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PANCYCLIC ARCS IN HAMILTONIAN CYCLES OF HYPERTOURNAMENTS

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Abstract. A k-hypertournament H on n vertices, where $2 \leq k \leq n$, is a pair $H = (V, A)$, where V is the vertex set of H and A is a set of k-tuples of vertices, called arcs, such that for all subsets $S \subseteq V$ with $|S| = k$, A contains exactly one permutation of S as an arc. Recently, Li et al. showed that any strong k-hypertournament H on n vertices, where $3 \leq k \leq n-2$, is vertex-pancyclic, an extension of Moon's theorem for tournaments.

In this paper, we prove the following generalization of another of Moon's theorems: If H is a strong k-hypertournament on n vertices, where $3 \leq k \leq n-2$, and C is a Hamiltonian cycle in H, then C contains at least three pancyclic arcs.

1. Introduction and terminology

A *directed k-hypergraph D on n vertices*, for integers n and $k \geq 2$, is a pair $D = (V, A)$, where the cardinality of the vertex set V of D is n and the arc set A of D is a subset of V^k, such that no arc in A contains the same vertex in V twice. If not otherwise specified, we will denote the vertex set (arc set, respectively) of an arbitrary directed k-hypergraph D by $V(D)$ ($A(D)$, respectively).

For the rest of this section, let $D = (V, A)$ be a directed k-hypergraph on n vertices. For two distinct vertices $x, y \in V$, $A_D(x, y) \subseteq A(D)$ denotes the set of all arcs $a = (x_1, \ldots, x_k) \in A$, such that there are indices $1 \leq i_0 < i_1 \leq k$ with $x_{i_0} = x$ and $x_{i_1} = y$. An arc $a = (x_1, \ldots, x_k) \in A$ is called an *out-arc* of the vertex x_1.

Let $X \subseteq V$. Then $D[X] := (X, A \cap X^k)$ is the *subhypergraph of D induced by* X (note that $A(D[X]) = \emptyset$ if $|X| < k$) and $D - X$ denotes the subhypergraph $D[V(D) \setminus X]$. We write $D - x$ instead of $D - \{x\}$.

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A (v_1, v_{l+1})*-path of length* l or an l*-path from* v_1 *to* v_{l+1} *in* D is a sequence $P = v_1 a_1 v_2 \cdots a_l v_{l+1}$, such that $v_1, \ldots, v_{l+1} \in V$ are pairwise distinct vertices, $a_1, \ldots, a_l \in A$ are pairwise distinct arcs and $a_i \in A_D(v_i, v_{i+1})$ holds for all $1 \leq i \leq l$. An l*-cycle in* D is defined analogously, with the exception of $v_1 = v_{l+1}$. For convenience, we will consider v_{l+1} to be v_1 in the context of an l-cycle $C = v_1 \cdots v_l v_1$. Let $P = x_1 a_1 \cdots a_{l-1} x_l$ be a path in D and let $x_i, x_j \in V(P)$ be two vertices with $i \leq j$. Then $x_i P x_j$ denotes the unique (x_i, x_j)-subpath of P. xCy is defined analogously for a cycle C in D and vertices $x, y \in V(C)$. In the case $k = 2$, if P is an (x, y)-path and Q is an (v, w)-path in D such that $V(P) \cap V(Q) = \emptyset$ and $A_D(y, v) \neq \emptyset$, then PQ is the path obtained by appending the path Q to P. An n-cycle ($(n-1)$-path, respectively) in D is called *Hamiltonian* or *Hamiltonian cycle* (*Hamiltonian path*, respectively) in D.

A vertex (an arc, respectively) of D is *pancyclic*, if it is contained in an l-cycle for all $l \in \{3, \ldots, n\}$. D is called *pancyclic*, if it contains an l-cycle for all $l \in \{3, \ldots, n\}$ and *vertex-pancyclic*, if all of its vertices are pancyclic. A vertex is called *out-arc pancyclic*, if all of its out-arcs are pancyclic.

A digraph D is *strongly connected* or *strong*, if there is an (x, y)-path in D for all distinct vertices $x, y \in V$. A *strong component* D' *of* D is a maximal induced subhypergraph of D which is strong.

A digraph D is called d-strong, if $|V(D)| \geq d + 1$ holds and $D - U$ is strong for all $U \subseteq V(D)$ with $|U| < d$. Two paths in D are *edge-disjoint*, if they do not have a shared arc. A digraph D is called d*-edge-connected*, if there are d edge-disjoint (x, y)-paths in D for all distinct vertices $x, y \in V$.

A k*-hypertournament* H is a directed k-hypergraph, such that for all subsets $S \subseteq V(H)$ with $|S| = k$, $A(H)$ contains exactly one permutation of S as an arc. A *tournament* is a 2-hypertournament.

It is the strong structure of tournaments which has made them the best studied class of digraphs. It is only natural to try to reduce this structure to its core properties necessary to maintain at least most of the results for tournaments, while broadening the scope of considered directed hypergraphs. One of the generalizations of tournaments is the class of directed 2-hypergraphs which contain a spanning tournament as a subhypergraph, called *semicomplete digraphs*. In other words, every pair of distinct vertices of a semicomplete digraph is connected by at least one arc. Many results for tournaments also hold for the larger class of semicomplete digraphs. Because of the similarities in their definition, one would hope that the same is true for the class of hypertournaments. But there are some obstacles which arise from the loosened structure of hypertournaments and the fact that an arc no longer connects exactly two vertices. To give an example, we will first add some notation for the case $k = 2$.

In this case, we will omit the arcs in our notation of a path or a cycle, since the sequence of vertices imply the arcs connecting them. Furthermore, we will use $xy \in A(D)$ and sometimes $x \to y$ instead of $(x, y) \in A(D)$. For two disjoint sets $X, Y \subseteq V(D)$, $X \Rightarrow Y$ denotes that there is no arc from a vertex in Y to one in X in D. By $X \to Y$, we denote that $xy \in A(D)$ for all $x \in X$ and all $y \in Y$.

A strong property of tournaments (and semicomplete digraphs) and integral part of many proofs is the fact that the strong components $D_1, \ldots, D_r$ of D are pairwise disjoint and can be ordered, such that both $D_i \Rightarrow D_j$ and $D_i \to D_j$ hold for all $1 \le i < j \le r$. This unique order is called the *strong decomposition of* D.

Example 1.1. Let the 4-hypertournament $H_4 := (V, A)$ be defined through

$$
\begin{aligned}
V \;:=\; \{x_1, \ldots, x_5\} \text{ and } A \;&:=\; \{a_1, a_2, a_3, a_4, a_5\}, \text{ where}\\
a_1 \;&:=\; (x_3, x_4, x_1, x_2),\\
a_2 \;&:=\; (x_5, x_3, x_2, x_1),\\
a_3 \;&:=\; (x_4, x_5, x_2, x_1),\\
a_4 \;&:=\; (x_4, x_5, x_3, x_1),\\
\text{and } a_5 \;&:=\; (x_4, x_5, x_2, x_3).
\end{aligned}
$$

H_4 is not strong, since, for example, there is no path from x_1 to $\{x_4, x_5\}$. Suppose that there is such a path P. Obviously, P starts with the subpath $P' = x_1a_1x_2a_5x_3$. Now we see that a_1, the only arc from x_3 to $\{x_4, x_5\}$, is already contained in P' and thus, we cannot extend P', a contradiction. Furthermore, for all $X \subseteq V$ such that $2 \le |X| \le 4$, $A(H_4[X])$ contains at most one arc. Therefore, $H_4[X]$ is not strong. Consequently, the strong components of H_4 are its vertices and since there are arcs from x_1 to x_2 and vice versa, there is no strong decomposition of H_4.

