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INTERIORS AND CLOSURES IN A SET WITH AN OPERATION

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Abstract. A set with an operation defined on a family of subsets is studied. The operation is used to generalize the topological space itself. The operation defines the operation-open subsets in the set. Relations are studied among two types of the interiors and the closures of subsets. Some properties of maximal operation-open sets are obtained. Semi-open sets and pre-open sets are defined in the sets with operations and some relations among them are proved.

1. Introduction

Let $\mathcal{P}(X)$ be the power set of a set X. Kasahara [2] defined an operation α on a family τ of *sets* to the power set of $\cup\tau$ (the union the sets in τ), namely a function $\alpha : \tau \to \mathcal{P}(\cup\tau)$ such that $G \subset G^\alpha$ for any $G \in \tau$, where $G^\alpha = \alpha(G)$, and studied the theory of operations on topologies τ of topological spaces (X, τ). However, Kasahara's original definition, which requires the target of the operation to be $\mathcal{P}(\cup\tau)$, seems to be too restrictive for our purpose when we study operations for any family of subsets of a set X. In this paper we adopt the following definition: Let $\mathcal{F} \subset \mathcal{P}(X)$. An *operation* κ on $\mathcal{F}$ is a function

$$\kappa : \mathcal{F} \to \mathcal{P}(X)$$

such that $U \subset U^\kappa$ for each $U \in \mathcal{F}$, where $U^\kappa = \kappa(U)$; we assume that $\mathcal{F}$ contains at least one non-empty set in this paper (see Section 2).

The operation κ satisfies one of the properties of the Kuratowski closure axioms for the sets in $\mathcal{F}$ (see Kelley [3], p. 43; κ satisfies the property (b) stated there; but κ does not satisfy the properties (a), (c), (d) there in general), although the operation κ is defined only for the sets in $\mathcal{F}$. However, we do not intend to generalize closure operation by κ. Kasahara [2] used the operation $\alpha : \tau \to \mathcal{P}(X)$ for topological spaces X to generalize some methods to sort out special kinds of open sets such as θ-open sets; we propose in this paper to use

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the operation $\kappa : \mathcal{F} \to \mathcal{P}(X)$ for sets X to generalize the topological space itself. Moreover, if we define an operation $\kappa : \mathcal{F} \to \mathcal{P}(X)$ in this form, the theory of the operations can be applied to various cases where topological structures or their weaker forms, namely 'generalized open sets', are not defined. The operation $\kappa : \mathcal{F} \to \mathcal{P}(X)$ defined above enables us to define the *κ-open set* as usual; a subset A of X is called a *κ-open* set of X if for each $x \in A$ there exists a set $U \in \mathcal{F}$ such that $x \in U \subset U^\kappa \subset A$ (Definition 2.2). Let $\mathcal{F}_\kappa$ be the family of κ-open sets. It follows that $\emptyset \in \mathcal{F}_\kappa$; and $X \in \mathcal{F}_\kappa$ if and only if $\cup\mathcal{F} = X$ by Proposition 2.5. Moreover, if $A_\lambda \in \mathcal{F}_\kappa$ for any $\lambda \in \Lambda$, then $\cup_{\lambda\in\Lambda} A_\lambda \in \mathcal{F}_\kappa$ by Proposition 2.8. We call the triple $(X, \mathcal{F}, \kappa)$ a *space*, which is a generalization of a topological space; if $\mathcal{F} = \tau$ for a topological space (X, τ) and $U^\kappa = U$ for any $U \in \mathcal{F}$, then $\mathcal{F}_\kappa = \mathcal{F} = \tau$. If $(X, \mathcal{F}, \kappa)$ is a space, then we call X a *$(\mathcal{F}, \kappa)$-space* or a *κ-space*. Generalizations of the topological space were in retrospect studied by choosing some parts of the axioms of the topological space (that is, the axioms in Chapter 1 of Kelley [3] for example); we use the operation $\kappa : \mathcal{F} \to \mathcal{P}(X)$ as a foundation of a generalization of the topological space.

In Section 3 we define and study two types of interiors and closures of subsets in κ-spaces, namely Int_κ, $\mathcal{F}_\kappa$-Int, Cl_κ and $\mathcal{F}_\kappa$-Cl, and prove relations among them. Different interiors and closures are necessary because $\mathcal{F}_\kappa$-Int(A) is κ-open, but $\mathrm{Int}_\kappa(A)$ is not always κ-open for subsets A of $(X, \mathcal{F}, \kappa)$; dually, $\mathcal{F}_\kappa$-Cl(A) is κ-closed, but $\mathrm{Cl}_\kappa(A)$ is not always κ-closed. We obtain relations between maximal κ-open sets and the $\mathcal{F}_\kappa$-closure (see Remarks 3.23 and 4.6). We define two kinds of 'neighborhoods' for each point x, namely *κ-neighborhoods* and *κ-open neighborhoods*. Since we do not assume $\cup\mathcal{F} = X$ in general, the points $x \in X - \cup\mathcal{F}$ have *no* neighborhoods. However, some standard formulas are proved as in Propositions 3.13, 3.16, 3.18 and Theorem 3.26. If we assume $\cup\mathcal{F} = X$, then any point $x \in X$ has at least one $V \in \mathcal{F}$ such that $x \in V \subset V^\kappa$ and at least one neighborhood of the type $x \in U \in \mathcal{F}_\kappa$ (see Remark 3.1).

In Section 4 we study relations among maximal κ-open sets and the $\mathcal{F}_\kappa$-closure.

In Section 5 we study some properties of pre-κ-open sets, pre-$\mathcal{F}_\kappa$-open sets, semi-κ-open sets and semi-$\mathcal{F}_\kappa$-open sets.

Janković [1] generalized the closures and the closed sets making use of the operation $\alpha : \tau \to \mathcal{P}(X)$ of [2] for topological spaces (X, τ) and defined Cl_α. Ogata [6] introduced the *γ-open sets* defined by any operation $\gamma : \tau \to \mathcal{P}(X)$ (which is the operation α in [2]) for any topological space (X, τ); he denoted the set of all γ-open sets by τ_γ, which is a 'generalized topology' of X by Proposition 2.3 of [6], and furthermore, he defined and studied the τ_γ-closures, τ_γ-Cl. Rehman and Ahmad [8] defined Int_γ. Some 'open sets' in a topological space (X, τ), such as θ-open sets and δ-open sets, are considered as γ-open sets for some operation $\gamma : \tau \to \mathcal{P}(X)$; θ-open sets are γ-open sets for $\gamma = \mathrm{Cl}$, the closure operation, and δ-open sets for $\gamma = \mathrm{IntCl}$, the interior-closure operation (cf. [1]). We also note that generalized closures and interiors were considered in "supratopological spaces" by Mashhour, Allam, Mahmoud and Khedr [4]; more general family of sets by Maki, Chandrasekhara Rao and Nagoor Gani [5] and Popa and Noiri [7] (the m-structure). We see that $\mathcal{F}_\kappa$-Cl and $\mathcal{F}_\kappa$-Int satisfy their definition by Propositions 3.4 and 3.10; however, Cl_κ and Int_κ do not have the properties. In this paper, an operation is not a function $\gamma : \tau \to \mathcal{P}(X)$ for a topological space (X, τ) or some generalization of it such as $\gamma : m_X \to \mathcal{P}(X)$ for a m-structure in [7]; a feature of our approach is that the definition of our operation $\kappa : \mathcal{F} \to \mathcal{P}(X)$ enables us to use any families $\mathcal{F} \subset \mathcal{P}(X)$ as well as, for example, any family of regular open sets, pre-open sets, semi-open sets, α-open sets, β-open sets, δ-open sets and so on for $\mathcal{F}$ (see Popa and Noiri [7] for definitions). Thus, the definition of the operation in this paper will extend the scope of applications of the theory of operations.

