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THE LEFSCHETZ CONDITION ON PROJECTIVIZATIONS OF COMPLEX VECTOR BUNDLES

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Abstract. We consider a condition under which the projectivization $P(E^k)$ of a complex k-bundle $E^k \to M$ over an even-dimensional manifold M can have the hard Lefschetz property, affected by [10]. It depends strongly on the rank k of the bundle E^k. Our approach is purely algebraic by using rational Sullivan minimal models [5]. We will give some examples.

1. Introduction

A Poincaré duality space Y of the formal dimension

$$fd(Y) = \max\{i; H^i(Y;\mathbb{Q}) \neq 0\} = 2m$$

is said to be *cohomologically symplectic* (c-symplectic) if $u^m \neq 0$ for some $u \in H^2(Y;\mathbb{Q})$ and, furthermore, is said to have the *hard Lefschetz property* (or simply the Lefschetz property) with respect to the c-symplectic class u, if the maps

$$\cup u^j : H^{m-j}(Y;\mathbb{Q}) \to H^{m+j}(Y;\mathbb{Q}), \quad 0 \leq j \leq m$$

are monomorphisms (then called the *Lefschetz maps*) [17]. For example, a compact Kähler manifold has the hard Lefschetz property [17], [6, Theorem 4.35]. Recall the Thurston-Weinstein problem [17, p. 198]: "*Describe symplectic compact manifolds with no Kähler structure*". Conversely, what conditions on a symplectic manifold imply the existence of a Kähler structure or, more generally, that the manifold satisfies the hard Lefschetz property?

Let M be an even-dimensional manifold and $\xi : E^k \to M$ be a complex k-bundle over M. The projectivization of the bundle ξ

$$P(\xi) : \ \mathbb{C}P^{k-1} \overset{j}{\to} P(E^k) \to M$$

satisfies the rational cohomology algebra condition $(*)$:

$$H^*(P(E^k);\mathbb{Q}) = H^*(M;\mathbb{Q})[x]/(x^k + c_1x^{k-1} + \cdots + c_{k-j}x^j + \cdots + c_{k-1}x + c_k)$$

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where c_i are the i-th Chern classes of ξ and x is a degree 2 class generating the cohomology of the complex projective space fiber (Leray-Hirsch theorem) [3], [10], [17, p. 122]. The manifold $P(E^k)$ appears as the exceptional divisor in blow-up construction for a certain embedding of M [11], [17, Chap. 4]. When M is a non-toral symplectic nilmanifold of dimension $2n$, there is a bundle E^n such that $P(E^n)$ is not Lefschetz [18], [10, Example 4.4]. In general, for a $2k$-dimensional manifold M and a fibration $\mathbb{C}P^{k-1} \to E \to M$, the total space E is Lefschetz if and only if M is Lefschetz [10, Remark 4.2]. We consider the following:

Problem 1.1. Suppose that the projectivization $P(E^k)$ of a k-dimensional vector bundle $E^k \to M$ is c-symplectic with respect to $\tilde{x}$ where $j^*(\tilde{x}) = x$; i.e., $\tilde{x}^m \neq 0$ when $\dim P(E^k) = 2m$. What rational homotopical conditions on M are necessary for $P(E^k)$ to have the Lefschetz property with respect to $\tilde{x}$?

Proposition 1.2. *Let M be an even dimensional manifold.*

(1) *For a sufficiently large k, there is a k-dimensional vector bundle $E^k \to M$ such that $P(E^k)$ is c-symplectic with respect to x.*

(2) *If $P(E^k)$ is c-symplectic with respect to x, then there is a vector bundle $E^m \to M$ such that $P(E^m)$ is c-symplectic with respect to x for any $m > k$.*

Definition 1.3. An even-dimensional manifold (or more general Poincaré duality space) M is said to be *projective* (k)*-Lefschetz* if there exists a complex k-bundle E^k such that the projectivization $P(E^k)$ is c-symplectic with respect to $\tilde{x}$ and has the Lefschetz property with respect to $\tilde{x}$. Then we often say simply that M is *projective Lefschetz*. In particular, we say that M is *projective non-Lefschetz* if $P(E^k)$ cannot have the Lefschetz property for any k and E^k.

In this paper, we recall D.Sullivan's rational model in §2 and we give some examples that indicate how the rational cohomology algebra of M determines the projective (n)-Lefschetzness of M when M is the product of at most four spheres in §3.

