid δ-ideal of R.*

Proof. Suppose that R is a (σ,δ)-compatible NI ring. We first show that $N^*(R)=N(R)$ is a σ-ideal of R. Let $a\in N^*(R)$ for $a\in R$. Then $a^n=0$ for some $n\geq 2$ implies $(\sigma(a))^n=\sigma(a^n)=0$ and so $\sigma(a)\in N^*(R)$. Thus $N^*(R)$ is a σ-ideal.

Now, let $a\sigma(a)\in N^*(R)$ for $a\in R$. Then $(a\sigma(a))^n=0$ for some $n\geq 2$. Using Lemma 2.2(1), we have

$$\begin{aligned}(a\sigma(a))a(\sigma(a)(a\sigma(a))^{n-2})=0 \ &\Rightarrow\ a\sigma(a)\sigma(a)\sigma(a)(a\sigma(a))^{n-2}=0\\ &\Rightarrow\ a\sigma(a^3)(a\sigma(a))^{n-2}=0.\end{aligned}$$

Continuing this process, $a\sigma(a^{2n-1})=0$ implies $\sigma(a^{2n})=0$ by Lemma 2.2(1). Since σ is a monomorphism, we have $a^{2n}=0$ and so $a\in N^*(R)$. Thus $N^*(R)$ is a σ-rigid ideal.

Finally, we show that $N^*(R)$ is a δ-ideal. Let $a\in N^*(R)$ for $a\in R$. Then $a^n=0$ for some $n\geq 2$ and so $\delta(a)a^{n-1}=0$ by Lemma 2.2(2). Thus

$$\begin{aligned}\delta(a)a^{n-1}=0 \ &\Rightarrow\ \delta(a)\delta(a^{n-1})=0\\ &\Rightarrow\ \delta(a)(\delta(a)a^{n-2}+\sigma(a)\delta(a^{n-2}))=0\\ &\Rightarrow\ (\delta(a))^2a^{n-2}=0\end{aligned}$$

by the δ-compatibility of R and Lemma 2.2(1). Then we inductively have $(\delta(a))^n=0$ and so $\delta(a)\in N^*(R)$, completing the proof. □

Following the literature, a ring R is called *reversible* if $ab=0$ implies $ba=0$ for $a,b\in R$. It is well-known that reduced rings are reversible, and that reversible rings are semicommutative but the converse does not hold in either case.

The nil-semicommutative property between $R[x;\sigma,\delta]$ and R is studied by Ouyang and Chen [21], when R is a (σ,δ)-compatible reversible ring. In this section, we continue to study for the nil-semicommutative property of $R[x;\sigma,\delta]$ and $R[[x;\sigma]]$.

Notice that if R is a (σ,δ)-compatible reversible ring, then $N^*(R)$ is clearly a σ-rigid δ-ideal of R by Proposition 2.3, but not conversely in general by the following example.

Example 2.4. We apply the example in [11]. Let $R = \left(\begin{smallmatrix} F & F \\ 0 & F \end{smallmatrix}\right)$, where F is a field. Then

$$N^*(R) = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix} = N(R).$$

Let $\sigma : R \to R$ be defined by $\sigma\left(\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)\right) = \left(\begin{smallmatrix} a & 0 \\ 0 & c \end{smallmatrix}\right)$. Then σ is obviously not a monomorphism. Thus R is not σ-compatible. Moreover, it is easy to check that R is not reversible (and hence R is not a σ-rigid ring).

Now we show that $N^*(R)$ is a σ-rigid ideal of R. Clearly $N^*(R)$ is a σ-ideal of R. If

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \sigma\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) \in N^*(R), \text{ for} \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in R,$$

then

$$\begin{pmatrix} a^2 & bc \\ 0 & c^2 \end{pmatrix} \in N^*(R) = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}.$$

This implies that $a^2 = 0$ and $c^2 = 0$, and so $a = 0 = c$. Thus $\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & b \\ 0 & 0 \end{smallmatrix}\right) \in N^*(R)$. Therefore $N^*(R)$ is a σ-rigid ideal of R.