2. The space $(X, \mathcal{F}, \kappa)$

Let $\mathcal{P}(X)$ be the power set of a set X and $\mathcal{F} \subset \mathcal{P}(X)$ such that $\mathcal{F}$ contains at least one non-empty set. An *operation* κ on $\mathcal{F}$ is a function

$$\kappa : \mathcal{F} \to \mathcal{P}(X)$$

such that $U \subset U^\kappa$ for each $U \in \mathcal{F}$, where $U^\kappa = \kappa(U)$. We call the triple $(X, \mathcal{F}, \kappa)$ a *space*. If $(X, \mathcal{F}, \kappa)$ is a space, then we call X a $(\mathcal{F}, \kappa)$-*space*, or simply, a κ-*space*.

Remark 2.1. We note that Kasahara considered the operation $\alpha : \tau \to \mathcal{P}(\cup\tau)$ such that $G \subset G^\alpha$ for any $G \in \tau$ in [2], where $\cup\tau = \cup_{U \in \tau} U$. However, we do not assume that $\cup\mathcal{F} = X$ in the above definition and we will state explicitly when the assumption $\cup\mathcal{F} = X$ is necessary. We follow the idea of Kasahara [2] which impose the relation '$U \subset U^\kappa$ for each $U \in \mathcal{F}$' to define the operation in this paper.

Definition 2.2. Let $\kappa : \mathcal{F} \to \mathcal{P}(X)$ be an operation. A subset A of X is called a κ-*open* set of X if for each $x \in A$ there exists a set $U \in \mathcal{F}$ such that $x \in U \subset U^\kappa \subset A$. The family of all κ-open sets is denoted by $\mathcal{F}_\kappa$. A subset F of X is called a κ-*closed* set of X if its complement $X - F$ is a κ-open set in X.

Remark 2.3. The set U^κ itself is not necessarily κ-open for any $U \in \mathcal{F}$.

(1) Let $(\mathbb{R}, \tau)$ be the real line with the usual topology τ. If we define $\mathcal{F} = \tau$ and $\kappa = \mathrm{Cl} : \tau \to \mathcal{P}(\mathbb{R})$, then we see U^κ is not necessarily κ-open.

(2) Let $X = \{a, b, c\}$ and $\mathcal{F} = \tau = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\}$. Then (X, τ) is a connected topological space. If we define $\mathcal{F} = \tau$ and $\kappa = \mathrm{Cl} : \tau \to \mathcal{P}(X)$, then we see $\emptyset^\kappa = \emptyset$, $\{a\}^\kappa = \{a, b\}^\kappa = \{a, c\}^\kappa = X^\kappa = X$. It follows that U^κ is κ-open for any $U \in \mathcal{F}$.

We have the following characterization of κ-open sets by the definition.

Proposition 2.4. *A subset A of X is κ-open if and only if there exists an index set Λ and $U_\lambda \in \mathcal{F}$ for each $\lambda \in \Lambda$ such that $A = \cup_{\lambda \in \Lambda} U_\lambda = \cup_{\lambda \in \Lambda} U_\lambda^\kappa$. It* *follows that $A \subset \cup\mathcal{F}$ for any κ-open set A, and $F \supset X - \cup\mathcal{F}$ for any κ-closed set F. Therefore, the relation $\cup\mathcal{F}_\kappa \subset \cup\mathcal{F}$ holds.*

If $\kappa : \mathcal{F} \to \mathcal{P}(X)$ is the inclusion, namely $\kappa(U) = U$ for any $U \in \mathcal{F}$, then we define 'κ-open sets' making use of the family $\mathcal{F}$ itself.

The following result is obtained by Definition 2.2:

Proposition 2.5. (1) $\emptyset \in \mathcal{F}_\kappa$.

(2) *$X \in \mathcal{F}_\kappa$ if and only if $\cup\mathcal{F} = X$.*

Example 2.6. Let $X = \{a, b, c\}$ and $\mathcal{F} = \{\{a\}, \{b\}, \{a, b\}\}$.

(1) If we define an operation $\kappa : \mathcal{F} \to \mathcal{P}(X)$ by $\{a\}^\kappa = \{a, c\}$, $\{b\}^\kappa = \{b, c\}$ and $\{a, b\}^\kappa = X$, then we see $\mathcal{F}_\kappa = \{\emptyset\}$.

(2) If we define an operation $\kappa : \mathcal{F} \to \mathcal{P}(X)$ by $\{a\}^\kappa = \{a, c\}$, $\{b\}^\kappa = \{b, c\}$ and $\{a, b\}^\kappa = \{a, b\}$, then we see $\mathcal{F}_\kappa = \{\emptyset, \{a, b\}\}$.

Example 2.7. Let $X = \{a, b, c\}$ and $\mathcal{F} = \{\{a\}, \{b\}, \{c\}\}$. We define an operation $\kappa : \mathcal{F} \to \mathcal{P}(X)$ by $\{a\}^\kappa = \{a\}$, $\{b\}^\kappa = \{a, b\}$ and $\{c\}^\kappa = \{a, c\}$. Then we see $\mathcal{F}_\kappa = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\}$.

If (X, τ) is a topological space and $\mathcal{F} = \tau$ and the operation κ is $\gamma : \tau \to \mathcal{P}(X)$, then $\mathcal{F}_\kappa = \tau_\gamma$ of Ogata [6]. However, without the assumption $\mathcal{F} = \tau$ for a topological space (X, τ), we have the following result as a direct consequence of Definition 2.2 (cf. Proposition 2.3 of [6]); the proof is obtained by a routine argument and it is omitted.

Proposition 2.8. *If $A_\lambda \in \mathcal{F}_\kappa$ for any $\lambda \in \Lambda$, then $\cup_{\lambda\in\Lambda} A_\lambda \in \mathcal{F}_\kappa$.*

The family $\mathcal{F}_\kappa$ of κ-open sets is not closed under finite intersections in general (cf. Remark 2.10 of Ogata [6], which is also quoted in Example 3.19).

Remark 2.9. Let $\gamma : \tau \to \mathcal{P}(X)$ be any operation for a topological space (X, τ). Janković [1] called a subset A of (X, τ) a *γ-closed* set if $\mathrm{Cl}_\gamma(A) \subset A$; this definition coincides with the κ-closed set in Definition 2.2 when $\kappa = \gamma : \tau \to \mathcal{P}(X)$ by Theorem 3.7 of Ogata [6] or its generalization Proposition 3.22. See also Definition 3.7.

3. Interiors and closures in κ-spaces

Let $(X, \mathcal{F}, \kappa)$ be a space. Considering well-known concepts in general topology, we define κ-interior points, $\mathcal{F}_\kappa$-interior points, κ-adherent points, $\mathcal{F}_\kappa$-adherent points, and introduce $\mathrm{Int}_\kappa(A)$, $\mathcal{F}_\kappa$-$\mathrm{Int}(A)$, $\mathrm{Cl}_\kappa(A)$ and $\mathcal{F}_\kappa$-$\mathrm{Cl}(A)$ for any subset A of X. Some relations among them are obtained.

Let x be any point of X. A set U^κ such that $x \in U \in \mathcal{F}$ is called a *κ-neighborhood of x*. A κ-open set W such that $x \in W$ is called a *κ-open neighborhood of x*.