Even if we weaken the definition of a strong component of a hypertournament, we still do not obtain a suitable structure. A *strong** *component of* D is a maximal induced subhypergraph D' such that there is an (x, y)-path in D for all distinct vertices $x, y \in V(D')$.

Since H_4 contains the cycles $x_1a_1x_2a_5x_3a_2x_1$ and $x_2a_5x_3a_1x_4a_4x_5a_2x_2$ but no path from x_1 to $\{x_4, x_5\}$, the vertex sets of the strong* components of H_4 are $\{x_1, x_2, x_3\}$ and $\{x_2, x_3, x_4, x_5\}$ and are therefore not disjoint, much less is there a strong* decomposition of H_4.

To account for this fact and to restore some of the structure, in 1997, Gutin and Yeo [3] introduced the majority digraph of a hypertournament.

For a k-hypertournament $H = (V, A)$ on n vertices, the *majority digraph* $M(H) = (V, A_{\mathrm{maj}}(H))$ *of* H is a digraph on the same vertex set and for a pair $x, y \in V$ of distinct vertices, xy is in $A_{\mathrm{maj}}(H)$ if and only if $|A_H(x, y)| \ge |A_H(y, x)|$, which is equivalent to

$$|A_H(x, y)| \ge \frac{1}{2}\binom{n-2}{k-2}.$$

By definition, there is an arc between every pair of distinct vertices, thus $M(H)$ is a semicomplete digraph.

This substructure allowed for Gutin and Yeo to prove the following generalizations of Redei's [9] and Camion's [1] theorem, respectively, two of the most fundamental results on tournaments.

Theorem 1.2 ([3]). *Every k-hypertournament on $n > k \geq 2$ vertices contains a Hamiltonian path.*

Theorem 1.3 ([3]). *Every strong k-hypertournament on n vertices, where $3 \leq k \leq n-2$, contains a Hamiltonian cycle.*

Furthermore, Gutin and Yeo posed the question whether Moon's theorem [6], which states that every strong tournament is vertex-pancyclic, could be extended to hypertournaments as well. In addition to giving some sufficient conditions for a hypertournament to be vertex-pancyclic, in 2006, Petrovic and Thomassen showed the following.

Theorem 1.4 ([8]). *Let H be a d-edge-connected k-hypertournament on n vertices. If $k = 3$ and $n \geq 30d + 2$ or $k \geq 4$ and $n \geq k + 1 + 24d$, then H contains d edge-disjoint Hamiltonian cycles.*

Amongst other results, in 2009, Yang gave an improvement of this theorem.

Theorem 1.5 ([10]). *Let H be a d-edge-connected k-hypertournament on n vertices. If $k = 3$ and $n \geq 14d + 1$ or $k \geq 8$ and $n \geq k + 2d + 1$, then H is d-edge-disjoint vertex-pancyclic, i.e., every vertex of H is contained in d edge-disjoint l-cycles for all $l \in \{3, \ldots, n\}$.*

Recently, Li et al. showed the following generalization of Moon's theorem and that its bound is best possible, thereby answering Gutin and Yeo's initial question.

Theorem 1.6 ([5]). *Every strong k-hypertournament with n vertices, where $3 \leq k \leq n-2$, is vertex-pancyclic.*

Goal of this paper, is the generalization of another of Moon's theorems on tournaments.

Theorem 1.7 ([7]). *Let T be a strong tournament. Then there is a Hamiltonian cycle in T that contains at least three pancyclic arcs.*

In fact, we will show that *every* Hamiltonian cycle of a hypertournament contains at least three pancyclic arcs.

Since the majority digraph of a strong hypertournament H is not necessarily strong, much less contains a specific Hamiltonian cycle corresponding to one in H, we will introduce a modified substructure, better suited for our own purposes.

Definition 1.8. Let $H = (V, A)$ be a strong k-hypertournament on $n \geq k \geq 3$ vertices and let $C = y_1 a_1 y_2 \cdots y_n a_n y_1$ be a Hamiltonian cycle in H. We define the *C-majority-digraph* $M(H, C) := (V, A^C_{\mathrm{maj}}(H))$ of H through

$$A^C_{\mathrm{maj}}(H) := (A_{\mathrm{maj}}(H) \setminus \{y_{i+1}y_i, y_1y_n \mid 1 \leq i < n\}) \cup \{y_iy_{i+1}, y_ny_1 \mid 1 \leq i < n\}.$$

For an $i \in \{1,\ldots,n-1\}$ we call $a_{y_iy_{i+1}} := a_i$ the *C-arc corresponding to y_iy_{i+1}*. $a_{y_ny_1} := a_n$ corresponds to y_ny_1.

Remark 1.9. In general, the C-majority-digraph of H does not have the property $A_{\mathrm{maj}}(H) \subseteq A^C_{\mathrm{maj}}(H)$. It still is semicomplete by definition and strong, since it contains the Hamiltonian cycle $C = y_1 \cdots y_n y_1$.

Let us now consider the following preliminaries.

2. Preliminaries

First of all, we note that Moon's theorem holds for semicomplete digraphs.

Corollary 2.1. *Every strong semicomplete digraph is vertex-pancyclic.*

Before we show the generalized version for hypertournaments, we will prove a stronger version of Theorem 1.7 for semicomplete digraphs. We will use the following results in the process.

Theorem 2.2 ([2])**.** *Let T be a 2-strong tournament. Then T contains at least three out-arc pancyclic vertices.*

Theorem 2.3 ([11])**.** *Let T be a non-strong tournament and let $T_1,\ldots,T_r$ be the strong decomposition of T. Then there is an (x,y)-path of length l in T for all $1 \le l \le |V(T)|-1$, $x \in V(T_1)$ and $y \in V(T_r)$.*

Corollary 2.4. *Let $D = (V,A)$ be a non-strong semicomplete digraph, let $D_1,\ldots,D_r$ be the strong decomposition of D, $1 \le i < j \le r$, $x \in V(D_i)$, $y \in V(D_j)$ and $l \in \{1,\ldots,|\bigcup_{i\le s\le j} V(D_r)|-1\}$. Then there is an (x,y)-path of length l in D.*

Theorem 2.5. *Let $D = (V,A)$ be a strong semicomplete digraph and C a Hamiltonian cycle in D. Then C contains at least three pancyclic arcs.*

Proof. Let $C = x_1x_2\cdots x_nx_1$. Without loss of generality, we may assume that D is a tournament, since we can destroy all 2-cycles in D such that the resulting tournament still contains the Hamiltonian cycle C. If D is 2-strong, D contains at least three out-arc pancyclic vertices, by Theorem 2.2. Suppose that D is not 2-strong, x_1 is a cut-vertex and $D_1,\ldots,D_r$ is the strong decomposition of $D - x_1$. Since $x_2x_3\cdots x_n$ is a path in $D - x_1$, x_2 is obviously contained in D_1 and x_n in D_r. By Corollary 2.4, there is an (x_2,x_n)-path $P^l_{x_2,x_n}$ of length l in $D - x_1$ for all $l \in \{1,\ldots,n-2\}$. Thus, x_1x_2 and x_nx_1 are contained in the l-cycle $x_1P^{l-2}_{x_2,x_n}x_1$ in D for all $l \in \{3,\ldots,n\}$ and are therefore pancyclic. Without loss of generality, we may assume that $|V(D_r)| \le \lfloor \frac{n-1}{2} \rfloor$.