Following [21], for integers i, j with $0 \le i \le j$, let $f_i^j \in \mathrm{End}(R,+)$ be the map which is the sum of all possible words in σ, δ built with i letters σ and $j-i$ letters δ. For example, $f_0^0 = 1, f_j^j = \sigma^i, f_0^j = \delta^j$ and

$$f_{j-1}^j = \sigma^{j-1}\delta + \sigma^{j-2}\delta\sigma + \cdots + \delta\sigma^{j-1}.$$

Proposition 2.5. *For a ring R, assume that $N^*(R)$ is a σ-rigid δ-ideal of R.*

(1) *$ab \in N(R)$ implies $af_i^j(b) \in N(R)$ for all $j \ge i \ge 0$ and $a, b \in R$.*

(2) *$N(R[x;\sigma,\delta]) \subseteq N(R)[x;\sigma,\delta]$.*

Proof. (1) If $ab \in N(R)$ for $a, b \in R$, then $a\sigma^i(b), a\delta^j(b) \in N(R)$ for all $i, j \ge 0$ by Lemma 2.1(2), (4), and thus $af_i^j(b) \in N(R)$ for all $j \ge i \ge 0$.

(2) We apply the proof of [21, Lemma 2.10]. Suppose that $p(x) \in N(R[x;\sigma,\delta])$ for $p(x) = a_0 + a_1x + \cdots + a_nx^n \in R[x;\sigma,\delta]$. We show that $a_i \in N(R)$ for all $0 \le i \le n$. We proceed by induction on n. Let $(p(x))^k = 0$ for some positive integer k. Then

$$a_n\sigma^n(a_n)\sigma^{2n}(a_n)\cdots\sigma^{(k-1)n}(a_n) = 0 \in N(R)$$

and so $a_n \in N(R)$ by Lemma 2.1(3). Then $f_s^t(a_n) \in N(R)$ for all $t \ge s \ge 0$ by (1). Let $q(x) = a_0 + a_1x + \cdots + a_{n-1}x^{n-1}$. Then

$$0 = (p(x))^k = (q(x) + a_nx^n)^k = q(x)^k + r(x) \in N(R)[x;\sigma,\delta],$$

where $r(x) \in R[x;\sigma,\delta]$. The coefficients of $r(x)$ can be written as sums of monomials in a_i and $f_u^v(a_j)$ for $i,j \in \{0,1,\ldots,n\}$ and $v \geq u \geq 0$, and each monomial has a_n or $f_u^v(a_n)$ and hence $r(x) \in N(R)[x;\sigma,\delta]$. Thus $(q(x))^k \in N(R)[x;\sigma,\delta]$, and it implies that

$$a_{n-1}\sigma^{n-1}(a_{n-1})\cdots\sigma^{(k-1)(n-1)}(a_{n-1}) \in N(R)$$

and so $a_{n-1} \in N(R)$ by Lemma 2.1(3). Using induction on n, we get $a_i \in N(R)$ for all i. Therefore $N(R[x;\sigma,\delta]) \subseteq N(R)[x;\sigma,\delta]$. □

Corollary 2.6. *For a ring R, let $N^*(R)$ be a σ-rigid δ-ideal of R. Then we have the following result.*

(1) [11, Proposition 3.8] $N^*(R[x;\sigma,\delta]) \subseteq N^*(R)[x;\sigma,\delta]$.

(2) $N^*(R)[x;\sigma,\delta] \subseteq N^*(R[x;\sigma,\delta])$ *if and only if* $R[x;\sigma,\delta]$ *is NI.*

Proof. (1) By Lemma 2.1(1) and Proposition 2.5(2), we have

$$N^*(R[x;\sigma,\delta]) \subseteq N(R[x;\sigma,\delta]) \subseteq N(R)[x;\sigma,\delta] = N^*(R)[x;\sigma,\delta].$$

(2) This comes from (1) and [11, Theorem 3.10]. □

Note that the class of NI rings is closed under subrings by [13, Proposition 2.4], and we will freely use this fact without reference.

Theorem 2.7. *For a ring R, let $N^*(R)$ be a σ-rigid δ-ideal of R. Then $N(R)[x;\sigma,\delta] \subseteq N(R[x;\sigma,\delta])$ if and only if $R[x;\sigma,\delta]$ is NI.*

Proof. Suppose that $N(R)[x;\sigma,\delta] \subseteq N(R[x;\sigma,\delta])$. Then

$$N(R)[x;\sigma,\delta] = N(R[x;\sigma,\delta])$$

by Proposition 2.5(2). This shows that $N(R[x;\sigma,\delta])$ is an ideal of $R[x;\sigma,\delta]$ because $N(R)$ is a σ-rigid δ-ideal, and so $R[x;\sigma,\delta]$ is NI.