Remark 3.1. We notice that if $\cup\mathcal{F} \neq X$ and $x \in X - \cup\mathcal{F}$, then x has no κ-neighborhood or κ-open neighborhood; this never happens in general topology.

If $\cup\mathcal{F} = X$, then for each point x of X there exists at least one $U \in \mathcal{F}$ such that $x \in U \subset U^\kappa \subset X$; it follows then that $\emptyset, X \in \mathcal{F}_\kappa$ and each point $x \in X$ has at least one κ-open neighborhood X.

3.1. Some properties of Int_κ, $\mathcal{F}_\kappa$-Int, Cl_κ and $\mathcal{F}_\kappa$-Cl

Definition 3.2. Let A be a subset of X and x a point of X. A point x is called a *κ-interior point* of A if there exists $U \in \mathcal{F}$ such that $x \in U \subset U^\kappa \subset A$. The *$\kappa$-interior* of A is defined by

$$\mathrm{Int}_\kappa(A) = \{x \mid x \text{ is a } \kappa\text{-interior point of } A\}.$$

A point x is called a *$\mathcal{F}_\kappa$-interior point* of A if there exists $V \in \mathcal{F}_\kappa$ such that $x \in V \subset A$. The *$\mathcal{F}_\kappa$-interior* of A is defined by

$$\mathcal{F}_\kappa\text{-Int}(A) = \{x \mid x \text{ is a } \mathcal{F}_\kappa\text{-interior point of } A\}.$$

Example 3.3. We consider the examples (1) and (2) in Example 2.6, where $X = \{a, b, c\}$: In (1) we see $\mathrm{Int}_\kappa(X) = \{a, b\}$ and $\mathcal{F}_\kappa\text{-Int}(X) = \emptyset$; in (2) we see $\mathrm{Int}_\kappa(X) = \{a, b\} = \mathcal{F}_\kappa\text{-Int}(X)$.

Proposition 3.4. $\mathcal{F}_\kappa\text{-Int}(A) = \cup\{V \mid V \subset A$ *and* $V \in \mathcal{F}_\kappa\}$.

Proof. We see $x \in \mathcal{F}_\kappa\text{-Int}(A)$ if and only if there exists $V \in \mathcal{F}_\kappa$ such that $x \in V \subset A$, which is equivalent to $x \in \cup\{V \mid V \subset A \text{ and } V \in \mathcal{F}_\kappa\}$. □

The following are immediate consequences of the definitions.

Proposition 3.5. *The following hold for any subset A of X.*

(1) $A \supset \mathrm{Int}_\kappa(A) \supset \mathcal{F}_\kappa\text{-Int}(A)$.

(2) *If A is a κ-open set, then* $A = \mathrm{Int}_\kappa(A) = \mathcal{F}_\kappa\text{-Int}(A)$.

Corollary 3.6. *Assume that (X, τ) is a topological space and $\mathcal{F} \subset \tau$. The following hold for any subset A of X.*

(1) $A \supset \mathrm{Int}(A) \supset \mathrm{Int}_\kappa(A) \supset \mathcal{F}_\kappa\text{-Int}(A)$.

(2) *If A is a κ-open set, then* $A = \mathrm{Int}(A) = \mathrm{Int}_\kappa(A) = \mathcal{F}_\kappa\text{-Int}(A)$.

Definition 3.7. Let A be a subset of X and x a point of X. A point x is called a *κ-adherent point* of A if $U^\kappa \cap A \neq \emptyset$ for any $U \in \mathcal{F}$ with $x \in U$, or there exists no $U \in \mathcal{F}$ with $x \in U$. The *κ-closure* of A is defined by

$$\mathrm{Cl}_\kappa(A) = \{x \mid x \text{ is a } \kappa\text{-adherent point of } A\}.$$

A point x is called a *$\mathcal{F}_\kappa$-adherent point* of A if $V \cap A \neq \emptyset$ for any $V \in \mathcal{F}_\kappa$ with $x \in V$, or there exists no $V \in \mathcal{F}_\kappa$ with $x \in V$. The *$\mathcal{F}_\kappa$-closure* of A is defined by

$$\mathcal{F}_\kappa\text{-Cl}(A) = \{x \mid x \text{ is a } \mathcal{F}_\kappa\text{-adherent point of } A\}.$$

Although some points of X may have no κ-neighborhood or κ-open neighborhood, the following results holds by Definition 3.7.

Proposition 3.8. *The inclusions $A \subset \mathrm{Cl}_\kappa(A)$ and $A \subset \mathcal{F}_\kappa\text{-Cl}(A)$ hold for any $A \subset X$.*

Remark 3.9. The following relations hold for any subset A of X.

$$\mathrm{Int}_\kappa(A) \subset \cup\mathcal{F} \text{ and } \mathcal{F}_\kappa\text{-Int}(A) \subset \cup\mathcal{F}_\kappa\,;$$

$$\mathrm{Cl}_\kappa(A) \supset X - \cup\mathcal{F} \text{ and } \mathcal{F}_\kappa\text{-Cl}(A) \supset X - \cup\mathcal{F}_\kappa.$$

The above relations hold even in the case where $A = \emptyset$ or X. It follows that

$$\mathrm{Int}_\kappa(X) = \cup\mathcal{F} \text{ and } \mathcal{F}_\kappa\text{-Int}(X) = \cup\mathcal{F}_\kappa\,;$$

$$\mathrm{Cl}_\kappa(\emptyset) = X - \cup\mathcal{F} \text{ and } \mathcal{F}_\kappa\text{-Cl}(\emptyset) = X - \cup\mathcal{F}_\kappa;$$

$$\mathrm{Int}_\kappa(\emptyset) = \emptyset = \mathcal{F}_\kappa\text{-Int}(\emptyset) \text{ and } \mathrm{Cl}_\kappa(X) = X = \mathcal{F}_\kappa\text{-Cl}(X).$$

Proposition 3.10. $\mathcal{F}_\kappa$-$\mathrm{Cl}(A) = \cap\{F \mid A \subset F$ *and* $X - F \in \mathcal{F}_\kappa\}$.

Proof. We see $x \notin \mathcal{F}_\kappa$-$\mathrm{Cl}(A)$ if and only if $V \cap A = \emptyset$ for some $V \in \mathcal{F}_\kappa$ with $x \in V$, which is equivalent to $x \notin \cap\{F \mid A \subset F$ and $X - F \in \mathcal{F}_\kappa\}$. □

The following results are obtained as immediate consequences of the definitions and Proposition 3.10; they are known for special cases where operations are $\kappa = \gamma : \tau \to \mathcal{P}(X)$ for topological spaces (X, τ) by (3.4) of Ogata [6].

Proposition 3.11. *The following hold for any subset A of X.*
(1) $A \subset \mathrm{Cl}_\kappa(A) \subset \mathcal{F}_\kappa$-$\mathrm{Cl}(A)$.
(2) *If A is a κ-closed set, then* $A = \mathrm{Cl}_\kappa(A) = \mathcal{F}_\kappa$-$\mathrm{Cl}(A)$.

Corollary 3.12. *Assume that (X, τ) is a topological space and $\mathcal{F} \subset \tau$. The following hold for any subset A of X.*
(1) $A \subset \mathrm{Cl}(A) \subset \mathrm{Cl}_\kappa(A) \subset \mathcal{F}_\kappa$-$\mathrm{Cl}(A)$.
(2) *If A is a κ-closed set, then* $A = \mathrm{Cl}(A) = \mathrm{Cl}_\kappa(A) = \mathcal{F}_\kappa$-$\mathrm{Cl}(A)$.