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2. Sullivan model

Let $\mathcal{M}(Y) = (\Lambda V, d)$ be the *Sullivan minimal model* of a nilpotent space Y. It is a freely generated $\mathbb{Q}$-commutative differential graded algebra (abbr. DGA) with a $\mathbb{Q}$-graded vector space $V = \bigoplus_{i \geq 1} V^i$ where $\dim V^i < \infty$, V admits a basis $\{v_\alpha\}$ indexed by a well-ordered set $\{\alpha\}$ such that $\deg(v_\alpha) \leq \deg(v_\beta)$ if $\alpha < \beta$ and $d(v_\alpha) \in \Lambda(v_\beta)_{\beta<\alpha}$. The differential d is a decomposable; i.e., $d(V^i) \subset (\Lambda^+ V \cdot \Lambda^+ V)^{i+1}$. Here $\Lambda^+ V$ is the ideal of ΛV generated by elements of positive degree. Denote the degree of a homogeneous element f of a graded algebra as $|f|$. Then $xy = (-1)^{|x||y|} yx$ and $d(xy) = d(x)y + (-1)^{|x|} x d(y)$. Note that $\mathcal{M}(Y)$ determines the rational homotopy type of Y. In particular, it is known that

$$H^*(\Lambda V, d) \cong H^*(Y;\mathbb{Q}) \ \text{ and } \ V^i \cong \mathrm{Hom}(\pi_i(Y),\mathbb{Q}).$$

See [5, §12∼§15] for details. When $\pi_*(Y)\otimes\mathbb{Q} < \infty$ and $\dim H^*(Y;\mathbb{Q}) < \infty$, Y is said to be *elliptic*. It is known that

$$fd(Y) = fd(\Lambda V, d) = \sum_i |y_i| - \sum_i (|x_i| - 1)$$

for $V^{odd} = \mathbb{Q}(y_i)_i$ and $V^{even} = \mathbb{Q}(x_i)_i$ when Y is elliptic [5, §32].

Proposition 2.1. *Let M be an even dimensional manifold. Then there is a graded algebra $A = H^*(M;\mathbb{Q})[x]/(x^k + c_1x^{k-1} + \cdots + c_{k-j}x^j + \cdots + c_{k-1}x + c_k)$ with $|x| = 2$ and $c_i \in H^{2i}(M;\mathbb{Q})$ if and only if there is a complex k-bundle $\xi : E^k \to M$ such that c_i are the Chern classes of ξ by suitable scalar multiplying and A is the rational cohomology of $P(E^k)$.*

Proof. The set of equivalence classes of complex k-vector bundles over M is identified as the homotopy set from M to the complex Grassmanian $G(k,N)$ of k-planes in $\mathbb{C}^N$ for a sufficiently large N [2, IV]. Then the Chern classes of a k-bundle are given as $f^*(c_1(\gamma)), \ldots, f^*(c_k(\gamma))$ for the classifying map f and the universal bundle γ over $G(k,N)$. Conversely, for given elements $c_1, \ldots, c_k$, a rational map $M \to M_{(0)} \to G(k,N)_{(0)}$ induced by $\Pi_i c_i : M \to \Pi_i K(\mathbb{Q}, 2i) \simeq BU(k)_{(0)}$ is factored through a map $f : M \to G(k,N)$ [12, Theorem 5.3] because $G(k,N) = U(N)/U(k)\times U(N-k)$ is 0-universal [1, Proposition 3.7]. Here $BU(k)$ is the classifying space of the unitary group $U(k)$ and $Y_{(0)}$ is the rationalization of a space Y [8]. Thus we obtain the appropriate k-bundle as the pullback of γ by f. □

Corollary 2.2. *The projective Lefschetzness of an even-dimensional manifold M depends only on the graded algebra $H^*(M;\mathbb{Q})$.*

Let $\mathcal{M}(\mathbb{C}P^{k-1}) = (\mathbb{Q}[x]\otimes\Lambda(y), d)$ with $d(y) = x^k$ and $d(x) = 0$. From Corollary 2.2, the information of $P(E^k)$ that we need in this note is given as the relative Sullivan model [5, §14] :

$$(H^*(M;\mathbb{Q}), 0) \to (H^*(M;\mathbb{Q})\otimes\mathbb{Q}[x]\otimes\Lambda(y), D) \to (\mathbb{Q}[x]\otimes\Lambda(y), d)$$

with $D(f) = 0$ for $f \in H^*(M;\mathbb{Q})$, $D(x) = 0$ and

$$(**) \qquad D(y) = x^k + c_1x^{k-1} + \cdots + c_{k-j}x^j + \cdots + c_{k-1}x + c_k,$$

where $c_i \in H^{2i}(M;\mathbb{Q})$ are the Chern classes of ξ. Especially, we don't need the assumption that M is nilpotent. Remark that $H^*(P(E^k);\mathbb{Q}) \cong H^*(H^*(M;\mathbb{Q})\otimes\mathbb{Q}[x]\otimes\Lambda(y), D)$ as a $\mathbb{Q}$-graded algebra and then

$$H^j(P(E^k);\mathbb{Q}) = H^j(M;\mathbb{Q}) \oplus H^{j-2}(M;\mathbb{Q})x \oplus \cdots \oplus H^{j-2k+2}(M;\mathbb{Q})x^{k-1}.$$