Conversely, assume that $R[x;\sigma,\delta]$ is NI. Note that R is NI, and so we have

$$N(R)[x;\sigma,\delta] = N^*(R)[x;\sigma,\delta] \subseteq N^*(R[x;\sigma,\delta]) = N(R[x;\sigma,\delta])$$

by Corollary 2.6(2). □

Corollary 2.8 ([21, Theorem 2.12]). *Let R be a reversible ring. If R is (σ,δ)-compatible, then R is nil-semicommutative if and only if $R[x;\sigma,\delta]$ is nil-semicommutative.*

Proof. Notice that if R is a (σ,δ)-compatible reversible ring, then $N^*(R)$ is a σ-rigid δ-ideal of R and $N(R)[x;\sigma,\delta] \subseteq N(R[x;\sigma,\delta])$ by Proposition 2.3 and [21, Lemma 2.10], respectively. Hence $R[x;\sigma,\delta]$ is NI by Theorem 2.7. This yields that both R and $R[x;\sigma,\delta]$ are nil-semicommutative. □

Let R_0 be the nil K-algebra (where K is any countable field) constructed by Smoktunowicz [24]. Smoktunowicz showed that $R_0[x]$ is not nil in [24, Theorem 12]. Let $R = K + R_0$. Then $N(R) = R_0$ and $R/R_0 \cong K$, entailing that R is NI (hence nil-semicommutative). Note that NI property does not go up to polynomial rings by help of Smoktunowicz. Consider the polynomial ring $R[x, y]$ with two commuting indeterminates x, y. Smoktunowicz showed that $a + bx + cy \notin N(R[x, y])$ in spite of $a, b, c \in N(R)$ (equivalently, $a, bx, cy \in N(R[x, y])$) in [24, Theorem 12]. This implies that nil-semicommutative property also does not go up to polynomial rings.

But we have the following result from Theorem 2.7, recalling that the class of NI rings is closed under subrings.

Corollary 2.9. *For a ring R, the following conditions are equivalent*:
(1) *R is NI and $N^*(R)[x] \subseteq N^*(R[x])$.*
(2) *R is NI and $N(R)[x] \subseteq N(R[x])$.*
(3) *$R[x]$ is NI.*

Corollary 2.9 and Proposition 1.5(2) provide the next result, noting that the class of nil-semicommutative rings is closed under subrings.

Corollary 2.10. *For a ring R with $N(R)[x] \subseteq N(R[x])$, the following conditions are equivalent*:
(1) *R is nil-semicommutative.*
(2) *R is NI.*
(3) *$R[x]$ is NI.*
(4) *$R[x]$ is nil-semicommutative.*

Recall that $N(R)[x] = N(R[x])$ when R is an Armendariz by [4, Corollary 5.2]. Hence, for an Armendariz ring R, the ring R is nil-semicommutative if and only if $R[x]$ is, and moreover we have the following corollary by Corollary 2.9.

Corollary 2.11 ([14, Proposition 18])**.** *If a ring R is both NI and Armendariz, then $R[x]$ is NI.*

Let $C_{p(x)}$ also denote the set of all coefficients of $p(x)$ for $p(x) \in R[x; \sigma, \delta]$ (or, $p(x) \in R[[x; \sigma]]$).

Theorem 2.12. *For a ring R, let $N^*(R)$ be a σ-rigid δ-ideal of R. For $p_1(x), p_2(x), \ldots, p_n(x) \in R[x; \sigma, \delta]$,*

$$p_1(x)p_2(x) \cdots p_n(x) \in N(R)[x; \sigma, \delta] \text{ if and only if } a_1a_2 \cdots a_n \in N(R),$$

where $a_i \in C_{p_i(x)}$ for $i = 1, 2, \ldots, n$.