Proposition 3.13. *The following hold for any subset A of X:*
(1) $X - (\mathcal{F}_\kappa\text{-Int}(A)) = \mathcal{F}_\kappa\text{-Cl}(X - A)$; $X - (\mathcal{F}_\kappa\text{-Cl}(A)) = \mathcal{F}_\kappa\text{-Int}(X - A)$.
(2) $X - \mathrm{Int}_\kappa(A) = \mathrm{Cl}_\kappa(X - A)$; $X - \mathrm{Cl}_\kappa(A) = \mathrm{Int}_\kappa(X - A)$.

Proof. (1) We prove the first formula. Let $x \in X - (\mathcal{F}_\kappa\text{-Int}(A))$.

Case 1; $x \notin \cup\mathcal{F}_\kappa$: $x \notin \mathcal{F}_\kappa$-$\mathrm{Int}(A)$ is equivalent to $x \in \mathcal{F}_\kappa$-$\mathrm{Cl}(X - A)$ by definitions.

Case 2; $x \in \cup\mathcal{F}_\kappa$: By the definitions of $\mathcal{F}_\kappa$-Int and $\mathcal{F}_\kappa$-Cl, the condition $x \notin \mathcal{F}_\kappa$-$\mathrm{Int}(A)$ is equivalent to the condition that there exists no $V \in \mathcal{F}_\kappa$ such that $x \in V \subset A$, that is, $V \cap (X - A) \neq \emptyset$ for any $V \in \mathcal{F}_\kappa$ with $x \in V$ or $x \in \mathcal{F}_\kappa$-$\mathrm{Cl}(X - A)$.

(2) We prove the first formula. Let $x \in X - \mathrm{Int}_\kappa(A)$.

Case 1; $x \notin \cup\mathcal{F}$: The condition $x \notin \mathrm{Int}_\kappa(A)$ is equivalent to the condition that $x \in \mathrm{Cl}_\kappa(X - A)$ by definitions.

Case 2; $x \in \cup\mathcal{F}$: The condition $x \notin \mathrm{Int}_\kappa(A)$ is equivalent to the condition that there exists no $U \in \mathcal{F}$ such that $x \in U \subset U^\kappa \subset A$, that is, $U^\kappa \cap (X - A) \neq \emptyset$ for any $U \in \mathcal{F}$ with $x \in U$ or $x \in \mathrm{Cl}_\kappa(X - A)$. □

Proposition 3.14. *Let A be any subset of X. Then $\mathcal{F}_\kappa$-$\mathrm{Int}(A)$ is a κ-open set and $\mathcal{F}_\kappa$-$\mathrm{Cl}(A)$ is a κ-closed set.*

Proof. By Propositions 2.8 and 3.4, the set $\mathcal{F}_\kappa$-Int(A) is κ-open. By Proposition 3.13(1), we have $\mathcal{F}_\kappa\text{-Cl}(A) = X - (\mathcal{F}_\kappa\text{-Int}(X - A))$, and hence $\mathcal{F}_\kappa$-Cl(A) is a κ-closed set. □

Remark 3.15. (1) We proved that $\mathcal{F}_\kappa$-Int(A) is a κ-open set and $\mathcal{F}_\kappa$-Cl(A) is a κ-closed set by Proposition 3.14. But, $\mathrm{Int}_\kappa(A)$ is not necessarily a κ-open set, and $\mathrm{Cl}_\kappa(A)$ is not necessarily a κ-closed set even in the case where $\kappa = \gamma : \tau \to \mathcal{P}(X)$ for a topological space (X, τ). For example: Let $X = \{a, b, c\}$ and $\mathcal{F} = \tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. Then (X, τ) is a topological space. Let $\kappa = \gamma$ be an operation defined by $\kappa(A) = \gamma(A) = \mathrm{Cl}(A)$ for any $A \in \tau$; we see $\mathcal{F}_\kappa = \tau_\gamma = \{\emptyset, X\}$ (cf. Example 2.7 of [6]). It follows that $\mathrm{Int}_\kappa(\{a, c\}) = \{a\}$ is not a κ-open set and

$$\{a\} = \mathrm{Int}_\kappa(\{a, c\}) \neq \mathcal{F}_\kappa\text{-Int}(\{a, c\}) = \emptyset.$$

Moreover, $\mathrm{Cl}_\kappa(\{a\}) = \{a, c\}$ is not a κ-closed set and $\mathcal{F}_\kappa\text{-Cl}(\{a\}) = X$ is a κ-closed set.

(2) Ogata [6] defined the τ_γ*-closure* of a subset A of X by

$$\tau_\gamma\text{-Cl}(A) = \cap\{F \mid A \subset F \text{ and } X - F \in \tau_\gamma\}$$

for any operation $\gamma : \tau \to \mathcal{P}(X)$ and any topological space (X, τ).

3.2. Some formulas and examples

By Propositions 3.4, 3.10 and 3.14, the following results for $\mathcal{F}_\kappa$-Int and $\mathcal{F}_\kappa$-Cl are obtained (cf. Theorem 1.1 of [4], Lemmas 2.2 and 2.3 of [5] and Lemmas 3.1 and 3.2 of [7]).

Proposition 3.16. *Let A and B be subsets of X.*

(1) $\mathcal{F}_\kappa\text{-Int}(\mathcal{F}_\kappa\text{-Int}(A)) = \mathcal{F}_\kappa\text{-Int}(A)$. *If $A \subset B$, then* $\mathcal{F}_\kappa\text{-Int}(A) \subset \mathcal{F}_\kappa\text{-Int}(B)$.

(2) $\mathcal{F}_\kappa\text{-Cl}(\mathcal{F}_\kappa\text{-Cl}(A)) = \mathcal{F}_\kappa\text{-Cl}(A)$. *If $A \subset B$, then* $\mathcal{F}_\kappa\text{-Cl}(A) \subset \mathcal{F}_\kappa\text{-Cl}(B)$.

Corollary 3.17. *For any subsets A and B of X, the following hold.*

$$\mathcal{F}_\kappa\text{-Int}(A \cap B) \subset \mathcal{F}_\kappa\text{-Int}(A) \cap \mathcal{F}_\kappa\text{-Int}(B);$$
$$\mathcal{F}_\kappa\text{-Int}(A \cup B) \supset \mathcal{F}_\kappa\text{-Int}(A) \cup \mathcal{F}_\kappa\text{-Int}(B);$$
$$\mathcal{F}_\kappa\text{-Cl}(A \cap B) \subset \mathcal{F}_\kappa\text{-Cl}(A) \cap \mathcal{F}_\kappa\text{-Cl}(B);$$
$$\mathcal{F}_\kappa\text{-Cl}(A \cup B) \supset \mathcal{F}_\kappa\text{-Cl}(A) \cup \mathcal{F}_\kappa\text{-Cl}(B).$$

Proof. By Proposition 3.16, the results follow. □

However, the formulas for Cl_κ and Int_κ are not obtained as special cases of these formulas. The following inclusions hold by the definitions, and the proof is omitted.

Proposition 3.18. (1) *If $A \subset B \subset X$, then* $\mathrm{Int}_\kappa(A) \subset \mathrm{Int}_\kappa(B)$ *and* $\mathrm{Cl}_\kappa(A) \subset \mathrm{Cl}_\kappa(B)$.