Notice that $(**)$ is equivalent to $(*)$ of §1 and also equivalent to

$$[x^k] = -[c_1x^{k-1} + \cdots + c_{k-j}x^j + \cdots + c_{k-1}x + c_k]$$

in $H^*(P(E^k);\mathbb{Q})$, which is the only relation between the elements of $H^*(M;\mathbb{Q})$ and x. Then, for example, $[x^{k+1}] = -[c_1x^k + \cdots + c_{k-j}x^{j+1} + \cdots + c_{k-1}x^2 + c_kx] = [c_1^2x^{k-1} + \cdots + (c_1c_{k-j} - c_{k-j+1})x^j + \cdots + (c_1c_{k-1} - c_k)x + c_1c_k]$. In particular,

$$(***) \qquad [a] \neq 0 \text{ in } H^*(M;\mathbb{Q}) \text{ if and only if } [ax^j] \neq 0 \text{ in } H^*(P(E^k);\mathbb{Q})$$

for any $0 \leq j < k$.

Lemma 2.3. *Let* $A = (H^*(M;\mathbb{Q}) \otimes \mathbb{Q}[x] \otimes \Lambda(y), D)$ *with* $D(y) = x^k + c_1x^{k-1} + \cdots + c_{k-1}x + c_k$ *and let* $B = (H^*(M;\mathbb{Q}) \otimes \mathbb{Q}[x] \otimes \Lambda(y'), D')$ *with* $D'(y') = x^{m-k}D(y) = x^m + c_1x^{m-1} + \cdots + c_{k-1}x^{m-k+1} + c_kx^{m-k}$ *for* $k < m$. *If* $[f] \neq 0$ *in* $H^*(A)$, *then* $[fx^{m-k}] \neq 0$ *in* $H^*(B)$.

Proof. Notice that an element of $H^*(A)$ is identified as one of $H^*(B)$ since $H^*(A)$ is a submodule of $H^*(B)$ over $H^*(M;\mathbb{Q})$. Suppose that $[f] = [a_1x^{k-1} + \cdots + a_{k-1}x + a_k] \neq 0$ in $H^*(A)$ for $[a_*] \in H^*(M;\mathbb{Q})$. Then there is an index i with $[a_i] \neq 0$ in $H^*(M;\mathbb{Q})$. Thus, in $H^*(B)$, $[fx^{m-k}] = [a_1x^{m-1} + \cdots + a_{k-1}x^{m-k+1} + a_kx^{m-k}] = [a_1]x^{m-1} + \cdots + [a_{k-1}]x^{m-k+1} + [a_k]x^{m-k} \neq 0$ from $(***)$. □

Proof of Proposition 1.2. From Proposition 2.1, it is sufficient to construct a certain DGA $(H^*(M;\mathbb{Q}) \otimes \mathbb{Q}[x] \otimes \Lambda(y), D)$. Let $\dim M = 2n$.

(1) Let Ω be the fundamental class of M. Then we can define $(H^*(M;\mathbb{Q}) \otimes \mathbb{Q}[x] \otimes \Lambda(y), D)$ by $D(y) = \Omega x^{k-n} + x^k$ for $k \geq n$. Notice $\dim P(E^k) = \dim M + \dim \mathbb{C}P^{k-1} = 2n + 2k - 2$. Then we have $[x^{n+k-1}] = -[(\Omega x^{k-n})x^{n-1}] = -[\Omega x^{k-1}] \neq 0$ from $(***)$.

(2) Suppose that the DGA $(H^*(M;\mathbb{Q}) \otimes \mathbb{Q}[x] \otimes \Lambda(y), D)$ makes $P(E^k)$ c-symplectic; i.e., $[x^{n+k-1}] \neq 0$. Then, for $m > k$, the DGA $(H^*(M;\mathbb{Q}) \otimes \mathbb{Q}[x] \otimes \Lambda(y'), D')$ with $|y'| = 2m - 1$ and $D'(y') := x^{m-k}D(y)$ makes a $2n + 2m - 2$-dimensional manifold $P(E^m)$ c-symplectic. Indeed, $[x^{n+m-1}] = [x^{n+k-1} \cdot x^{m-k}] \neq 0$ in cohomology from Lemma 2.3. □

In (2) in Proposition 1.2, the bundle E^m is geometrically realized as the Whitney sum $E^k \oplus \theta^{m-k}$ where θ^{m-k} is the trivial $m-k$-bundle over M, in the manner of Proposition 2.1. Thus, if $P(E^k)$ is c-symplectic with respect to x, then $P(E^k \oplus \theta^m)$ is c-symplectic with respect to x for any $m > 0$.

3. Examples

In this section, let M be a 2-connected even-dimensional manifold and $\dim P(E^k) = 2m$.