We define the following indices.

$i_0 := \max\{i \mid 2 \le i \le n-1,\ x_1x_j \in A \text{ for all } 2 \le j \le i\}$.
$i_1 := \min\{i \mid 2 \le i \le n,\ x_i \in V(D_r)\}$.

We obviously have $V(D_r) = \{x_j \mid i_1 \leq j \leq n\}$ and $i_1 \geq \lceil \frac{n-1}{2} \rceil + 2$. $x_{i_0}x_{i_0+1}$ is contained in the l-cycle $x_1 x_{i_0+3-l} \cdots x_{i_0} x_{i_0+1} x_1$ in D for all $l \in \{3, \ldots, i_0+1\}$. If $i_0 \geq \lfloor \frac{n-1}{2} \rfloor + 1$, then we have $n+2-i_0 \leq \lceil \frac{n-1}{2} \rceil + 1 \leq i_0+2$. Thus, $x_{i_0}x_{i_0+1}$ is contained in the l-cycle $x_1 x_{n+2-l} \cdots x_n x_1$ in D for all $l \in \{n+2-i_0, \ldots, n\} \supseteq \{i_0+2, \ldots, n\}$ and is therefore pancyclic.

Suppose that $i_0 \leq \lfloor \frac{n-1}{2} \rfloor$. Then we have $i_0 + 1 < i_1$ and hence $x_{i_0+1} \notin V(D_r)$. Consequently, $D - \{x_1, \ldots, x_{i_0}\}$ is not strong. Furthermore, x_{i_0+1} is contained in the first and x_n is contained in the last component of the strong decomposition of $D - \{x_1, \ldots, x_{i_0}\}$, since $x_{i_0+1} \cdots x_n$ is a path in $D - \{x_1, \ldots, x_{i_0}\}$. By Corollary 2.4, there is an (x_{i_0+1}, x_n)-path $P^l_{x_{i_0+1},x_n}$ of length l in $D - \{x_1, \ldots, x_{i_0}\}$ for all $l \in \{1, \ldots, n-i_0-1\}$. Hence, $x_{i_0}x_{i_0+1}$ is contained in the l-cycle $x_1 \cdots x_{i_0} P^{l-(i_0+1)}_{x_{i_0+1},x_n} x_1$ in D for all $l \in \{i_0+2, \ldots, n\}$ and is therefore pancyclic. □

Lemma 2.6. *Let $k \geq 4$ and $n \geq k+2$.*

- *If $(n,k) \notin \{(6,4),(7,4),(7,5)\}$, then $\binom{n-2}{k-2} \geq 2n-1$ holds.*
- *If $(n,k) \neq (6,4)$, then $\binom{n-2}{k-2} \geq 2n-4$ holds.*

Theorem 2.7 ([4]). *Let S be a set, let J be a finite index set and let $(T_i)_{i \in J}$ be a family of subsets of S. Then there is an injective function $r : J \to S$ with $r(i) \in T_i$ for all $i \in J$ if and only if $|I| \leq |\bigcup_{i \in I} T_i|$ for all $I \subseteq J$ holds.*

Corollary 2.8. *Let H be a k-hypertournament $(k \geq 3)$, C a Hamiltonian cycle in H, C_M a cycle in $M(H,C)$ and $A_{vw} \subseteq A_H(v,w)$ for all $vw \in A(C_M)$. If $|I| \leq |\bigcup_{vw \in I} A_{vw}|$ for all $I \subseteq A(C_M)$, then every arc in $\bigcup_{vw \in A(C_M)} A_{vw}$ is contained in a cycle C_H in H on the same vertex set as C_M.*

Lemma 2.9. *Let H be a strong 3-hypertournament on $n \geq 5$ vertices, let D be a strong semicomplete digraph on the vertex set of H, $B_D \subseteq A(D)$ with $A(D) \setminus B_D \subseteq A_{\mathrm{maj}}(H)$ and $r : B_D \to A(H)$ an injective function, such that $r(xy) \in A_H(x,y)$ holds for all $xy \in B_D$. Then for every cycle C in D, there is a cycle C_H in H on the same vertex set. Furthermore, if C contains an arc $xy \in B_D$, then C_H can be chosen such that $r(xy)$ is contained in C_H.*

Proof. Let $C = x_1 \cdots x_l x_1$ be an l-cycle in D with $l \in \{2, \ldots, n\}$. If C contains an arc $xy \in B_D$ (without loss of generality, we may assume that $xy = x_1x_2$) we define $a^0_1 := r(xy)$. Otherwise, all arcs of C are contained in $A(D) \setminus B_D \subseteq A_{\mathrm{maj}}(H)$, in particular $|A_H(x_1,x_2)| \geq \frac{1}{2}\binom{n-2}{1} \geq \frac{3}{2}$ holds, by the definition of $A_{\mathrm{maj}}(H)$ and therefore, there is an $a^0_1 \in A_H(x_1,x_2)$. An l-cycle C_H in H on the vertex set of C, which contains a^0_1, can be constructed as follows. We start with a 1-path $z_1a_1z_2 := x_1a^0_1x_2$ in H. Let $z_1a_1z_2 \cdots a_{i-1}z_i$ be an $(i-1)$-path in H for an $i \in \{2, \ldots, l\}$ such that the following conditions are met:

(1) $z_1, \ldots, z_i \in \{x_1, \ldots, x_i\}$.
(2) $z_1 = x_1$, $z_i = x_i$ and $a_1 = a^0_1$.
(3) If $x_ix_{i+1} \in B_D$, then $a_{i-1} \neq r(x_ix_{i+1})$.

Suppose that $i \le l-2$. If $x_i x_{i+1} \in B_D$, we define $a_i := r(x_i x_{i+1})$ and $z_{i+1} := x_{i+1}$ and gain an i-path $z_1 a_1 z_2 \cdots a_i z_{i+1}$ in H, because for all $j \in \{1,\dots,i-2\}$ we have $a_i \neq a_j$, since $a_i \in A_H(x_i, x_{i+1})$, $a_j \in A_H(x_j, x_{j+1})$ and x_i, x_{i+1}, x_j and x_{j+1} are pairwise distinct. Furthermore, $a_{i-1} \neq r(x_i x_{i+1}) = a_i$ holds by condition (3) for $z_1 a_1 z_2 \cdots a_{i-1} z_i$. Obviously, $z_1 a_1 z_2 \cdots a_i z_{i+1}$ meets conditions (1) and (2). Condition (3) is met, since r is injective by assumption.