Proof. We partially refer to the method in the proof of [21, Theorem 2.11]. Let $p_1(x)p_2(x) \cdots p_n(x) \in N(R)[x; \sigma, \delta]$ for $p_1(x), p_2(x), \ldots, p_n(x) \in R[x; \sigma, \delta]$. We proceed by induction on n. The case $n = 2$ comes from [11, Theorem 2.5]. Let $n \geq 3$.

`Claim 1:` *$p(x)q(x)d \in N(R)[x; \sigma, \delta]$ for $d \in R$ if and only if $abd \in N(R)$ for any $a \in C_{p(x)}$ and $b \in C_{q(x)}$.*

Suppose that $p(x)q(x)d \in N(R)[x; \sigma, \delta]$ for $d \in R$. Let $q(x) = \sum_{j=0}^{n} b_j x^j$ and $q(x)d = \sum_{s=0}^{n} q_s x^s$ where $q_s = \sum_{j=s}^{n} b_j f_s^j(d)$ for $0 \leq s \leq n$. We show that

$ab_jd \in N(R)$ for any $a \in C_{p(x)}$ and all j. We proceed by induction on n. Since $p(x)q(x)d \in N(R)[x;\sigma,\delta]$, $aq_s = a(\sum_{j=s}^{n} b_j f_s^j(d)) \in N(R) = N^*(R)$ for any $a \in C_{p(x)}$ and $0 \leq s \leq n$.

If $s = n$, then $aq_s = ab_n\sigma^n(d) \in N(R)$ and so $ab_nd \in N(R)$ by Lemma 2.1(2).

If $s = n-1$, then

$$aq_s = ab_{n-1}f_{n-1}^{n-1}(d) + ab_nf_{n-1}^{n}(d) = ab_{n-1}\sigma^{n-1}(d) + ab_nf_{n-1}^{n}(d) \in N(R).$$

Since $ab_nd \in N(R)$, we have $ab_nf_{n-1}^n(d) \in N(R)$ by Proposition 2.5(1) and so $ab_{n-1}\sigma^{n-1}(d) \in N(R)$. Thus $ab_{n-1}d \in N(R)$.

Now assume that we have $ab_jd \in N(R)$ for all $j > k$ and any $a \in C_{p(x)}$. Let $s = k$, then

$$aq_k = a(b_kf_k^k(d) + b_{k+1}f_k^{k+1}(d) + \cdots + b_nf_k^n(d)) \in N(R).$$

Since $ab_jd \in N(R)$ for $j > k$, $ab_jf_k^j(d) \in N(R)$. By the similar method to above, we have $ab_jf_k^k(d) = ab_j\sigma^k(d) \in N(R)$ and hence $ab_kd \in N(R)$. By induction hypothesis, we have $ab_jd \in N(R)$ for any $a \in C_{p(x)}$ and all j.

Conversely, suppose that $abd \in N(R)$ for $d \in R$ and any $a \in C_{p(x)}, b \in C_{q(x)}$. Then $abf_s^t(d) \in N(R)$ for all $t \geq s \geq 0$ by Proposition 2.5(1). Then all coefficients of $p(x)q(x)d$ are in $N(R)$, i.e., $p(x)q(x)d \in N(R)[x;\sigma,\delta]$.

`Claim 2:` $p_1(x)p_2(x)\cdots p_n(x) \in N(R)[x;\sigma,\delta]$ *if and only if* $a_1a_2\cdots a_n \in N(R)$ *where* $a_i \in C_{p_i(x)}$ *for* $i = 1, 2, \ldots, n$.

Let $h(x) = p_1(x)p_2(x)\cdots p_{n-1}(x)$. Then $h(x)p_n(x) \in N(R)[x;\sigma,\delta]$ and so $a_ha_n \in N(R)$ for any $a_h \in C_{h(x)}$ and $a_n \in C_{p_n(x)}$ by [11, Theorem 2.5]. Thus for all $a_n \in C_{p_n(x)}$,

$$(p_1(x)p_2(x)\cdots p_{n-2}(x))p_{n-1}(x)a_n \in N(R)[x;\sigma,\delta].$$

Let $p(x) = p_1(x)p_2(x)\cdots p_{n-2}(x)$ and $q(x) = p_{n-1}(x)$. Then for all $a_n \in C_{p_n(x)}$, we have $p(x)q(x)a_n \in N(R)[x;\sigma,\delta]$ and so $a_pa_{n-1}a_n \in N(R)$ for any $a_p \in C_{p(x)}$ and $a_{n-1} \in C_{p_{n-1}(x)}$ by `Claim 1`. Then for all $a_{n-1} \in C_{p_{n-1}(x)}$ and $a_n \in C_{p_n(x)}$,

$$p_1(x)p_2(x)\cdots p_{n-3}(x)p_{n-2}(x)(a_{n-1}a_n) \in N(R)[x;\sigma,\delta].$$