Theorem 3.1. *The* $2n$*-dimensional sphere* S^{2n} *is projective* (k)*-Lefschetz for any* $k \geq n$.

Proof. Let $H^*(S^{2n};\mathbb{Q}) = \mathbb{Q}[v]/(v^2)$ with $|v| = 2n$. Consider $P(E^k)$ such that $\dim P(E^k) = 2m$ and $D(y) = vx^{k-n} + x^k$ for $k \geq n$. Then $m = n + k - 1$ from $2m = \dim \mathbb{C}P^{k-1} + \dim S^{2n} = 2n + 2k - 2$. Since $[x^m] = -[vx^{k-n} \cdot$

$x^{m-k}] = -[vx^{k-1}] \neq 0$ from $(***)$, $P(E^k)$ is c-symplectic with respect to x. Furthermore, $\cup x^{k-n-1-2i}(vx^i) = vx^{k-n-1-i} \neq 0$ in

$$\cup x^{k-n-1-2i} : H^{m-(k-n-1-2i)}(P(E^k);\mathbb{Q}) \to H^{m+(k-n-1-2i)}(P(E^k);\mathbb{Q})$$

for $i \geq 0$ Thus S^{2n} is projective (k)-Lefschetz. □

Proposition 3.2. *When M has the rational homotopy type of the product of odd spheres such that $H^*(M;\mathbb{Q}) \cong \Lambda(v_1, v_2, \ldots, v_n)$ with all $|v_i|$ odd and $1 < |v_1| \leq |v_2| \leq \cdots \leq |v_n|$ (n even), then there exists a bundle E^k such that $P(E^k)$ is c-symplectic if and only if $|v_1| + |v_n| \leq 2k$, $|v_2| + |v_{n-1}| \leq 2k, \ldots, |v_{n/2}| + |v_{n/2+1}| \leq 2k$.*

Proof. (sketch) The minimal DGA $(\mathbb{Q}[x] \otimes \Lambda(v_1, v_2, \ldots, v_n, y), D)$ with $|y| = 2k-1$ is c-symplectic if $D(v_1) = \cdots = D(v_n) = 0$ and

$$D(y) = v_1v_nx^{a_1} + v_2v_{n-1}x^{a_2} + \cdots + v_{n/2}v_{n/2+1}x^{a_{n/2}} + x^k$$

for $a_i = (2k - |v_i| - |v_{n-i+1}|)/2 \geq 0$. Then we have the "if" part from Proposition 2.1 and [14, Theorem 1.2]. The "only if" part is obvious from [14, Theorem 1.2]. □

Theorem 3.3. *Let $M = S^a \times S^b$ with $a \leq b$.*
(i) *When $a = b$, it is projective (k)-Lefschetz for $k \geq b$.*
(ii) *When a and b are even, it is projective $(\frac{b}{2})$-Lefschetz.*
(iii) *When a and b are odd with $a < b$, it is projective non-Lefschetz.*

Proof. Note that $H^*(M;\mathbb{Q}) = \mathbb{Q}[v_1, v_2]/(v_1^2, v_2^2) = \mathbb{Q}(1, v_1, v_2, v_1v_2)$ as a $\mathbb{Q}$-graded vector space with $|1| = 0$, $|v_1| = a$, $|v_2| = b$ and $|v_1v_2| = a+b$. Consider $P(E^k)$ such that $\dim P(E^k) = 2m$ and

$$D(y) = v_1v_2x^{k-\frac{a+b}{2}} + x^k$$

for $k \geq (a+b)/2$. Then $m = \frac{a+b}{2} + k - 1$ from $2m = a+b+2k-2$ and $\cup x^{m-a}(v_1) = v_1x^{m-a} = v_1x^{\frac{a+b}{2}+k-1-a}$ in

$$\cup x^{m-a} : H^a(P(E^k);\mathbb{Q}) \to H^{2m-a}(P(E^k);\mathbb{Q})$$

for $0 \leq a \leq m$. In cohomology, this element has the form $v_1x^{\geq k} = 0$ if and only if $a < b$. Thus, when $a < b$, $\cup x^{m-a}(v_1) = 0$; i.e., $\cup x^{m-a}$ is not the Lefschetz map. On the other hand, when $a = b$, we have from $(***)$

$$\begin{aligned}
&\cup x^{m-2i}(x^i) = x^{m-i},\\
&\cup x^{m-a-2i}(v_1x^i) = v_1x^{m-a-i},\\
&\cup x^{m-b-2i}(v_2x^i) = v_2x^{m-b-i},\\
&\cup x^{m-a-b-2i}(v_1v_2x^i) = v_1v_2x^{m-a-b-i},
\end{aligned}$$

whose linear combination can not be zero in cohomology. Thus M is projective (k)-Lefschetz for $k \geq b$ when $a = b$.

