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A SIMPLY CONNECTED MANIFOLD WITH TWO SYMPLECTIC DEFORMATION EQUIVALENCE CLASSES WITH DISTINCT SIGNS OF SCALAR CURVATURES

Jongsu Kim

Abstract. We present a smooth simply connected closed eight dimensional manifold with distinct symplectic deformation equivalence classes $[[\omega_i]]$, $i = 1, 2$ such that the symplectic Z invariant, which is defined in terms of the scalar curvatures of almost Kähler metrics in [5], satisfies $Z(M, [[\omega_1]]) = \infty$ and $Z(M, [[\omega_2]]) < 0$.

1. Introduction

Kazdan and Warner classified closed smooth manifolds of dimension > 2 into three classes according to what the scalar curvature functions can be on a manifold [2, Chapter 4].

Recently, we studied an analogous problem on symplectic manifolds with almost Kähler metrics. An almost Kähler metric is a Riemannian metric compatible with a symplectic structure, see the beginning of Section 2. Two symplectic forms ω_0 and ω_1 on M are called *deformation equivalent*, if there exists a diffeomorphism ψ of M such that $\psi^*\omega_1$ and ω_0 can be joined by a smooth homotopy of sympelctic forms, [6]. For a symplectic form ω, its deformation equivalence class shall be denoted by $[[\omega]]$. We denote by $\Omega_{[[\omega]]}$ the set of all almost Kähler metrics compatible with a symplectic form in $[[\omega]]$.

We recall the symplectic Z invariant from [5]. For a smooth closed manifold M of dimension $2n \geq 4$ which admits a symplectic structure, we defined

$$Z(M, [[\omega]]) = \sup_{g \in \Omega_{[[\omega]]}} \frac{\int_M s_g d\mathrm{vol}_g}{(\mathrm{Vol}_g)^{\frac{n-1}{n}}},$$

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where dvol_g, s_g, Vol_g are the volume form, the scalar curvature and the volume of g respectively, and also defined

$$Z(M) = \sup_{[[\omega]]} Z(M, [[\omega]]).$$

Then we have a basic inequality;

$$Z(M, [[\omega]]) \leq \sup_{\omega \in [[\omega]]} \frac{4\pi c_1(\omega) \cdot \frac{[\omega]^{n-1}}{(n-1)!}}{\left(\frac{[\omega]^n}{n!}\right)^{\frac{n-1}{n}}}, \tag{1}$$

where $c_1(\omega)$ is the first Chern class of ω.

With Z invariants we have posed the following question;

Question 1.1. Let M be a smooth closed manifold of dimension $2n \geq 4$ admitting a symplectic structure.

Is the (necessary and sufficient) condition for a smooth function f on M to be the scalar curvature of some smooth almost-Kähler metric on M as follows?

(a) f is arbitrary, if $0 < Z(M) \leq \infty$,

(b) f is identically zero or somewhere negative, if $Z(M) = 0$ and M admits a scalar-flat almost-Kähler metric,

(c) f is negative somewhere, if otherwise.

Also, is the condition for a smooth function f on M to be the scalar curvature of some smooth almost-Kähler metric in $\Omega_{[[\omega]]}$ as follows?

(a$'$) f is arbitrary, if $0 < Z(M, [[\omega]]) \leq \infty$,

(b$'$) f is identically zero or somewhere negative, if $Z(M, [[\omega]]) = 0$ and M admits a scalar-flat almost-Kähler metric in $\Omega_{[[\omega]]}$,

(c$'$) f is negative somewhere, if otherwise.

This question in turn supplies a motivation to study Z invariants. In previous work [5], we presented a six dimensional non-simply connected closed manifold which admits two symplectic deformation classes $[[\omega_i]]$, $i = 1, 2$, such that their Z values have distinct signs.

The main result in this article is to present a simply connected manifold with two symplectic deformation equivalence classes with similar properties.