(2) *If $A, B \subset X$, then*

$$\mathrm{Int}_\kappa(A\cap B)\subset \mathrm{Int}_\kappa(A)\cap \mathrm{Int}_\kappa(B);\quad \mathrm{Int}_\kappa(A\cup B)\supset \mathrm{Int}_\kappa(A)\cup \mathrm{Int}_\kappa(B);$$
$$\mathrm{Cl}_\kappa(A\cap B)\subset \mathrm{Cl}_\kappa(A)\cap \mathrm{Cl}_\kappa(B);\quad \mathrm{Cl}_\kappa(A\cup B)\supset \mathrm{Cl}_\kappa(A)\cup \mathrm{Cl}_\kappa(B).$$

Example 3.19. Well-known formulas:

$$\mathrm{Cl}(A\cup B)=\mathrm{Cl}(A)\cup\mathrm{Cl}(B);\quad \mathrm{Int}(A\cap B)=\mathrm{Int}(A)\cap\mathrm{Int}(B)$$

for subsets A, B of a topological space X do not hold in general for Cl_γ, Int_γ, τ_γ-Cl and τ_γ-Int even in the case where $\kappa=\gamma:\tau\to\mathcal{P}(X)$ for a topological space (X,τ). For example, we consider the topological space (X,τ) and the operation γ of Example 2.8 of Ogata [6]: Let $X=\{a,b,c\}$ and

$$\mathcal{F}=\{\emptyset,\{a\},\{b\},\{a,b\},\{a,c\},X\}.$$

Define an operation $\kappa:\mathcal{F}\to\mathcal{P}(X)$ by $A^\kappa=A$ if $b\in A$ and $A^\kappa=\mathrm{Cl}(A)$ if $b\notin A$. In this case, $\mathcal{F}_\kappa=\{\emptyset,\{b\},\{a,b\},\{a,c\},X\}$ and $\{a,b\}\cap\{a,c\}=\{a\}\notin\mathcal{F}_\kappa$ by Remark 2.10 of [6]. We have

$$\mathrm{Cl}_\kappa(\{b\})\cup\mathrm{Cl}_\kappa(\{c\})=\{b\}\cup\{c\}\subsetneq X=\mathrm{Cl}_\kappa(\{b,c\});$$
$$\mathrm{Int}_\kappa(\{a\})\cup\mathrm{Int}_\kappa(\{b\})=\emptyset\cup\{b\}\subsetneq\{a,b\}=\mathrm{Int}_\kappa(\{a,b\});$$
$$\mathcal{F}_\kappa\text{-}\mathrm{Cl}(\{b\})\cup\mathcal{F}_\kappa\text{-}\mathrm{Cl}(\{c\})=\{b\}\cup\{c\}\subsetneq X=\mathcal{F}_\kappa\text{-}\mathrm{Cl}(\{b,c\});$$
$$\mathcal{F}_\kappa\text{-}\mathrm{Int}(\{a\})\cup\mathcal{F}_\kappa\text{-}\mathrm{Int}(\{b\})=\emptyset\cup\{b\}\subsetneq\{a,b\}=\mathcal{F}_\kappa\text{-}\mathrm{Int}(\{a,b\});$$
$$\mathrm{Int}_\kappa(\{a,b\}\cap\{a,c\})=\mathrm{Int}_\kappa(\{a\})=\emptyset\subsetneq\{a\}=\mathrm{Int}_\kappa(\{a,b\})\cap\mathrm{Int}_\kappa(\{a,c\});$$
$$\mathcal{F}_\kappa\text{-}\mathrm{Int}(\{a,b\}\cap\{a,c\})=\mathcal{F}_\kappa\text{-}\mathrm{Int}(\{a\})=\emptyset$$
$$\subsetneq\{a\}=\mathcal{F}_\kappa\text{-}\mathrm{Int}(\{a,b\})\cap\mathcal{F}_\kappa\text{-}\mathrm{Int}(\{a,c\}).$$

Theorem 3.20. *The following hold for any subset A of X.*

$$\mathcal{F}_\kappa\text{-}\mathrm{Int}(\mathcal{F}_\kappa\text{-}\mathrm{Cl}(\mathcal{F}_\kappa\text{-}\mathrm{Int}(\mathcal{F}_\kappa\text{-}\mathrm{Cl}(A))))=\mathcal{F}_\kappa\text{-}\mathrm{Int}(\mathcal{F}_\kappa\text{-}\mathrm{Cl}(A));$$
$$\mathcal{F}_\kappa\text{-}\mathrm{Cl}(\mathcal{F}_\kappa\text{-}\mathrm{Int}(\mathcal{F}_\kappa\text{-}\mathrm{Cl}(\mathcal{F}_\kappa\text{-}\mathrm{Int}(A))))=\mathcal{F}_\kappa\text{-}\mathrm{Cl}(\mathcal{F}_\kappa\text{-}\mathrm{Int}(A)).$$

Proof. By Proposition 3.16, we have the results. □

Proposition 3.21. *The following hold for any subset A of X.*

$$\mathrm{Cl}_\kappa(\mathrm{Cl}_\kappa(\mathrm{Int}_\kappa(A)))\supset\mathrm{Cl}_\kappa(\mathrm{Int}_\kappa(\mathrm{Cl}_\kappa(\mathrm{Int}_\kappa(A))))\supset\mathrm{Cl}_\kappa(\mathrm{Int}_\kappa(\mathrm{Int}_\kappa(A)));$$
$$\mathrm{Cl}_\kappa(\mathrm{Int}_\kappa(\mathrm{Cl}_\kappa(A)))\supset\mathrm{Cl}_\kappa(\mathrm{Int}_\kappa(\mathrm{Cl}_\kappa(\mathrm{Int}_\kappa(A))))\supset\mathrm{Int}_\kappa(\mathrm{Cl}_\kappa(\mathrm{Int}_\kappa(A)));$$
$$\mathrm{Int}_\kappa(\mathrm{Cl}_\kappa(\mathrm{Cl}_\kappa(A)))\supset\mathrm{Int}_\kappa(\mathrm{Cl}_\kappa(\mathrm{Int}_\kappa(\mathrm{Cl}_\kappa(A))))\supset\mathrm{Int}_\kappa(\mathrm{Int}_\kappa(\mathrm{Cl}_\kappa(A)));$$
$$\mathrm{Cl}_\kappa(\mathrm{Int}_\kappa(\mathrm{Cl}_\kappa(A)))\supset\mathrm{Int}_\kappa(\mathrm{Cl}_\kappa(\mathrm{Int}_\kappa(\mathrm{Cl}_\kappa(A))))\supset\mathrm{Int}_\kappa(\mathrm{Cl}_\kappa(\mathrm{Int}_\kappa(A))).$$

Proof. By Proposition 3.18, we have the results. □

The following result generalizes Theorem 3.7 of Ogata [6].

Proposition 3.22. *The following statements are equivalent for any subset A of X.*

(1) *A is κ-open.*
(2) $A=\mathrm{Int}_\kappa(A)$.

(3) $A = \mathcal{F}_\kappa\text{-Int}(A)$.
(4) $\text{Cl}_\kappa(X - A) = X - A$.
(5) $\mathcal{F}_\kappa\text{-Cl}(X - A) = X - A$.
(6) $X - A$ *is* κ*-closed.*

Proof. (1) and (2) are equivalent by the definitions of κ-open sets and κ-interior points. (1) and (3) are equivalent by the definition of $\mathcal{F}_\kappa$-interior points and Propositions 2.8 and 3.4. (1) and (6) are equivalent by the definition of κ-closed sets. (2) and (4) are equivalent by Proposition 3.13(2). (3) and (5) are equivalent by Proposition 3.13(1). □

Remark 3.23. (1) For any subset $A \subset X$, we have

$$\text{Cl}_\kappa(\mathcal{F}_\kappa\text{-Cl}(A)) = \mathcal{F}_\kappa\text{-Cl}(A) \quad \text{and} \quad \text{Int}_\kappa(\mathcal{F}_\kappa\text{-Int}(A)) = \mathcal{F}_\kappa\text{-Int}(A)$$

by Propositions 3.14 and 3.22.