Let $a \le b$ be even.Consider $P(E^k)$ such that $\dim P(E^k) = 2m$ and

$$D(y) = v_1 x^{\frac{b-a}{2}} + v_2 + x^{\frac{b}{2}}. \qquad (k = \frac{b}{2})$$

Then $m = \frac{a}{2} + b - 1$ and we have from $(***)$

$$\cup\, x^{m-2i}(x^i) = x^{m-i},$$

$$\cup\, x^{m-a-2i}(v_1 x^i) = v_1 x^{m-a-i} = \begin{cases} v_1 v_2 x^{m-a-\frac{b}{2}-i} & (i < \frac{b-a}{2}) \\ v_1 x^{m-a-i} & (\frac{b-a}{2} \le i < \frac{-a+2b}{4}), \end{cases}$$

$$\cup\, x^{m-b-2i}(v_2 x^i) = v_2 x^{m-b-i},$$

$$\cup\, x^{m-a-b-2i}(v_1 v_2 x^i) = v_1 v_2 x^{m-a-b-i},$$

whose linear combination can not be zero in cohomology; i.e., $\cup x^j$ are the Lefschetz maps. Thus M is projective $(\frac{b}{2})$-Lefschetz. □

Remark 3.4. Even if M is projective (k)-Lefschetz, it is not projective (m)-Lefschetz for $m > k$, in general. For example, when $M = S^4 \times S^6$, M is projective (3)-Lefschetz from Theorem 3.3 but not projective (4)-Lefschetz. Indeed, in the proof of Theorem 3.3, $\cup x^2 : H^{m-2}(P(E^4)) \to H^{m+2}(P(E^4))$ is not a monomorphism since $\cup x^2([v_1 x + v_2 + x^3]) = [v_1 x^3 + v_2 x^2 + x^5] = 0$, when $Dy = v_1 x^2 + v_2 x + x^4$ $(m = 8)$.

Theorem 3.5. *Let $M = S^a \times S^b \times S^c$ with $a \le b \le c$. We have the following*:

(i) *When a, b and c are even, M is projective $(\frac{c}{2})$-Lefschetz.*

(ii) *When a and c are odd, b is even, M is projective non-Lefschetz.*

(iii) *When a is even, b and c are odd, M is projective Lefschetz if and only if $b = c$. Then M is projective (b)-Lefschetz.*

(iv) *When a and b are odd, c is even, M is projective Lefschetz if and only if $a = b$. Then M is projective $(\max\{a, \frac{c}{2}\})$-Lefschetz.*

Proof. Then $\dim M = a + b + c$ and $H^*(M;\mathbb{Q}) = \Lambda(v_1, v_2, v_3)/(v_1^2, v_2^2, v_3^2)$ with $|v_1| = a$, $|v_2| = b$, $|v_3| = c$.

(i) When $k = \frac{c}{2}$, $\dim P(E^k) = a + b + 2c - 2$ and $m = \frac{a+b+2c-2}{2}$. Then $|y| = c-1$ and $d(y) = x^{\frac{c}{2}}$. Let $D(y) = v_1 x^{\frac{c-a}{2}} + v_2 x^{\frac{c-b}{2}} + v_3 + x^{\frac{c}{2}}$. Then $P(E^k)$ is c-symplectic by x since $[x^m] = -[6 v_1 v_2 v_3 x^{\frac{c-2}{2}}] \ne 0$. Moreover, we have from $(***)$

$$\cup\, x^{m-2i}(x^i) = x^{m-i},$$

$$\cup\, x^{m-a-2i}(v_1 x^i) = \begin{cases} 2 v_1 v_2 v_3 x^{\frac{-a+c-2}{2}-i} & (0 \le i < \frac{-a+b}{2}) \\ v_1 v_2 v_3 x^{\frac{-a+c-2}{2}-i} - v_1 v_3 x^{\frac{-a+b+c-2}{2}-i} & (\frac{-a+b}{2} \le i < \frac{-a+c}{2}) \\ -v_1 v_2 x^{\frac{-a+2c-2}{2}-i} - v_1 v_3 x^{\frac{-a+b+c-2}{2}-i} & (\frac{-a+c}{2} \le i < \frac{-a+b+2c}{4}), \end{cases}$$

$$\cup\, x^{m-b-2i}(v_2 x^i) = \begin{cases} v_1 v_2 v_3 x^{\frac{-b+c-2}{2}-i} - v_2 v_3 x^{\frac{a-b+c-2}{2}-i} & (0 \le i < \frac{-b+c}{2}) \\ -v_1 v_2 x^{\frac{-b+2c-2}{2}-i} - v_2 v_3 x^{\frac{a-b+c-2}{2}-i} & (\frac{-b+c}{2} \le i < \frac{a-b+c}{2}) \\ v_2 x^{\frac{a-b+2c-2}{2}-i} & (\frac{a-b+c}{2} \le i < \frac{a-b+2c}{4}), \end{cases}$$