2. Catanese-LeBrun example

An *almost-Kähler* metric on a smooth manifold M^{2n} of real dimension $2n$ is a Riemannian metric g compatible with a symplectic structure ω, i.e., $\omega(X, Y) = g(X, JY)$ for an almost complex structure J, where X, Y are tangent vectors at a point of the manifold; [3]. We call a Riemannian metric g ω-almost Kähler if g is compatible with ω. An almost-Kähler metric (g, ω, J) is Kähler if and only if J is integrable. We shall prove the following:

Theorem 2.1. *There exists a smooth closed simply connected 8-dimensional manifold N with symplectic deformation equivalence classes $[[\omega_i]]$, $i = 1, 2$ such that $Z(N, [[\omega_1]]) = \infty$ and $Z(N, [[\omega_2]]) < 0$.*

The manifold N in the theorem will be the one studied by Catanese and LeBrun [4]. In fact, N is (diffeomorphic to) the product of two copies of a complex surface of general type with ample canonical line bundle which is homeomorphic to R_8, the blow up of the complex projective plane $\mathbb{CP}_2$ at 8 points in general position. This general type complex surface is obtained as a small deformation of Barlow's explicit complex surfaces [1].

In their work, they showed that N admits two distinct holomorpohic deformation classes. But it was not seen whether N admits two distinct symplectic deformation classes. Examples of smooth manifolds with more than one symplectic deformation class have been an interesting subject to study; refer to [7], [9] or [10]. To prove this theorem, we need the following:

Proposition 2.2. *Let W be a complex surface of general type with ample canonical line bundle, homeomorphic to R_8, the blow up of $\mathbb{CP}_2$ at eight points in general position. Consider a Kähler Einstein metric of negative scalar curvature on W with Kähler form ω_W on W. Set $N := W \times W$.*

Then $Z(N, [[\omega_W + \omega_W]]) = -8\sqrt{2}\pi$, and it is attained by a Kähler Einstein metric.

Proof. The argument here follows the scheme in [5, Section 3]. We recall a few known facts about W from [9, Section 4]; there is a homeomorphism of W onto R_8 which preserves the Chern class c_1 and there is a diffeomorphism of $W \times W$ onto $R_8 \times R_8$. Note that R_8 admits a Kähler Einstein metric of positive scalar curvature obtained by Calabi-Yau solution.

Then, the first Chern class of W can be written as $c_1(W) = 3E_0 - \sum_{i=1}^{8} E_i \in H^2(W, \mathbb{R}) \cong \mathbb{R}^9$, where E_i, $i = 0, \ldots, 8$, is the Poincare dual of a homology class $\tilde{E}_i$, $i = 0, \ldots, 8$ so that $\tilde{E}_i$, $i = 0, \ldots, 8$, form a basis of $H_2(W, \mathbb{Z}) \cong \mathbb{Z}^9$ and their intersections satisfy $\tilde{E}_i \cdot \tilde{E}_j = \epsilon_i \delta_{ij}$, where $\epsilon_0 = 1$ and $\epsilon_i = -1$ for $i \geq 1$. So, in this basis the intersection form becomes

$$I = \begin{bmatrix} 1 & 0 & \cdot & \cdot & 0 \\ 0 & -1 & \cdot & \cdot & 0 \\ . & . & \cdot & \cdot & 0 \\ . & . & \cdot & \cdot & 0 \\ 0 & 0 & & 0 & -1 \end{bmatrix}.$$

We have the orientation of W induced by the complex structure and the fundamental class $[W] \in H_4(W, \mathbb{Z}) \cong \mathbb{Z}$. As ω_W is Kähler Einstein of negative scalar curvature, we may get $[\omega_W] = -3E_0 + \sum_{i=1}^{8} E_i$ by scaling if necessary.

With $N = W \times W$, by Künneth theorem

$$H^2(N, \mathbb{R}) \cong \pi_1^* H^2(W) \oplus \pi_2^* H^2(W) \cong \mathbb{R}^9 \oplus \mathbb{R}^9,$$

where π_{i} are the projection of N onto the i-th factor. Then,

$$c_1(N) = \pi_1^* c_1(W) + \pi_2^* c_1(W) = \pi_1^*(3E_0 - \sum_{i=1}^{8} E_i) + \pi_2^*(3E_0 - \sum_{i=1}^{8} E_i).$$