(2) Let $A \subset X$. Then there are examples such that

$$\text{Cl}_\kappa(\text{Cl}_\kappa(A)) \neq \text{Cl}_\kappa(A), \quad \mathcal{F}_\kappa\text{-Cl}(\text{Cl}_\kappa(A)) \neq \text{Cl}_\kappa(A),$$

$$\text{Int}_\kappa(\text{Int}_\kappa(A)) \neq \text{Int}_\kappa(A) \quad \text{or} \quad \mathcal{F}_\kappa\text{-Int}(\text{Int}_\kappa(A)) \neq \text{Int}_\kappa(A)$$

even in the case where $\kappa = \gamma : \tau \to \mathcal{P}(X)$ for a topological space (X, τ): Let (X, τ) be the space in Remark 3.15(1). Then we have $\text{Cl}_\kappa(\{a\}) = \{a, c\}$, $\text{Cl}_\kappa(\{a, c\}) = X$ and $\mathcal{F}_\kappa\text{-Cl}(\{a, c\}) = X$. Therefore (cf. Remark 3.9 of [6])

$$\text{Cl}_\kappa(\text{Cl}_\kappa(\{a\})) = X \neq \{a, c\} = \text{Cl}_\kappa(\{a\});$$

$$\mathcal{F}_\kappa\text{-Cl}(\text{Cl}_\kappa(\{a\})) = X \neq \{a, c\} = \text{Cl}_\kappa(\{a\}).$$

Next, we have the following relations, since $\text{Int}_\kappa(\{a, c\}) = \{a\}$:

$$\text{Int}_\kappa(\text{Int}_\kappa(\{a, c\})) = \text{Int}_\kappa(\{a\}) = \emptyset \neq \{a\} = \text{Int}_\kappa(\{a, c\});$$

$$\mathcal{F}_\kappa\text{-Int}(\text{Int}_\kappa(\{a, c\})) = \mathcal{F}_\kappa\text{-Int}(\{a\}) = \emptyset \neq \{a\} = \text{Int}_\kappa(\{a, c\}).$$

Remark 3.24. If $\kappa = \gamma : \tau \to \mathcal{P}(X)$ for some topological space (X, τ), then Proposition 3.13(2) is the result of Theorem 2(2.1) of Rehman and Ahmad [8]. The relations in Proposition 3.18 are also remarked in [8].

3.3. Regular operations

An operation $\kappa : \mathcal{F} \to \mathcal{P}(X)$ is said to be *regular* if for any $a \in X$ and any sets $U, V \in \mathcal{F}$ with $a \in U \cap V$, there exists a set $W \in \mathcal{F}$ such that $a \in W \subset W^\kappa \subset U^\kappa \cap V^\kappa$ (cf. p. 98 of Kasahara [2]). Then the following result, which is a generalization of Proposition 2.9(1) of Ogata [6], is obtained by a standard argument.

Proposition 3.25. *If* $\kappa : \mathcal{F} \to \mathcal{P}(X)$ *is regular, then* $A_1 \cap A_2 \in \mathcal{F}_\kappa$ *for any* A_1, $A_2 \in \mathcal{F}_\kappa$.

Theorem 3.26. *Assume that the operation* $\kappa : \mathcal{F} \to \mathcal{P}(X)$ *is regular. Let* A *be a* κ*-open set and* B *any subset of* X*. Then the following relations hold.*
(1) $A \cap \mathcal{F}_\kappa\text{-Cl}(B) \subset \mathcal{F}_\kappa\text{-Cl}(A \cap B)$.
(2) $A \cap \text{Cl}_\kappa(B) \subset \text{Cl}_\kappa(A \cap B)$.

Proof. (1) If $x \in A \cap \mathcal{F}_\kappa$-Cl$(B)$, then for any $U \in \mathcal{F}_\kappa$ such that $x \in U$ we see $U \cap (A \cap B) = (U \cap A) \cap B \neq \emptyset$, since $x \in U \cap A \in \mathcal{F}_\kappa$ by Proposition 3.25.

(2) Suppose that $x \in A \cap \mathrm{Cl}_\kappa(B)$ and $U \in \mathcal{F}$ is any set such that $x \in U$. Since A is a κ-open set and $x \in A$, there exists $V \in \mathcal{F}$ such that $x \in V \subset V^\kappa \subset A$. Since κ is regular, there exists a neighborhood $W \in \mathcal{F}$ of x such that $x \in W^\kappa \subset U^\kappa \cap V^\kappa$. Since $x \in \mathrm{Cl}_\kappa(B)$, we have

$$U^\kappa \cap (A \cap B) = (U^\kappa \cap A) \cap B \supset (U^\kappa \cap V^\kappa) \cap B \supset W^\kappa \cap B \neq \emptyset.$$

It follows that $x \in \mathrm{Cl}_\kappa(A \cap B)$. □

If the operation $\kappa : \mathcal{F} \to \mathcal{P}(X)$ is regular, then the following hold (cf. Example 3.19).

Proposition 3.27. *If the operation $\kappa : \mathcal{F} \to \mathcal{P}(X)$ is regular, then the following equalities hold for any subsets A and B of X.*

$$\mathrm{Int}_\kappa(A \cap B) = \mathrm{Int}_\kappa(A) \cap \mathrm{Int}_\kappa(B);$$
$$\mathrm{Cl}_\kappa(A \cup B) = \mathrm{Cl}_\kappa(A) \cup \mathrm{Cl}_\kappa(B);$$
$$\mathcal{F}_\kappa\text{-Int}(A \cap B) = \mathcal{F}_\kappa\text{-Int}(A) \cap \mathcal{F}_\kappa\text{-Int}(B);$$
$$\mathcal{F}_\kappa\text{-Cl}(A \cup B) = \mathcal{F}_\kappa\text{-Cl}(A) \cup \mathcal{F}_\kappa\text{-Cl}(B).$$

Proof. The first formula is obtained by the definition of Int_κ and the definition of regularity. The second is then obtained by Proposition 3.13(2). The third and the fourth formulas are obtained by Propositions 3.25 and 3.13(1). □

4. Maximal κ-open sets and the $\mathcal{F}_\kappa$-closure.

Let $(X, \mathcal{F}, \kappa)$ be a space.

Definition 4.1. A proper non-empty κ-open set U of X is said to be a *maximal κ-open set* if any κ-open set which contains U is X or U.