$$\cup\, x^{m-c-2i}(v_3x^i) = \begin{cases} -v_1v_3x^{\frac{b-2}{2}-i} - v_2v_3x^{\frac{a-2}{2}-i} \ (0 \le i < \frac{a+b-2}{2}) \\ v_3x^{\frac{a+b-2}{2}} \ (\frac{a+b-c}{2} \le i < \frac{a+b}{4}), \end{cases}$$

$$\cup\, x^{m-(a+b)-2i}(v_1v_2x^i) = \begin{cases} -v_1v_2v_3x^{\frac{-a-b+c-2}{2}-i} \ (0 \le i < \frac{-a-b+c}{2}) \\ v_1v_2x^{\frac{-a-b+2c-2}{2}-i} \ (\frac{-a-b+c}{2} \le i < \frac{-a-b+2c}{4}), \end{cases}$$

$$\cup\, x^{m-(a+c)-2i}(v_1v_3x^i) = v_1v_3x^{\frac{-a+b-2}{2}-i} \ (a < b),$$

whose linear combination can not be zero in cohomology. Thus M is projective $(\frac{c}{2})$-Lefschetz.

(ii) For $|y| = 2k-1$ and $m = \frac{a+b+c+2k-2}{2}$, there are two types of c-symplectic models as follows:

(1) $$D(y) = v_1v_3x^{k-\frac{a+c}{2}} + v_2x^{k-\frac{b}{2}} + x^k.$$

Then $\cup x^{m-a}(v_1) = -v_1v_2x^{\frac{-a+c+2k-2}{2}} = -v_1v_2x^{\ge k} = 0$ from $a < c$.

(2) $$D(y) = v_1v_2v_3x^{k-\frac{a+b+c}{2}} + x^k.$$

Then $\cup x^{m-a}(v_1) = v_1x^{\frac{-a+b+c+2k-2}{2}} = v_1x^{\ge k} = 0$. Thus the Lefschetz maps do not exist in both cases (1) and (2).

(iii) Let $b < c$. For $|y| = 2k-1$ and $m = \frac{a+b+c+2k-2}{2}$, there are two types of c-symplectic models as follows:

(1) $$D(y) = v_1x^{k-\frac{a}{2}} + v_2v_3x^{k-\frac{b+c}{2}} + x^k.$$

Then $\cup x^{m-b}(v_2) = -v_1v_2x^{\frac{-b+c+2k-2}{2}} = -v_1v_2x^{\ge k} = 0$ from $b < c$.

(2) $$D(y) = v_1v_2v_3x^{k-\frac{a+b+c}{2}} + x^k.$$

Then $\cup x^{m-a}(v_1) = v_1x^{\frac{-a+b+c+2k-2}{2}} = v_1x^{\ge k} = 0$. Thus the Lefschetz maps do not exist in both cases (1) and (2).

Let $b = c$. Then $M = S^a \times S^b \times S^b$, $\dim M = a + 2b$ and $H^*(M;\mathbb{Q}) = \mathbb{Q}[v_1]/(v_1^2) \otimes \Lambda(v_2, v_3)$ with $|v_1| = a$, $|v_2| = |v_3| = b$. When $k = b$, $\dim P(E^k) = a + 4b - 2$ and $m = \frac{a+4b-2}{2}$. Then $|y| = 2b-1$ and $d(y) = x^b$. Let $D(y) = v_1x^{b-\frac{a}{2}} + v_2v_3 + x^b$. Then $P(E^k)$ is c-symplectic with respect to x. Moreover, we have from $(***)$

$$\cup\, x^{m-2i}(x^i) = x^{m-i},$$

$$\cup\, x^{m-a-2i}(v_1x^i) = \begin{cases} -v_1v_2v_3x^{\frac{-a+2b-2}{2}-i} \ (0 \le i < \frac{-a+2b}{2}) \\ v_1x^{\frac{-a+4b-2}{2}-i} \ (\frac{-a+2b}{2} \le i < \frac{-a+4b}{4}), \end{cases}$$

$$\cup\, x^{m-b-2i}(v_2x^i) = \begin{cases} -v_1v_2x^{b-1-i} \ (0 \le i < \frac{a}{2}) \\ v_2x^{\frac{a+2b-2}{2}-i} \ (\frac{a}{2} \le i < \frac{a+2b}{4}), \end{cases}$$

$$\cup\, x^{m-b-2i}(v_3x^i) = \begin{cases} -v_1v_3x^{b-1-i} \ (0 \le i < \frac{a}{2}) \\ v_3x^{\frac{a+2b-2}{2}-i} \ (\frac{a}{2} \le i < \frac{a+2b}{4}), \end{cases}$$

$$\cup\, x^{m-(a+b)-2i}(v_1v_2x^i) = v_1v_2x^{\frac{-a+2b-2}{2}-i},$$

$$\cup\, x^{m-(a+b)-2i}(v_1v_3x^i) = v_1v_3x^{\frac{-a+2b-2}{2}-i},$$
$$\cup\, x^{m-2b-2i}(v_2v_3x^i) = v_2v_3x^{\frac{a-2}{2}-i},$$

whose linear combination can not be zero in cohomology. Thus M is projective (b)-Lefschetz. The proof of (iv) is similar to that of (iii). □