If $x_i x_{i+1} \in A(D) \setminus B_D$ and $x_{i+1} x_{i+2} \in A(D) \setminus B_D$, then $|A_H(x_i, x_{i+1})| \ge 2$ and thus, there is an arc $a_i \in A_H(x_i, x_{i+1}) \setminus \{a_{i-1}\}$. With $z_{i+1} := x_{i+1}$, the path $z_1 a_1 z_2 \cdots a_i z_{i+1}$ is a suitable i-path in H, since condition (3) obviously holds.

If $x_i x_{i+1} \in A(D) \setminus B_D$, but $x_{i+1} x_{i+2} \in B_D$, we define $a := r(x_{i+1} x_{i+2})$. If there is an $a_i \in A_H(x_i, x_{i+1}) \setminus \{a_{i-1}, a\}$, we proceed as in the case where $x_{i+1} x_{i+2} \in A(D) \setminus B_D$. Otherwise, we consequently have $A_H(x_i, x_{i+1}) = \{a_{i-1}, a\}$ and hence $|A_H(x_{i+1}, x_i)| = 1$. Then $a_{i-1} = (x_{i-1}, x_i, x_{i+1})$ and $a = (x_i, x_{i+1}, x_{i+2})$ hold and there exists an arc $b \in A_H(x_{i+1}, x_i)$. Therefore, we have $a \neq a_j$ for all $j \in \{1,\dots,i-1\}$, by representation of a, $b \neq a_j$ for all $j \in \{1,\dots,i-2\}$, since $b \in A_H(x_{i+1}, x_i)$, $a_j \in A_H(x_j, x_{j+1})$ and x_i, x_{i+1}, x_j and x_{j+1} are pairwise distinct, and $b \notin \{a_{i-1}, a\}$, since $b \in A_H(x_{i+1}, x_i)$ and $a_{i-1}, a \in A_H(x_i, x_{i+1})$. We gain an $(i+1)$-path $z_1 a_1 \cdots z_{i-1} a_{i-1} x_{i+1} b x_i a x_{i+2}$ in H, which obviously meets conditions (1) and (2). Condition (3) holds, since $a = r(x_{i+1} x_{i+2})$ and r is injective by assumption.

Suppose that $i = l-1$. Then the same arguments give us an $(l-1)$-path $z_1 a_1 z_2 \cdots a_{l-1} z_l$ in H, which meets the conditions above, or a suitable l-cycle $C_H = z_1 a_1 \cdots z_{i-1} a_{i-1} x_l b x_{l-1} a z_1$ in H. Note that in the latter case, we have $x_l x_1 \in B_D$. As a direct consequence, we have $x_1 x_2 \in B_D$ and therefore, $a = r(x_l x_1) \neq r(x_1 x_2) = a_1^0 = a_1$, since r is injective by assumption. In the case where $i = l$, we find an arc $a_l \in A_H(z_l, z_1) \setminus \{a_{l-1}, a_1\}$ and thereby a suitable l-cycle $C_H = z_1 a_1 \cdots z_l a_l z_1$ in H, or otherwise, a corresponding l-cycle $C_H = z_2 a_2 \cdots z_{l-1} a_{l-1} z_1 b z_l a_1 z_2$ in H, analogously. □

3. Main results

Theorem 3.1. *Let $H = (V, A)$ be a strong k-hypertournament on $n \ge k+2 \ge 5$ vertices and let C be a Hamiltonian cycle in H. Then C contains at least three pancyclic arcs.*

We will give the proof of Theorem 3.1 in form of four lemmas, where Lemmas 3.4 and 3.5 cover almost all hypertournaments and in Lemmas 3.7 and 3.6 the result is shown for a finite number of rather tedious exceptions. But first, let us consider the following corollaries to Theorem 3.1.

Corollary 3.2. *Let $H = (V, A)$ be a strong k-hypertournament on $n \ge k+2 \ge 5$ vertices. Then H contains at least three pancyclic arcs.*

Furthermore, Theorems 1.4 and 1.5 allow for a better bound of the pancyclic arcs contained in a d-edge-connected hypertournament.

Corollary 3.3. *Let $H = (V, A)$ be a d-edge-connected k-hypertournament on n vertices, with $k = 3$ and $n \geq 14d + 1$ or $4 \leq k \leq 7$ and $n \geq 24d + 1 + k$ or $k \geq 8$ and $n \geq 2d + k + 1$. Then H contains at least $3d$ pancyclic arcs.*

Lemma 3.4. *Let $H = (V, A)$ be a strong 3-hypertournament on $n \geq 5$ vertices and let C be a Hamiltonian cycle in H. Then C contains at least three pancyclic arcs.*

Proof. Let $C = x_1a_1x_2 \cdots x_na_nx_1$. We consider the C-majority-digraph $D := M(H, C)$ of H. By Theorem 2.5, $\tilde{C} := x_1x_2 \cdots x_nx_1$ contains at least three arcs that are pancyclic in D. Let $x_{i_0}x_{i_0+1}$ be such an arc for an i_0 in $\{1, \ldots, l\}$. We will show that a_{i_0} is pancyclic in H. Let $\tilde{C_l} = y_1 \cdots y_ly_1$ be an l-cycle in D that contains $x_{i_0}x_{i_0+1}$ for an $l \in \{3, \ldots, n-1\}$. We define $B_D := \{x_1x_2, x_2x_3, \ldots, x_{n-1}x_n, x_nx_1\}$ and $r : B_D \to A(H)$, $x_ix_{i+1} \mapsto a_i$ for all i in $\{1, \ldots, n\}$. By Definition 1.8 of D, the conditions of Lemma 2.9 are met and thus, there is an l-cycle C_H in H, that contains a_{i_0}. Since $x_{i_0}x_{i_0+1}$ and $l \in \{3, \ldots, n-1\}$ were arbitrarily chosen, C contains at least three pancyclic arcs. $\square$

Lemma 3.5. *Let $H = (V, A)$ be a strong k-hypertournament on $n \geq k+2 \geq 6$ vertices, with $(n, k) \notin \{(6, 4), (7, 4), (7, 5)\}$ and let C be a Hamiltonian cycle in H. Then C contains at least three pancyclic arcs.*

Proof. Let $C = x_1a_1x_2a_2 \cdots x_n$. We consider the C-majority-digraph $D := M(H, C)$ of H. By Lemma 2.6 we have $|A_H(x, y)| \geq \lceil \frac{1}{2}\binom{n-2}{k-2} \rceil \geq n$ for all $xy \in A_{\text{maj}}(H)$.