By the similar computation to above, we inductively obtain $a_1a_2\cdots a_n \in N(R)$ where $a_i \in C_{p_i(x)}$ for $i = 1, 2, \ldots, n$.

Conversely, suppose that $a_1a_2\cdots a_n \in N(R)$ where $a_i \in C_{p_i(x)}$ for $i = 1, 2, \ldots, n$. Then

$$a_1f_{u_1}^{v_1}(a_2)f_{u_2}^{v_2}(a_3)\cdots f_{u_{n-1}}^{v_{n-1}}(a_n) \in N(R)$$

for all $v_k, u_k \geq 0$ $(1 \leq k \leq n-1)$ by Proposition 2.5(1), completing the proof. □

As noted in the proof of Corollary 2.8, if R is a (σ,δ)-compatible reversible ring, then $R[x;\sigma,\delta]$ is NI. Hence, we can obtain the result of [21, Theorem 2.11] as a corollary, combining Proposition 2.5 and Theorem 2.7 with Theorem 2.12. Moreover,

Corollary 2.13. (1) [1, Theorem 3.6] *If R is a (σ,δ)-compatible semicommutative ring, then $a_ix^ib_jx_j \in N(R)[x;\sigma,\delta]$ whenever $p(x)q(x) = 0$ for any $p(x) = \sum_{i=0}^m a_ix^i, q(x) = \sum_{j=0}^n b_jx^j \in R[x;\sigma,\delta]$.*

(2) [2, Theorem 3.6] *If R is a σ-compatible NI ring, then $a_i\sigma^i(b_j) \in N(R)$ whenever $p(x)q(x) \in N(R)[x;\sigma]$ for any $p(x) = \sum_{i=0}^m a_ix^i, q(x) = \sum_{j=0}^n b_jx^j \in R[x;\sigma]$.*

(3) [4, Proposition 2.1] *NI rings are nil-Armendariz.*

Proof. (1) This comes from Proposition 2.3, Proposition 2.5(1) and Theorem 2.12.

(2) This follows from Lemma 2.1(2), Proposition 2.3 and Theorem 2.12.

(3) It directly follows from Theorem 2.12. □

Now we turn our attention to the nil-semicommutative property of the skew power series ring $R[[x;\sigma]]$. We first have the next proposition with the similar computation to the proof of Proposition 2.5(2), (3).

Proposition 2.14. *For a ring R, if $N^*(R)$ is a σ-rigid ideal of R, then we have the following result.*

(1) *$N(R[[x;\sigma]]) \subseteq N(R)[[x;\sigma]]$.*

(2) *For $p_1(x), p_2(x), \dots, p_n(x) \in R[[x;\sigma]]$,*

$$p_1(x)p_2(x)\cdots p_n(x) \in N(R)[[x;\sigma]] \text{ if and only if } a_1a_2\cdots a_n \in N(R),$$

where $a_i \in C_{p_i(x)}$ for $i = 1, 2, \dots, n$.

Proof. (1) Suppose that $p(x) \in N(R[[x;\sigma]])$ for $p(x) = \sum_{i=0}^\infty a_ix^i \in R[[x;\sigma]]$. Let $(p(x))^k = 0$ for some positive integer k. Then $a_0^k = 0$ and so $a_0 \in N(R)$. Let $q(x) = \sum_{i=1}^\infty a_ix^i$. Then

$$0 = (p(x))^k = (q(x) + a_0)^k = q(x)^k + r(x) \in N(R)[[x;\sigma]],$$

where $r(x) \in R[[x;\sigma]]$. The coefficients of $r(x)$ can be written as sums of monomials in a_i and $\sigma^m(a_j)$ for $i, j \geq 0$ and any $m \geq 0$, and each monomial has a_0 or $\sigma^m(a_0)$ and hence $r(x) \in N(R)[[x;\sigma]]$. Thus $(q(x))^k \in N(R)[[x;\sigma]]$, and it implies that

$$a_1\sigma(a_1)\sigma^2(a_1)\cdots\sigma^{k-1}(a_1) \in N(R)$$

and so $a_1 \in N(R)$ by Lemma 2.1(3). Continuing this process, we get $a_i \in N(R)$ for all i. Therefore $N(R[[x;\sigma]]) \subseteq N(R)[[x;\sigma]]$.