Consider any smooth path of symplectic forms ω_t, $0 \le t \le \delta$, on N such that $\omega_0 = \omega_W + \omega_W$. We may write

$$[\omega_t] = \sum_{i=0}^{8} \{n_i(t)\pi_1^* E_i + l_i(t)\pi_2^* E_i\} \in H^2(N, \mathbb{R})$$

for some smooth functions $n_i(t)$, $l_i(t)$, $i = 0, \dots, 8$. As they are connected, their first Chern class $c_1(\omega_t) = c_1(N)$. Using the intersection form we compute;

$$\begin{aligned} (2) \qquad [\omega_t]^4([W \times W]) &= [\sum_{i=0}^{8} \{n_i(t)\pi_1^* E_i + l_i(t)\pi_2^* E_i\}]^4([W \times W]) \\ &= 6\{n_0^2(t) - \sum_{i=1}^{8} n_i^2(t)\}\{l_0^2(t) - \sum_{i=1}^{8} l_i^2(t)\} > 0. \end{aligned}$$

As $n_0(0) = -3$ and $n_i(0) = 1$, $i = 1, \dots, 8$, so $n_0^2(t) > \sum_{i=1}^{8} n_i^2(t)$. We get $n_0(t) < 0$. Similarly we also have $l_0(0) = -3$, $l_i(0) = 1$, $i = 1, \dots, 8$, $l_0^2(t) > \sum_{i=1}^{8} l_i^2(t)$ and $l_0(t) < 0$.

$$\begin{aligned} c_1 \cdot [\omega_t]^3([W \times W]) = 3\{l_0^2(t) - \sum_{i=1}^{8} l_i^2(t)\}\{3n_0(t) + \sum_{i=1}^{8} n_i(t)\} \\ + 3\{n_0^2(t) - \sum_{i=1}^{8} n_i^2(t)\}\{3l_0(t) + \sum_{i=1}^{8} l_i(t)\}. \end{aligned}$$

Since $n_0^2(t) > \sum_{i=1}^{8} n_i^2(t)$ and $|\sum_{i=1}^{8} n_i(t)| \le \sqrt{8}\sqrt{\sum_{i=1}^{8} n_i^2(t)}$, we get

$$\begin{aligned} (3) \qquad 3n_0(t) + \sum_{i=1}^{8} n_i(t) &\le 3n_0(t) + 2\sqrt{2}\sqrt{\sum_{i=1}^{8} n_i^2(t)} \\ &< 3n_0(t) + 2\sqrt{2}\sqrt{n_0^2(t)} = (3 - 2\sqrt{2})n_0(t) < 0. \end{aligned}$$

So, $c_1 \cdot [\omega_t]^3([W \times W]) < 0$. Set $A_n = n_0^2(t) - \sum_{i=1}^{8} n_i^2(t)$, $A_l = l_0^2(t) - \sum_{i=1}^{8} l_i^2(t)$, $B_n = 3n_0(t) + \sum_{i=1}^{8} n_i(t)$ and $B_l = 3l_0(t) + \sum_{i=1}^{8} l_i(t)$. From above, $A_n, A_l > 0$ and $B_n, B_l < 0$. By the inequality of arithmetic and geometric means we have

$$\begin{aligned} \frac{c_1 \cdot [\omega_t]^3}{[\omega_t^4]^{3/4}} &= \frac{3}{6^{3/4}}\{\frac{A_n B_l + A_l B_n}{A_n^{3/4} A_l^{3/4}}\} = \frac{3}{6^{3/4}}\{(\frac{A_n}{A_l})^{\frac{1}{4}} \frac{B_l}{\sqrt{A_l}} + (\frac{A_n}{A_l})^{\frac{-1}{4}} \frac{B_n}{\sqrt{A_n}}\} \\ &\le -6^{1/4}\sqrt{\frac{B_l B_n}{\sqrt{A_l A_n}}}. \end{aligned}$$

From (3),

$$\frac{B_n^2}{A_n} \geq \frac{\{3n_0(t) + 2\sqrt{2}\sqrt{\sum_{i=1}^8 n_i^2(t)}\}^2}{n_0^2(t) - \sum_{i=1}^8 n_i^2(t)} = \frac{(3-2\sqrt{2}\sqrt{y})^2}{1-y}$$

where $y = \sum_{i=1}^8 \frac{n_i^2(t)}{n_0^2(t)}$. By calculus, $\frac{(3-2\sqrt{2}\sqrt{y})^2}{1-y} \geq 1$ for $y \in [0,1)$ with equality at $y = \frac{8}{9}$. So, we get $\frac{B_n^2}{A_n} \geq 1$ and similarly $\frac{B_l^2}{A_l} \geq 1$.