D is a strong semicomplete digraph. By Theorem 2.5, the Hamiltonian cycle $\tilde{C} = x_1x_2 \cdots x_n$ in D contains three pancyclic arcs. Let $x_{i_0}x_{i_0+1}$ be such an arc for an $i_0 \in \{1, \ldots, l\}$. We will show that a_{i_0} is pancyclic in H. Let $\tilde{C_l} = y_1 \cdots y_ly_1$ be an l-cycle in D that contains $x_{i_0}x_{i_0+1}$ for an $l \in \{3, \ldots, n-1\}$. Furthermore, let $I_0 \subseteq \{1, \ldots, l\}$ be the set of indices i such that $y_iy_{i+1} = x_{j(i)}x_{j(i)+1}$ for a $j(i) \in \{1, \ldots, n\}$. For an $i \in I_0$ we chose $b_i := a_{j(i)}$. By Definition 1.8 of D, these b_i are pairwise distinct and we have $|A_H(y_j, y_{j+1})| \geq \lceil \frac{1}{2}\binom{n-2}{k-2} \rceil \geq n$ for all $j \in \{1, \ldots, l\} \setminus I_0$. Thus, we can chose $b_j \in A_H(y_j, y_{j+1})$ for all $j \in \{1, \ldots, l\} \setminus I_0$, such that all b_i for $i \in \{1, \ldots, l\}$ are pairwise distinct and therefore, a_{i_0} is contained in the l-cycle $y_1b_1y_2b_2 \cdots y_lb_ly_1$ in H. $\square$

Lemma 3.6. *Let $H = (V, A)$ be a strong 4- or 5-hypertournament on 7 vertices and let C be a Hamiltonian cycle in H. Then C contains at least three pancyclic arcs.*

Proof. Let $C = x_1a_1 \cdots x_6a_6x_7a_7x_1$. We consider the C-majority-digraph $D := M(H, C)$. Let t_0 denote the smallest integer $t \in \{1, \ldots, 5\}$, such that $D - V(T)$ is strong for all $(t-2)$-subpaths of $x_1x_2x_3x_4x_5x_6x_7x_1$ but there exists such a $(t-1)$-subpath $\tilde{T}$ (without loss of generality, we may assume that $\tilde{T} = x_1 \cdots x_{t_0}$), such that $D - V(\tilde{T})$ is not strong.

($*$) If $x, y \in V$ are distinct vertices with $xy \notin A^C_{\mathrm{maj}}(H) \cup \{x_{i+1}x_i \mid 1 \leq i \leq 7\}$, then $|A_H(x,y)| \geq \frac{1}{2}\binom{n-2}{k-2} + 1 = 6$. This is particularly true for distinct vertices $x, y \in V(D) \setminus V(\tilde{T})$, such that $xy \notin \{x_{i+1}x_i \mid 1 \leq i \leq 7\}$ and y is contained in a component of the strong decomposition of $D - V(\tilde{T})$ that precedes x.

Case 1. $t_0 = 1$.

By Definition 1.8 of D, the strong decomposition of $D - x_1$ does not contain components of cardinality 2 and for all $2 \leq i < j \leq 6$, the vertex x_j is either contained in the same component as x_i or in one that succeeds it. Therefore, we only need to consider the following subcases.

Case 1.1. The first or the last component (without loss of generality, we may assume the last) of the strong decomposition of $D - x_1$ contains exactly 1 vertex. For all $i \in \{2, \ldots, 5\}$, the arcs a_1 and a_7 are contained in the $(i+1)$-cycle $x_1a_1 \cdots x_iax_7a_7x_1$ in H for an $a \in A_H(x_i, x_7) \setminus \{a_1, \ldots, a_{i-1}, a_7\}$ and therefore are pancyclic. Note, that $|A_H(x_5, x_7)| \geq 6$, by ($*$). Let $i_0 := \min\{i \mid 2 \leq i \leq 6,\ x_{i+1}x_1 \in A^C_{\mathrm{maj}}(H)\}$. For all $i \in \{2, \ldots, i_0\}$, the arc a_{i_0} is contained in the $(3 + i_0 - i)$-cycle $x_1ax_ia_i \cdots x_{i_0}a_{i_0}x_{i_0+1}bx_1$ for an arc $a \in A_H(x_1, x_{i_0}) \setminus \{a_i, \ldots, a_{i_0}\}$ and an arc $b \in A_H(x_{i_0+1}, x_1) \setminus \{a_i, \ldots, a_{i_0}, a\}$. Note, that $|A_H(x_1, x_3)| \geq \frac{1}{2}\binom{n-2}{k-2} + 1 = 6$, if $i_0 = 6$, since $x_3x_1 \notin A^C_{\mathrm{maj}}(H)$, by the definition of i_0. Furthermore, for all $i \in \{i_0+1, \ldots, 5\}$, a_{i_0} is contained in the $(i+1)$-cycle $x_1a_1 \cdots x_iax_7a_7x_1$ in H for an arc $a \in A_H(x_i, x_7) \setminus \{a_1, \ldots, a_{i-1}, a_7\}$, since $|A_H(x_i, x_7)| \geq 6$, by ($*$). Thus, a_{i_0} is pancyclic as well.

Case 1.2. The strong decomposition of $D - x_1$ contains two components of cardinality 3. a_1 and a_7 are contained in the 3-cycle $x_7a_7x_1a_1x_2ax_7$ in H for an arc $a \in A_H(x_2, x_7) \setminus \{a_1, a_7\}$. Since $D[\{x_5, x_6, x_7\}]$ is strong, we consequently have $|A_H(x_7, x_5)| \geq 5$. Thus, there is an arc $a \in A_H(x_7, x_5) \setminus \{a_5, a_6\}$ and a_6 is contained in the 3-cycle $x_5a_5x_6a_6x_7ax_5$ in H. For all $i \in \{2, \ldots, 4\}$, a_1, a_6 and a_7 are contained in the $(i+2)$-cycle $x_1a_1 \cdots x_iax_6a_6x_7a_7x_1$ for an arcs $a \in A_H(x_i, x_6) \setminus \{a_1, \ldots, a_{i-1}, a_6, a_7\}$, which exists by ($*$). Therefore, a_1, a_6 and a_7 are pancyclic.

Case 2. $t_0 = 2$.

Without loss of generality, we may assume that the last component of the strong decomposition of $D - \{x_1, x_2\}$ contains exactly 1 vertex. Since $D - x_1$ is strong, we have $x_7x_2 \in A^C_{\mathrm{maj}}(H)$. a_2 is contained in the 3-cycle $x_7ax_2a_2x_3bx_7$ in H for arcs $a \in A_H(x_7, x_2) \setminus \{a_2\}$ and $b \in A_H(x_3, x_7) \setminus \{a_2, a\}$. If $x_3x_1 \in A^C_{\mathrm{maj}}(H)$, then a_1 is contained in the 3-cycle $x_1a_1x_2a_2x_3ax_1$ in H for an $a \in A_H(x_3, x_1) \setminus \{a_1, a_2\}$. If $x_1x_3 \in A^C_{\mathrm{maj}}(H)$, then a_7 is contained in the 3-cycle $x_7a_7x_1ax_3bx_7$ in H for arcs $a \in A_H(x_1, x_3) \setminus \{a_7\}$ and $b \in \{a_7, a\}$. For all $i \in \{3, \ldots, 5\}$, a_1, a_2 and a_7 are contained in the $(i+1)$-cycle $x_1a_1 \cdots x_iax_7a_7x_1$ in H for an arc $a \in A_H(x_i, x_7) \setminus \{a_1, \ldots, a_{i-1}, a_7\}$, which exists by ($*$). Thus a_2 and a_1 or a_7 are pancyclic.