(2) We apply the proof of Theorem 2.12. We proceed by induction on n. The case $n = 2$ comes from [11, Proposition 2.7]. Let $n \geq 3$ and $q(x) = p_1(x)p_2(x)\cdots p_{n-1}(x)$. Then $q(x)p_n(x) \in N(R)[[x;\sigma]]$ and so $a_qa_n \in N(R)$ for any $a_q \in C_{q(x)}$ and $a_n \in C_{p_n(x)}$ by [11, Proposition 2.7]. By the same argument as in the proof of Theorem 2.12, we obtain $a_1\sigma^{v_1}(a_2)\sigma^{v_2}(a_3)\cdots\sigma^{v_{n-1}}(a_n) \in$

$N(R)$ for all $v_k \geq 0$ ($1 \leq k \leq n-1$) and any $a_i \in C_{p_i(x)}$ ($1 \leq i \leq n$), replacing f_u^v with σ^v. Thus $a_1 a_2 \cdots a_n \in N(R)$ by Lemma 2.1(2).

The converse can be obtained by Lemma 2.1(3). □

Corollary 2.15. *For a ring R, let $N^*(R)$ be a σ-rigid ideal of R.*

(1) [11, Proposition 3.12] $N^*(R[[x;\sigma]]) \subseteq N^*(R)[[x;\sigma]]$.

(2) $N^*(R)[[x;\sigma]] \subseteq N^*(R[[x;\sigma]])$ *if and only if* $R[[x;\sigma]]$ *is NI.*

Proof. (1) This comes from Lemma 2.1(1) and Proposition 2.14(1).

(2) This follows from (1) and [11, Proposition 3.14]. □

The property "$N^*(R)$ is a σ-rigid ideal" and the condition "$N(R)[[x;\sigma]] \subseteq N(R[[x;\sigma]])$" are independent of each other by [11, Examples 3.13 and 3.15(1)]. [11, Example 3.15(1)] also shows that the condition "$N^*(R)$ is σ-rigid ideal" cannot be replaced by "R is NI" in Propositions 2.5(2) and 2.14(1).

Proposition 2.16. (1) *For a ring R, let $N^*(R)$ be a σ-rigid ideal of R. Then $N(R)[[x;\sigma]] \subseteq N(R[[x;\sigma]])$ if and only if $R[[x;\sigma]]$ is NI.*

(2) *If R is a nil-semicommutative ring and $N(R)[[x]] \subseteq N(R[[x]])$, then R is NI.*

Proof. (1) This is similar to the proof of Theorem 2.7, combining Proposition 2.14(1) with Corollary 2.15(2).

(2) It follows from the same argument as the proof of Proposition 1.5(2). □

Corollary 2.17. (1) *For a ring R, R is NI and $N^*(R)[[x]] \subseteq N^*(R[[x]])$ if and only if R is NI and $N(R)[[x]] \subseteq N(R[[x]])$ if and only if $R[[x]]$ is NI.*

(2) *For a ring R with $N(R)[[x]] \subseteq N(R[[x]])$, R is nil-semicommutative if and only if R is NI if and only if $R[[x]]$ is NI if and only if $R[[x]]$ is nil-semicommutative.*

Proof. (1) It comes from Proposition 2.16(1).

(2) This follows from (1) and Proposition 2.16(2). □

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Nam Kyun Kim
Faculty of Liberal Arts and Sciences
Hanbat National University
Daejeon 305-719, Korea
E-mail address: nkkim@hanbat.ac.kr

Tai Keun Kwak
Department of Mathematics
Daejin University
Pocheon 487–711, Korea
E-mail address: tkkwak@daejin.ac.kr

Yang Lee
Department of Mathematics Education
Pusan National University
Pusan 609-735, Korea
E-mail address: ylee@pusan.ac.kr