Theorem 3.6. *Let $M = S^a \times S^b \times S^c \times S^d$ with $a \le b \le c \le d$. When a, b, c and d are odd, M is projective Lefschetz if and only if $a = b$ and $c = d$. Then M is projective (c)-Lefschetz.*

Proof. Let $a < b$. For $|y| = 2k-1$ and $m = \frac{a+b+c+d+2k-2}{2}$, there are four types of c-symplectic models as follows:

(1) $$D(y) = v_1v_2x^{k-\frac{a+b}{2}} + v_3v_4x^{k-\frac{c+d}{2}} + x^k.$$

Then $\cup x^{m-a}(v_1) = -v_1v_3v_4x^{\frac{-a+b+2k-2}{2}} = -v_1v_3v_4x^{\ge k} = 0$.

(2) $$D(y) = v_1v_3x^{k-\frac{a+c}{2}} + v_2v_4x^{k-\frac{b+d}{2}} + x^k.$$

Then $\cup x^{m-a}(v_1) = -v_1v_2v_4x^{\frac{-a+c+2k-2}{2}} = -v_1v_2v_4x^{\ge k} = 0$.

(3) $$D(y) = v_1v_4x^{k-\frac{a+d}{2}} + v_2v_3x^{k-\frac{b+c}{2}} + x^k.$$

Then $\cup x^{m-a}(v_1) = -v_1v_2v_3x^{\frac{-a+d+2k-2}{2}} = -v_1v_2v_3x^{\ge k} = 0$.

(4) $$D(y) = v_1v_2v_3v_4x^{k-\frac{a+b+c+d}{2}} + x^k.$$

Then $\cup x^{m-a}(v_1) = v_1x^{\ge k} = 0$. Thus, when $a < b$, M is projective non-Lefschetz.

Let $c < d$. For $|y| = 2k-1$ and $m = \frac{a+b+c+d+2k-2}{2}$, there are four types of c-symplectic models as follows:

(1) $$D(y) = v_1v_2x^{k-\frac{a+b}{2}} + v_3v_4x^{k-\frac{c+d}{2}} + x^k.$$

Then $\cup x^{m-c}(v_3) = -v_1v_2v_3x^{\frac{-c+d+2k-2}{2}} = -v_1v_2v_3x^{\ge k} = 0$.

(2) $$D(y) = v_1v_3x^{k-\frac{a+c}{2}} + v_2v_4x^{k-\frac{b+d}{2}} + x^k.$$

Then $\cup x^{m-b}(v_2) = v_1v_2v_3x^{\frac{-b+d+2k-2}{2}} = v_1v_2v_3x^{\ge k} = 0$.

(3) $$D(y) = v_1v_4x^{k-\frac{a+d}{2}} + v_2v_3x^{k-\frac{b+c}{2}} + x^k.$$

Then $\cup x^{m-a}(v_1) = -v_1v_2v_3x^{\frac{-a+d+2k-2}{2}} = -v_1v_2v_3x^{\ge k} = 0$.

(4) $$D(y) = v_1v_2v_3v_4x^{k-\frac{a+b+c+d}{2}} + x^k.$$

Then $\cup x^{m-a}(v_1) = v_1x^{\ge k} = 0$. Thus, when $c < d$, M is projective non-Lefschetz.

Let $a = b$ and $c = d$. Then $M = S^a \times S^a \times S^c \times S^c$, $\dim M = 2a + 2c$ and $H^*(M;\mathbb{Q}) = \Lambda(v_1, v_2, v_3, v_4)$ with $|v_1| = |v_2| = a$, $|v_3| = |v_4| = c$. When $k = c$, $\dim P(E^k) = 2a + 4c - 2$ and $m = a + 2c - 1$. Then $|y| = 2k - 1$ and $d(y) = x^c$.