Let $i_0 := \min\{i \mid 3 \leq i \leq 6,\ x_{i+1}x_2 \in A^C_{\mathrm{maj}}(H)\}$. Then a_{i_0} is contained in the 3-cycle $x_2ax_{i_0}a_{i_0}x_{i_0+1}bx_2$ in H, for arcs $a \in A_H(x_2, x_{i_0}) \setminus \{a_{i_0}\}$ and $b \in A_H(x_{i_0+1}, x_2) \setminus \{a_{i_0}, a\}$. If $i_0 = 3$, then a_{i_0} is contained in the 4-cycle $x_2a_2x_{i_0}a_{i_0}x_{i_0+1}ax_7bx_2$ in H for arcs $a \in A_H(x_{i_0+1}, x_7) \setminus \{a_2, a_{i_0}\}$ and $b \in A_H(x_7, x_2) \setminus \{a_2, a_{i_0}, a\}$. For all $i \in \{4, 5\}$, a_{i_0} is contained in the $(i + 1)$-cycle $x_2a_2x_{i_0}a_{i_0} \cdots x_iax_7a_7x_1a_1x_2$ in H for an arc $a \in A_H(x_i, x_7) \setminus \{a_1, \ldots, a_i, a_7\}$, which exists by $(*)$. Suppose that $i_0 > 3$. Then a_{i_0} is contained in the 4-cycle $x_2ax_{i_0-1}a_{i_0-1}x_{i_0}a_{i_0}x_{i_0+1}bx_2$ in H for arcs $a \in A_H(x_2, x_{i_0-1}) \setminus \{a_{i_0-1}, a_{i_0}\}$ and $b \in A_H(x_{i_0+1}, x_2) \setminus \{a_{i_0-1}, a_{i_0}, a\}$, in the 5-cycle $x_2cx_4a_4x_5a_5x_6a_6x_7dx_2$ in H for arcs $c \in A_H(x_2, x_4) \setminus \{a_4, a_5, a_6\}$ and $d \in A_H(x_7, x_2) \setminus \{a_4, a_5, a_6, c\}$ and in the 6-cycle $x_2ex_4a_4x_5a_5x_6a_6x_7a_7x_1a_1x_2$ in H for an arc $e \in A_H(x_2, x_4) \setminus \{a_1, a_4, a_5, a_6, a_7\}$, which exists, since there are no arcs from x_4 to x_2 in D, by the definition of i_0 and thus, $|A_H(x_2, x_4)| \geq \frac{1}{2}\binom{n-2}{k-2} + 1 = 6$. Therefore, $a_{i_0} \in \{a_3, a_4, a_5, a_6\}$ is the third pancyclic arc.

Case 3. $t_0 = 3$.

Without loss of generality, we may assume that the last component of the strong decomposition of $D - \{x_1, x_2, x_3\}$ contains exactly 1 vertex. Since $D - \{x_1, x_2\}$ is strong, we have $x_7x_3 \in A^C_{\mathrm{maj}}(H)$. For all $i \in \{4, \ldots, 6\}$, the arc a_3 is contained in the $(i-1)$-cycle $x_3a_3 \cdots x_iax_7bx_3$ in H for arcs $a \in A_H(x_i, x_7) \setminus \{a_3, \ldots, a_{i-1}\}$ and $b \in A_H(x_7, x_3) \setminus \{a_3, \ldots, a_{i-1}, a\}$. With the exception of $i = 4$, the arc a_4 is also contained in said cycles. Furthermore, a_3 and a_4 are contained in the 6-cycle $x_3a_3x_4a_4x_5ax_7a_7x_1a_1x_2a_2x_3$ in H for an arc $a \in A_H(x_5, x_7) \setminus \{a_1, a_2, a_3, a_4, a_7\}$, which exists by $(*)$. Therefore, a_3 is pancyclic.

Let $i_0 := \min\{i \mid 4 \leq i \leq 6,\ x_{i+1}x_3 \in A^C_{\mathrm{maj}}(H)\}$. Then a_{i_0} is contained in the 3-cycle $x_3ax_{i_0}a_{i_0}x_{i_0+1}bx_3$ in H for arcs $a \in A_H(x_3, x_{i_0}) \setminus \{a_{i_0}\}$ and $b \in A_H(x_{i_0+1}, x_3) \setminus \{a_{i_0}, a\}$. If $i_0 = 4$, then we have already shown a_{i_0} to be pancyclic. Suppose that $i_0 \in \{5, 6\}$. Then a_{i_0} is contained in the 4-cycle $x_3ax_5a_5x_6a_6x_7bx_3$ in H for arcs $a \in A_H(x_3, x_5) \setminus \{a_5, a_6\}$ and $b \in A_H(x_7, x_3) \setminus \{a_5, a_6, a\}$, it is contained in the 5-cycle $x_3a_3x_4a_4x_5a_5x_6a_6x_7cx_3$ in H for an arc $c \in A_H(x_7, x_3) \setminus \{a_3, a_4, a_5, a_6\}$ and finally, it is contained in the 6-cycle $x_3dx_5a_5x_6a_6x_7a_7x_1a_1x_2a_2x_3$ for an arc $d \in A_H(x_3, x_5) \setminus \{a_1, a_2, a_5, a_6, a_7\}$, which exists by the definition of i_0 and $(*)$. Thus, $a_{i_0} \in \{a_4, a_5, a_6\}$ is pancyclic.

Since $D - \{x_2, x_3\}$ is strong, we have $x_1x_{j_0} \in A^C_{\mathrm{maj}}(H)$ for an index $j_0 \in \{4, 5, 6\}$, such that x_{j_0} is contained in the first component of the strong decomposition of $D - \{x_1, x_2, x_3\}$. a_7 is contained in the 3-cycle $x_7a_7x_1ax_{j_0}bx_7$ in H for arcs $a \in A_H(x_1, x_{j_0}) \setminus \{a_7\}$ and $b \in A_H(x_{j_0}, x_7) \setminus \{a_7, a\}$. For $j_0 \in \{4, 5\}$, a_7 is contained in the 4-cycle $x_7a_7x_1ax_{j_0}a_{j_0}x_{j_0+1}bx_7$ in H for arcs $a \in A_H(x_1, x_{j_0}) \setminus \{a_7, a_{j_0}\}$ and $b \in A_H(x_{j_0+1}, x_7) \setminus \{a_7, a_{j_0}, a\}$. If $j_0 = 6$, then x_6 is contained in the first component of the strong decomposition of $D - \{x_1, x_2, x_3\}$ and thus, $x_6x_4 \in A^C_{\mathrm{maj}}(H)$. Therefore, a_7 is contained in the 4-cycle $x_7a_7x_1ax_{j_0}bx_4cx_7$ in H for arcs $a \in A_H(x_1, x_{j_0}) \setminus \{a_7\}$, $b \in A_H(x_{j_0}, x_4) \setminus \{a_7, a\}$ and $c \in A_H(x_4, x_7) \setminus \{a_7, a, b\}$. Furthermore, for all $i \in \{4, 5\}$, there exists an

$a \in A_H(x_i, x_7) \setminus \{a_1, \ldots, a_{i-1}, a_7\}$, such that a_7 is contained in the $(i+1)$-cycle $x_7a_7x_1a_1 \cdots x_iax_7$ in H, by $(*)$. Hence, a_7 is pancyclic as well.

Case 4. $t_0 = 4$.

We have $x_7x_4, x_1x_5 \in A^C_{\mathrm{maj}}(H)$, since $D - \{x_1, x_2, x_3\}$ and $D - \{x_2, x_3, x_4\}$ are strong, by the definition of t_0.