Theorem 4.2. *If U is a maximal κ-open set and x is an element of $X - U$, then $X - U \subset W$ for any κ-open neighborhood W of x.*

Proof. If $x \notin \cup\mathcal{F}$, then there is no κ-open neighborhood of x and hence the theorem holds. Now suppose that $x \in \cup\mathcal{F}$: Since $W \not\subset U$ for any κ-open neighborhood W of x, we see $W \cup U = X$ by the definition of maximal κ-open set and Proposition 2.8, or $X - U \subset W$. (*Remark.* If $W \cup U = X$ for some $W, U \in \mathcal{F}_\kappa$, then we see $\cup\mathcal{F} = X$.) □

Corollary 4.3. *If U is a maximal κ-open set, then either of the following* (1) *and* (2) *holds.*

(1) *For each $x \in X - U$ and each κ-open neighborhood W of x, the relation $X - U \subsetneq W$ holds.*

(2) *U is κ-closed.*

Proof. Assume that (2) does not hold. Then we must show that (1) holds: For each $x \in X - U$ and each κ-open neighborhood W of x, we see $X - U \subset W$ by Theorem 4.2. If $X - U = W$, then U is a κ-closed set, which contradicts our assumption. Hence, we must have $X - U \subsetneq W$. □

Theorem 4.4. *If U is a maximal κ-open set, then $\mathcal{F}_\kappa$-$\mathrm{Cl}(U) = X$ or $\mathcal{F}_\kappa$-$\mathrm{Cl}(U) = U$. If $\cup\mathcal{F} \neq X$, then $\mathcal{F}_\kappa$-$\mathrm{Cl}(U) = X$.*

Proof. Suppose $\mathcal{F}_\kappa$-$\mathrm{Cl}(U) \neq U$. Then U is not κ-closed by Proposition 3.11(2). Let x be any element of $X - U$ and W any κ-open neighborhood of x. By Corollary 4.3, we have $X - U \subsetneq W$ for any κ-open neighborhood W of x, and hence we get $W \cap U \neq \emptyset$ or $x \in \mathcal{F}_\kappa$-$\mathrm{Cl}(U)$. If $x \in X - U$ has no κ-open neighborhood, then $x \in \mathcal{F}_\kappa$-$\mathrm{Cl}(U)$. It follows that $X - U \subset \mathcal{F}_\kappa$-$\mathrm{Cl}(U)$. Since

$$X = U \cup (X - U) \subset U \cup \mathcal{F}_\kappa\text{-}\mathrm{Cl}(U) = \mathcal{F}_\kappa\text{-}\mathrm{Cl}(U) \subset X,$$

we have $\mathcal{F}_\kappa$-$\mathrm{Cl}(U) = X$.

If $\cup\mathcal{F} \neq X$, then $\mathcal{F}_\kappa$-$\mathrm{Cl}(U) \neq U$ and hence $\mathcal{F}_\kappa$-$\mathrm{Cl}(U) = X$ by the first assertion. □

Example 4.5. (1) In Example 2.7, maximal κ-open sets are $\{a, b\}$ and $\{a, c\}$. We see $\mathcal{F}_\kappa$-$\mathrm{Cl}(\{a, b\}) = X = \mathcal{F}_\kappa$-$\mathrm{Cl}(\{a, c\})$.

(2) In Example 3.19, maximal κ-open sets are $\{a, b\}$ and $\{a, c\}$. We see $\mathcal{F}_\kappa$-$\mathrm{Cl}(\{a, b\}) = X$ and $\mathcal{F}_\kappa$-$\mathrm{Cl}(\{a, c\}) = \{a, c\}$.

Remark 4.6. The result of Theorem 4.4 does not hold for Cl_κ in general even in the case where $\kappa = \gamma : \tau \to \mathcal{P}(X)$ for a topological space (X, τ). For example: Let $X = \{a, b, c, d\}$ and let

$$\mathcal{F} = \tau = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}, X\}.$$

Let $\kappa : \mathcal{F} \to \mathcal{P}(X)$ be an operation defined by

$$\emptyset^\kappa = \emptyset, \quad \{a\}^\kappa = \{a, d\}, \quad \{b\}^\kappa = \{c\}^\kappa = \{b, c\}^\kappa = \{b, c\},$$
$$\{a, b\}^\kappa = \{a, b, d\}, \quad \{a, c\}^\kappa = \{a, c, d\}, \quad \{a, b, c\}^\kappa = X^\kappa = X.$$

Then $\mathcal{F}_\kappa = \{\emptyset, \{b, c\}, X\}$ and $\{b, c\}$ is a maximal κ-open set. We see

$$\{b, c\} \neq \mathrm{Cl}_\kappa(\{b, c\}) = \{b, c, d\} \neq X = \mathcal{F}_\kappa\text{-}\mathrm{Cl}(\{b, c\}).$$

Finally, we note the following relations which are considered again in Corollary 4.8 and Example 5.3:

$$\mathrm{Cl}_\kappa(\{a, b, c\}) = X = \mathcal{F}_\kappa\text{-}\mathrm{Cl}(\{a, b, c\}); \quad \mathrm{Cl}_\kappa(\{b, c, d\}) = X = \mathcal{F}_\kappa\text{-}\mathrm{Cl}(\{b, c, d\}).$$

Theorem 4.7. *Let U be a maximal κ-open set and S a non-empty subset of $X - U$. Then $\mathcal{F}_\kappa$-$\mathrm{Cl}(S) = X - U$.*

Proof. We have $\mathcal{F}_\kappa$-$\mathrm{Cl}(S) \subset \mathcal{F}_\kappa$-$\mathrm{Cl}(X - U) = X - U$ by Proposition 3.11(2), since $X - U$ is a κ-closed set and $S \subset X - U$. Next, suppose that x is any element of $X - U$: If x has no κ-open neighborhood, then $x \in \mathcal{F}_\kappa$-$\mathrm{Cl}(S)$; if W is any κ-open neighborhood of x, then $W \cap S \neq \emptyset$ by Theorem 4.2, since $\emptyset \neq S \subset X - U$. Hence $X - U \subset \mathcal{F}_\kappa$-$\mathrm{Cl}(S)$. □

Corollary 4.8. *If U is a maximal κ-open set and M is a subset of X such that $U \subsetneq M$, then $\mathcal{F}_\kappa$-Cl$(M) = X$.*

Proof. Since $U \subsetneq M \subset X$, there exists a non-empty subset S of $X - U$ such that $M = U \cup S$. Hence we have

$$\mathcal{F}_\kappa\text{-Cl}(M) = \mathcal{F}_\kappa\text{-Cl}(S \cup U) \supset \mathcal{F}_\kappa\text{-Cl}(S) \cup \mathcal{F}_\kappa\text{-Cl}(U) \supset (X - U) \cup U = X$$

by Theorem 4.7 or $\mathcal{F}_\kappa$-Cl$(M) = X$. □

5. The pre-κ-open sets and the semi-κ-open sets

Let $(X, \mathcal{F}, \kappa)$ be a space. A subset M of a set X is called
a *pre-κ-open set* if $M \subset \mathrm{Int}_\kappa(\mathrm{Cl}_\kappa(M))$;
a *pre-$\mathcal{F}_\kappa$-open set* if $M \subset \mathcal{F}_\kappa$-Int$(\mathcal{F}_\kappa$-Cl$(M))$;
a *semi-κ-open set* if $M \subset \mathrm{Cl}_\kappa(\mathrm{Int}_\kappa(M))$;
a *semi-$\mathcal{F}_\kappa$-open set* if $M \subset \mathcal{F}_\kappa$-Cl$(\mathcal{F}_\kappa$-Int$(M))$.

The complement of a pre-κ-open (resp. pre-$\mathcal{F}_\kappa$-open, semi-κ-open, semi-$\mathcal{F}_\kappa$-open) set is called a pre-κ-closed (resp. pre-$\mathcal{F}_\kappa$-closed, semi-κ-closed, semi-$\mathcal{F}_\kappa$-closed) set.

Any κ-open (resp. κ-closed) set is pre-κ-open, pre-$\mathcal{F}_\kappa$-open, semi-κ-open and semi-$\mathcal{F}_\kappa$-open (resp. pre-κ-closed, pre-$\mathcal{F}_\kappa$-closed, semi-κ-closed and semi-$\mathcal{F}_\kappa$-closed) by Proposition 3.22.