We have $\frac{c_1[\omega_t]^3}{[\omega_t^4]^{3/4}} \leq -6^{\frac{1}{4}}$; the equality is achieved exactly when $n_0(t) = -3$, $n_i(t) = 1$, $i = 1, \ldots, 8$ modulo scaling, i.e., when $[\omega_t]$ is a positive multiple of $-c_1(N)$. The Kähler form of a product Kähler Einstein metric of negative scalar curvature on $N = W \times W$ belongs to this class.

As the expression $\frac{4\pi c_1(\omega)\cdot\frac{[\omega]^{n-1}}{(n-1)!}}{(\frac{[\omega]^n}{n!})^{\frac{n-1}{n}}}$ is invariant under a change $\omega \mapsto \phi^*(\omega)$ by any diffeomorphism ϕ, so from (1) the above inequality gives

$$Z(N, [[\omega_0]]) \leq \sup_{\omega\in[[\omega_0]]} \frac{4\pi}{6} \cdot 24^{3/4} \frac{c_1 \cdot [\omega]^3}{[\omega^4]^{3/4}} \leq -8\sqrt{2}\pi.$$

As the equality is attained by a Kähler Einstein metric, $Z(N, [[\omega_0]]) = -8\sqrt{2}\pi$. □

Proof of Theorem 2.1. Consider the positive Kähler Einstein metric on R_8 and let ω_1 be the Kähler form of the product positive Kähler Einstein metric on $R_8 \times R_8$, which is diffeomorphic to N. We have $Z(N, [[\omega_1]]) = \infty$ (scaling by different constants on each factor gives ∞). And let ω_2 be $\omega_W + \omega_W$. Then $Z(N, [[\omega_2]]) < 0$ from Proposition 2.2. From the fact that these values are different, we conclude that $[[\omega_1]]$ and $[[\omega_2]]$ are distinct symplectic deformation equivalence classes. This proves Theorem 2.1. □

In contrast to $Z(N, [[\omega_2]]) < 0$, for dimension $n \geq 5$ there are no examples known to have negative Yamabe invariant and Petean proved that the Yamabe invariant of any simply connected smooth closed manifold is nonnegative; [8]. Of course the Yamabe invariant $Y(N)$ is positive.

Remark 2.3. We get $Z(N, [[\omega_1]]) = \infty$, $Z(N, [[\omega_2]]) < 0$ and $Z(N) = \infty$ from Theorem 2.1. As led by Question 1.1, we therefore expect for N that a smooth function is the scalar curvature of some almost-Kähler metrics in $[[\omega_2]]$ if and only if it is somewhere negative, and that any smooth function is the scalar curvature of some almost-Kähler metrics.

In fact, we may need certain surjectivity of the derivative of a scalar curvature map at the Kähler negative Einstein metric as well as the Kähler positive Einstein metric. This kind of argument is already outlined in [5, Section 4].

Question 2.4. Does there exist a *simply connected* closed 6-dimensional smooth manifold with two symplectic deformation classes with distinct signs of $Z(\cdot, [[\omega]])$?

Question 2.5. Does there exist a closed 4-dimensional smooth manifold with two symplectic deformation classes $[[\omega_i]]$, $i = 1, 2$ such that $Z(\cdot, [[\omega_1]]) > 0$ (or $Z(\cdot, [[\omega_1]]) = 0$) and $Z(\cdot, [[\omega_2]]) < 0$?

Using further products, one may obtain, for each $n \geq 3$, examples of closed symplectic $2n$-dimensional manifolds admitting two symplectic deformation equivalence classes with distinct signs of $Z(\ , [[\cdot]])$ invariants.

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Department of Mathematics
Sogang University
Seoul 121-742, Korea
E-mail address: jskim@sogang.ac.kr