If $x_4x_1 \notin A^C_{\mathrm{maj}}(H)$, we also have $x_4x_2, x_3x_1 \in A^C_{\mathrm{maj}}(H)$, since $D - \{x_5, x_6, x_7\}$ is strong by the definition of t_0. a_4 is contained in the 3- and in the 4-cycle $x_4a_4x_5ax_7bx_4$ and $x_4a_4x_5a_5x_6a_6x_7bx_4$ in H, for arcs $a \in A_H(x_5, x_7) \setminus \{a_4\}$ and $b \in A_H(x_7, x_4) \setminus \{a_4, a_5, a_6, a\}$. a_7 is contained in the 3-cycle $x_7a_7x_1ax_5bx_7$ and in the 4-cycle $x_7a_7x_1ax_5a_5x_6a_6x_7$ in H for arcs $a \in A_H(x_1, x_5) \setminus \{a_5, a_6, a_7\}$ and $b \in A_H(x_5, x_7) \setminus \{a_7, a\}$. Furthermore, a_4 and a_7 are contained in the 5-cycle $x_4a_4x_5a_5x_6a_6x_7a_7x_1ax_4$ in H for an $a \in A_H(x_1, x_4) \setminus \{a_4, a_5, a_6, a_7\}$ and in the 6-cycle $x_4a_4x_5bx_7a_7x_1a_1x_2a_2x_3a_3x_4$ in H for an arc $b \in A_H(x_5, x_7) \setminus \{a_1, a_2, a_3, a_4, a_7\}$, which exists by $(*)$. Therefore, a_4 and a_7 are pancyclic.

If $x_6x_4 \notin A^C_{\mathrm{maj}}(H)$, then a_6 is contained in the 3-cycle $x_6a_6x_7ax_4bx_6$ and in the 4-cycle $x_6a_6x_7ax_4a_4x_5a_5x_6$ in H for arcs $a \in A_H(x_7, x_4) \setminus \{a_4, a_5, a_6\}$ and $b \in A_H(x_4, x_6) \setminus \{a_4, a_5, a_6, a\}$. Furthermore, a_6 is contained in the 5-cycle $x_6a_6x_7a_7x_1ax_4a_4x_5a_5x_6$ for an $a \in A_H(x_1, x_4) \setminus \{a_4, a_5, a_6, a_7\}$ and in the 6-cycle $x_6a_6x_7a_7x_1a_1x_2a_2x_3a_3x_4bx_6$ for some $b \in A_H(x_4, x_6) \setminus \{a_1, a_2, a_3, a_6, a_7\}$, which exists by $(*)$. Thus, a_6 is the third pancyclic arc.

Suppose that $x_6x_4 \in A^C_{\mathrm{maj}}(H)$. For all $i \in \{6, 7\}$, a_5 is contained in the $(i-3)$-cycle $x_5a_5 \cdots x_iax_4a_4x_5$ in H for an $a \in A_H(x_i, x_4) \setminus \{a_4, \ldots, a_{i-1}\}$. Furthermore, a_5 is contained in the 5-cycle $x_5a_5x_6a_6x_7x_1ax_4a_4x_5$ for an $a \in A_H(x_1, x_4) \setminus \{a_4, \ldots, a_7\}$. Since a_5 contains only four vertices, we have $a_5 \notin A_H(x_6, x_4) \cap A_H(x_4, x_2) \cap A_H(x_3, x_1) \cap A_H(x_1, x_5)$. Without loss of generality, we may assume that $a_5 \notin A_H(x_1, x_5)$. Then a_5 is contained in the 6-cycle $x_5a_5x_6ax_4bx_2a_2x_3cx_1dx_5$ in H for arcs $a \in A_H(x_6, x_4) \setminus \{a_2, a_5\}$, $b \in A_H(x_4, x_2) \setminus \{a_2, a_5, a\}$, $c \in A_H(x_3, x_1) \setminus \{a_2, a_5, a, b\}$ and $d \in A_H(x_1, x_5) \setminus \{a_2, a_5, a, b, c\}$. Therefore, a_5 is pancyclic.

Suppose now that $x_4x_1 \in A^C_{\mathrm{maj}}(H)$. For arcs $a \in A_H(x_5, x_7) \setminus \{a_4\}$ and $b \in A_H(x_7, x_4) \setminus \{a_4, a_5, a_6, a\}$, a_4 is contained in the 3-cycle $x_4a_4x_5ax_7bx_4$ and in the 4-cycle $x_4a_4x_5a_5x_6a_6x_7bx_4$ in H. a_7 is contained in the 3- and the 4-cycle $x_7a_7x_1ax_5bx_7$ and $x_7a_7x_1ax_5a_5x_6a_6x_7$ in H, respectively, for arcs $a \in A_H(x_1, x_5) \setminus \{a_5, a_6, a_7\}$ and $b \in A_H(x_5, x_7) \setminus \{a_7, a\}$.

If $x_1x_3 \in A^C_{\mathrm{maj}}(H)$, then a_3 is contained in the 3-cycle $x_3a_3x_4ax_1bx_3$ and in the 4-cycle $x_3a_3x_4ax_1a_1x_2a_2x_3$ in H for arcs $a \in A_H(x_4, x_1) \setminus \{a_1, a_2, a_3\}$ and $b \in A_H(x_1, x_3) \setminus \{a_3, a\}$. Furthermore, a_3, a_4 and a_7 are contained in the 5-cycle $x_3a_3x_4a_4x_5ax_7a_7x_1bx_3$ and in the 6-cycle $x_3a_3x_4a_4x_5ax_7a_7x_1a_1x_2a_2x_3$ in H for arcs $a \in A_H(x_5, x_7) \setminus \{a_1, a_2, a_3, a_4, a_7\}$, which exists by $(*)$, and $b \in A_H(x_1, x_3) \setminus \{a_3, a_4, a_7, a\}$. Thus, a_3, a_4 and a_7 are pancyclic. If $x_2x_4 \in A^C_{\mathrm{maj}}(H)$, then a_1, a_4 and a_7 are pancyclic by analogous arguments.

Suppose that $x_1x_3, x_2x_4 \notin A^C_{\mathrm{maj}}(H)$. Then a_1 and a_2 are contained in the 3-cycle $x_1a_1x_2a_2x_3ax_1$ and in the 4-cycle $x_1a_1x_2a_2x_3a_3x_4bx_1$ in H for arcs

$a \in A_H(x_3, x_1) \setminus \{a_1, a_2\}$ and $b \in A_H(x_4, x_1) \setminus \{a_1, a_2, a_3\}$. a_3 is contained in the 3-cycle $x_2a_2x_3a_3x_4ax_2$ and in the 4-cycle $x_2a_2x_3a_3x_4bx_1a_1x_2$ in H for an $a \in A_H(x_4, x_2) \setminus \{a_2, a_3\}$ and an arc $b \in A_H(x_4, x_1) \setminus \{a_1, a_2, a_3\}$.