Theorem 5.1. *Let U be a maximal κ-open set and M any subset of X with $U \subset M \subset \cup\mathcal{F}_\kappa$. Then M is pre-$\mathcal{F}_\kappa$-open.*

Proof. If $M = U$, then M is a κ-open set. Therefore M is a pre-$\mathcal{F}_\kappa$-open set. Otherwise $U \subsetneq M$, and we have $\mathcal{F}_\kappa$-Int$(\mathcal{F}_\kappa$-Cl$(M)) = \mathcal{F}_\kappa$-Int$(X) = \cup\mathcal{F}_\kappa \supset M$ by Corollary 4.8 and Remark 3.9. Therefore M is a pre-$\mathcal{F}_\kappa$-open set. □

Example 5.2. Let (X, τ) be a topological space and $\gamma : \tau \to \mathcal{P}(X)$ any operation. If U is a maximal γ-open set, then $X - \{a\}$ is a pre-τ_γ-open for any element a of $X - U$ by Theorem 5.1.

Example 5.3. Let (X, τ) and κ be the same as in Remark 4.6. Let $M = \{a, b, c\}$ which contains a maximal κ-open set $\{b, c\}$. Then by Theorem 5.1, the subset M is pre-$\mathcal{F}_\kappa$-open. However, we see

$$\mathrm{Int}_\kappa(M) = \{b, c\} = \mathcal{F}_\kappa\text{-Int}(M);$$

$$\mathcal{F}_\kappa\text{-Cl}(\mathcal{F}_\kappa\text{-Int}(M)) = X \supset M; \quad \mathrm{Cl}_\kappa(\mathrm{Int}_\kappa(M)) = \{b, c, d\} \not\supset M.$$

Hence M is not semi-κ-open, but M is semi-$\mathcal{F}_\kappa$-open. Moreover, we see

$$\mathrm{Cl}_\kappa(\{a, b, c\}) = X = \mathrm{Cl}_\kappa(\{a, c, d\}).$$

Hence $\{a, b, c\}(= M)$ and $\{a, c, d\}$ are pre-κ-open sets.

Example 5.4. A subset M which contains a maximal κ-open set is not necessarily pre-κ-open or semi-κ-open or semi-$\mathcal{F}_\kappa$-open: Let $X = \{a, b, c, d\}$ and $\mathcal{F} = \{\{a\}, \{b\}, \{c\}, \{d\}\}$. Define $\kappa : \mathcal{F} \to \mathcal{P}(X)$ by

$$\{a\}^\kappa = \{a\},\ \{b\}^\kappa = \{b, c\},\ \{c\}^\kappa = \{c, d\},\ \{d\}^\kappa = \{b, d\}.$$

It follows that $\mathcal{F}_\kappa = \{\emptyset, \{a\}, \{b,c,d\}, X\}$. We see $\{a\}$ and $\{b,c,d\}$ are maximal κ-open sets. Let $M = \{a,b\}$. Then we see $\mathrm{Cl}_\kappa(M) = M \cup \{d\}$ and $\mathrm{Int}_\kappa(M) = \{a\}$. It follows that

$$\mathrm{Int}_\kappa(\mathrm{Cl}_\kappa(M)) = \{a,d\} \not\supset M; \quad \mathrm{Cl}_\kappa(\mathrm{Int}_\kappa(M)) = \{a\} \not\supset M.$$

Hence M is not pre-κ-open or semi-κ-open. We see $\mathcal{F}_\kappa$-$\mathrm{Int}(M) = \{a\}$ and

$$\mathcal{F}_\kappa\text{-}\mathrm{Cl}(\mathcal{F}_\kappa\text{-}\mathrm{Int}(M)) = \mathcal{F}_\kappa\text{-}\mathrm{Cl}(\{a\}) = \{a\} \not\supset M.$$

It follows that M is not a semi-$\mathcal{F}_\kappa$-open set. However, M is a pre-$\mathcal{F}_\kappa$-open set by Theorem 5.1.

Example 5.5. If $M \subset X - \cup\mathcal{F}$, then M is semi-κ-open and semi-$\mathcal{F}_\kappa$-open, since the relation $\mathrm{Int}_\kappa(M) = \emptyset = \mathcal{F}_\kappa$-$\mathrm{Int}(M)$ and the facts in Remark 3.9 and Proposition 2.4 imply the following relations:

$$\mathrm{Cl}_\kappa(\mathrm{Int}_\kappa(M)) = \mathrm{Cl}_\kappa(\emptyset) = X - \cup\mathcal{F} \supset M;$$
$$\mathcal{F}_\kappa\text{-}\mathrm{Cl}(\mathcal{F}_\kappa\text{-}\mathrm{Int}(M)) = \mathcal{F}_\kappa\text{-}\mathrm{Cl}(\emptyset) = X - \cup\mathcal{F}_\kappa \supset X - \cup\mathcal{F} \supset M.$$

By the following theorem, we have relations between pre-κ-open sets and pre-$\mathcal{F}_\kappa$-open sets; or semi-κ-open sets and semi-$\mathcal{F}_\kappa$-open sets under some conditions.

Theorem 5.6. *Let M be any subset of X.*

(1) *If* $\mathrm{Int}_\kappa(\mathcal{F}_\kappa\text{-}\mathrm{Cl}(M)) = \mathcal{F}_\kappa\text{-}\mathrm{Int}(\mathrm{Cl}_\kappa(M))$, *then*

$$\mathrm{Int}_\kappa(\mathrm{Cl}_\kappa(M)) = \mathcal{F}_\kappa\text{-}\mathrm{Int}(\mathcal{F}_\kappa\text{-}\mathrm{Cl}(M)).$$

(2) *If* $\mathrm{Cl}_\kappa(\mathcal{F}_\kappa\text{-}\mathrm{Int}(M)) = \mathcal{F}_\kappa\text{-}\mathrm{Cl}(\mathrm{Int}_\kappa(M))$, *then*

$$\mathrm{Cl}_\kappa(\mathrm{Int}_\kappa(M)) = \mathcal{F}_\kappa\text{-}\mathrm{Cl}(\mathcal{F}_\kappa\text{-}\mathrm{Int}(M)).$$

Proof. (1) By Propositions 3.11 and 3.5, we have

$$\mathrm{Int}_\kappa(\mathcal{F}_\kappa\text{-}\mathrm{Cl}(M)) \supset \mathrm{Int}_\kappa(\mathrm{Cl}_\kappa(M)) \supset \mathcal{F}_\kappa\text{-}\mathrm{Int}(\mathrm{Cl}_\kappa(M));$$
$$\mathrm{Int}_\kappa(\mathcal{F}_\kappa\text{-}\mathrm{Cl}(M)) \supset \mathcal{F}_\kappa\text{-}\mathrm{Int}(\mathcal{F}_\kappa\text{-}\mathrm{Cl}(M)) \supset \mathcal{F}_\kappa\text{-}\mathrm{Int}(\mathrm{Cl}_\kappa(M)).$$

Hence the result follows.

(2) By Propositions 3.11 and 3.5, we have

$$\mathrm{Cl}_\kappa(\mathcal{F}_\kappa\text{-}\mathrm{Int}(M)) \subset \mathrm{Cl}_\kappa(\mathrm{Int}_\kappa(M)) \subset \mathcal{F}_\kappa\text{-}\mathrm{Cl}(\mathrm{Int}_\kappa(M));$$
$$\mathrm{Cl}_\kappa(\mathcal{F}_\kappa\text{-}\mathrm{Int}(M)) \subset \mathcal{F}_\kappa\text{-}\mathrm{Cl}(\mathcal{F}_\kappa\text{-}\mathrm{Int}(M)) \subset \mathcal{F}_\kappa\text{-}\mathrm{Cl}(\mathrm{Int}_\kappa(M)).$$

Hence the result follows. □

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