Let $D(y) = v_1v_2x^{c-a} + v_3v_4 + x^c$. Then $P(E^k)$ is c-symplectic with respect to x. Moreover, we have from $(***)$

$$\begin{aligned}
&\cup\, x^{m-2i}(x^i) = x^{m-i},\\
&\cup\, x^{m-a-2i}(v_1x^i) = -v_1v_3v_4x^{c-1-i},\\
&\cup\, x^{m-a-2i}(v_2x^i) = -v_2v_3v_4x^{c-1-i},\\
&\cup\, x^{m-c-2i}(v_3x^i) = \begin{cases} -v_1v_2v_3x^{c-1-i} \ (0 \le i < a) \\ v_3x^{a+c-1-i} \ (a \le i < \frac{a+c}{2}), \end{cases}\\
&\cup\, x^{m-c-2i}(v_4x^i) = \begin{cases} -v_1v_2v_4x^{c-1-i} \ (0 \le i < a) \\ v_4x^{a+c-1-i} \ (a \le i < \frac{a+c}{2}), \end{cases}\\
&\cup\, x^{m-2a-2i}(v_1v_2x^i) = \begin{cases} -v_1v_2v_3v_4x^{-a+c-1-i} \ (0 \le i < -a+c) \\ v_1v_2x^{-a+2c-1-i} \ (-a+c \le i < \frac{-a+2c}{2}), \end{cases}\\
&\cup\, x^{m-(a+c)-2i}(v_1v_3x^i) = v_1v_3x^{c-1-i},\\
&\cup\, x^{m-(a+c)-2i}(v_1v_4x^i) = v_1v_4x^{c-1-i},\\
&\cup\, x^{m-(a+c)-2i}(v_2v_3x^i) = v_2v_3x^{c-1-i},\\
&\cup\, x^{m-(a+c)-2i}(v_2v_4x^i) = v_2v_4x^{c-1-i},\\
&\cup\, x^{m-2c-2i}(v_3v_4x^i) = v_3v_4x^{a-1-i},\\
&\cup\, x^{m-(2a+c)-2i}(v_1v_2v_3) = v_1v_2v_3x^{-a+c-1-i} \ (a < c),\\
&\cup\, x^{m-(2a+c)-2i}(v_1v_2v_4) = v_1v_2v_4x^{-a+c-1-i} \ (a < c),
\end{aligned}$$

whose linear combination can not be zero in cohomology. Thus M is projective (c)-Lefschetz. □

A nilpotent space is said to be *formal* if there is a quasi-isomorphism from its Sullivan minimal model to its rational cohomology algebra thought of as a DGA with zero differential [15]([5]). For example, compact Kähler manifolds are formal [4]. Finally we give a non-formal example.

Theorem 3.7. *Let M be a simply connected 16-dimensional manifold such that $\mathcal{M}(M) = (\Lambda(v_1, v_2, v_3, v_4), d)$ with $|v_1| = |v_2| = 3$, $|v_3| = |v_4| = 5$, $d(v_1) = d(v_2) = 0$, $d(v_3) = v_1v_2$ and $d(v_4) = 0$. Then M is projective non-Lefschetz.*

Proof. There are only two cases for which $P(E^k)$ is c-symplectic.

First, let $D(y) = v_1v_4x^i + v_2v_3x^i + x^{i+4}$ with $|y| = 7 + 2i$. Then

$$\dim P(E^k) = 22 + 2i \text{ and } m = 11 + i.$$

Then $P(E^k)$ is c-symplectic from $[x^{11+i}] = -[v_1v_2v_3v_4x^{i+3}] \neq 0$. But $P(E^k)$ does not have the Lefschetz property since $[v_1x^{8+i}] = [v_1(-v_1v_4x^i - v_2v_3x^i)x^4]$ $= -[v_1v_2v_3x^{i+4}] = [v_1v_2v_3(v_1v_4x^i + v_2v_3x^i)] = 0$.

Secondly, let $D(y) = v_1v_2v_3v_4x^i + x^{i+8}$ with $|y| = 15 + 2i$. Then

$$\dim P(E^k) = 30 + 2i \text{ and } m = 15 + i.$$

Then $P(E^k)$ is c-symplectic from $[x^{15+i}] = -[v_1v_2v_3v_4x^{i+7}] \neq 0$. But $P(E^k)$ does not have the Lefschetz property since $[v_1x^{12+i}] = [v_1(-v_1v_2v_3v_4x^i)x^4] = 0$. □

Note that the manifold M of Theorem 3.7 is the product of S^5 with the pullback of the sphere bundle of the tangent bundle of S^6 by the canonical degree 1 map $S^3 \times S^3 \to S^6$. It is not formal since $H^*(M;\mathbb{Q})$ contains an indecomposable element $[v_1v_3]$ (or $[v_2v_3]$), which corresponds to a non-trivial Massey product $\langle v_1, v_2, v_1\rangle$ (or $\langle v_2, v_1, v_2\rangle$) [4]. Recall that $Y = (S^3 \times S^8)\sharp(S^3 \times S^8) \times S^5$ is formal and has the same rational cohomology as M. From Corollary 2.2, we see that Y is projective non-Lefschetz.

Remark 3.8. We know that $S^3 \times S^3 \times S^5 \times S^5$ is projective (5)-Lefschetz from Theorem 3.6. It has the same rational homotopy groups as the manifold M of Theorem 3.7. Thus projective Lefschetzness is not determined by the rational homotopy groups.

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