If $x_3x_6 \in A^C_{\mathrm{maj}}(H)$, then the three arcs a_1, a_2 and a_7 are contained in the 5-cycle $x_1a_1x_2a_2x_3ax_6a_6x_7a_7x_1$ and in the 6-cycle $x_1a_1x_2a_2x_3a_3x_4a_4x_5bx_7a_7x_1$ in H for an $a \in A_H(x_2, x_4) \setminus \{a_1, a_2, a_6, a_7\}$ and an arc $b \in A_H(x_5, x_7) \setminus \{a_1, a_2, a_3, a_4, a_7\}$, which exists by $(*)$. Hence, a_1, a_2 and a_7 are pancyclic. If $x_6x_2 \in A^C_{\mathrm{maj}}(H)$, then a_2, a_3 and a_4 are pancyclic by analogous arguments.

Thus, we may assume that $x_3x_6, x_6x_2 \notin A^C_{\mathrm{maj}}(H)$. Consequently, there are arcs $a \in A_H(x_2, x_6) \setminus \{a_1, a_3\}$, $b \in A_H(x_6, x_3) \setminus \{a_1, a_3, a\}$, $c \in A_H(x_4, x_1) \setminus \{a_1, a_3, a, b\}$ and $d \in A_H(x_5, x_7) \setminus \{a_1, a_2, a_3, a_4, a_7\}$, by $(*)$, such that a_1 and a_3 are contained in the 5-cycle $x_1a_1x_2ax_6bx_3a_3x_4cx_1$ as well as in the 6-cycle $x_1a_1x_2a_2x_3a_3x_4a_4x_5dx_7a_7x_1$ in H. Therefore, a_1 and a_3 are pancyclic.

If $x_7x_3 \in A^C_{\mathrm{maj}}(H)$, then, for an $a \in A_H(x_7, x_3) \setminus \{a_3, a_4, a_5, a_6\}$, a_4 is contained in the 5-cycle $x_3a_3x_4a_4x_5a_5x_6a_6x_7ax_3$ and it is contained in the 6-cycle $x_1a_1x_2a_2x_3a_3x_4a_4x_5bx_7a_7x_1$ in H for an arc $b \in A_H(x_5, x_7) \setminus \{a_1, a_2, a_3, a_4, a_7\}$, which exists by $(*)$. Hence, a_4 is pancyclic as well.

For $x_7x_3 \notin A^C_{\mathrm{maj}}(H)$, a_2 is contained in the 5-cycle $x_2a_2x_3ax_7bx_4cx_1a_1x_2$ and in the 6-cycle $x_1a_1x_2a_2x_3a_3x_4a_4x_5dx_7a_7x_1$ in H for an $a \in A_H(x_3, x_7) \setminus \{a_1, a_2\}$, an arc $b \in A_H(x_7, x_4) \setminus \{a_1, a_2, a\}$, an arc $c \in A_H(x_4, x_1) \setminus \{a_1, a_2, a, b\}$ and an arc $d \in A_H(x_5, x_7) \setminus \{a_1, a_2, a_3, a_4, a_7\}$, which exists by $(*)$. Thus, a_2 is the third pancyclic arc.

Case 5. $t_0 = 5$.

By the definition of t_0, we have x_3x_1, x_4x_2, x_5x_3, x_6x_4, x_7x_5, x_1x_6, $x_2x_7 \in A^C_{\mathrm{maj}}(H)$.

We will show the following: For all $i \in \{1, \ldots, 7\}$ and $l \in \{3, 4, 5\}$, the arc a_i is contained in an l-cycle in H. Without loss of generality, we may assume that $i = 1$. Then a_1 is contained in the 3-cycle $x_1a_1x_2a_2x_3ax_1$ for an $a \in A_H(x_3, x_1) \setminus \{a_1, a_2\}$. If $x_3x_7 \in A^C_{\mathrm{maj}}(H)$, then a_1 is contained in the 4-cycle $x_1a_1x_2a_2x_3ax_7a_7x_1$ in H for an $a \in A_H(x_3, x_7) \setminus \{a_1, a_2, a_7\}$. Otherwise, a_1 is contained in the 4-cycle $x_1a_1x_2ax_7bx_3cx_1$ in H for an $a \in A_H(x_2, x_7) \setminus \{a_1\}$, an arc $b \in A_H(x_7, x_3) \setminus \{a_1, a\}$ and an arc $c \in A_H(x_3, x_1) \setminus \{a_1, a, b\}$. Furthermore, a_1 is contained in the 5-cycle $x_1a_1x_2ax_7bx_5cx_3dx_1$ in H for arcs $a \in A_H(x_2, x_7) \setminus \{a_1\}$, $b \in A_H(x_7, x_5) \setminus \{a_1, a\}$, $c \in A_H(x_5, x_3) \setminus \{a_1, a, b\}$ and $d \in A_H(x_3, x_1) \setminus \{a_1, a, b, c\}$.

To prove the existence of suitable 6-cycles, we will show the following: If $x_ix_j \notin A^C_{\mathrm{maj}}(H)$ for a pair of indices $i, j \in \{1, \ldots, 7\}$, such that $j - i \in \{3, -4\}$, then a_i is contained in a 6-cycle in H. Without loss of generality, we may assume that $i = 1$ and $j = 4$. Then a_1 is contained in the 6-cycle $x_1a_1x_2ax_7bx_5cx_3a_3x_4dx_1$ for arcs $a \in A_H(x_2, x_7) \setminus \{a_1, a_3\}$, $b \in A_H(x_7, x_5) \setminus \{a_1, a_3, a\}$, $c \in A_H(x_5, x_3) \setminus \{a_1, a_3, a, b\}$ and $d \in A_H(x_4, x_1) \setminus \{a_1, a_3, a, b, c\}$, which exists by $(*)$. Thus, if there are three such pairs, we are finished.

Otherwise, there are indices $i_1, i_2, j_1, j_2 \in \{1, \ldots, 7\}$, such that $i_2 - i_1 \in \{1, -6\}$, $j_1 - i_1 \in \{3, -4\}$, $j_2 - i_2 \in \{3, -4\}$ and $x_{i_1}x_{j_1}, x_{i_2}x_{j_2} \in A^C_{\mathrm{maj}}(H)$. Without loss of generality, we may assume that $i_1 = 1, i_2 = 2, j_1 = 4$ and $j_2 = 5$. Furthermore, we have $a_6 \notin A_H(x_1, x_4) \cap A_H(x_4, x_2) \cap A_H(x_2, x_5)$, since a_6 would otherwise contain six vertices, a contradiction. Without loss of generality, we may assume that $a_6 \notin A_H(x_2, x_5)$. Therefore, a_5, a_6 and a_7 are contained in the 6-cycle $x_1ax_4bx_2cx_5a_5x_6a_6x_7a_7x_1$ in H for arcs $a \in A_H(x_1, x_4) \setminus \{a_5, a_6, a_7\}$, $b \in A_H(x_4, x_2) \setminus \{a_5, a_6, a_7, a\}$ and $c \in A_H(x_2, x_5) \setminus \{a_5, a_6, a_7, a, b\}$. Thus, we have found three pancyclic arcs. □

Lemma 3.7. *Let $H = (V, A)$ be a strong 4-hypertournament on 6 vertices and let C be a Hamiltonian cycle in H. Then C contains at least three pancyclic arcs.*

Since it is similar in structure to, but far exceeding the length of the proof of Lemma 3.6, we will omit our proof of Lemma 3.7.

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