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CONTINUITY OF BANACH ALGEBRA VALUED FUNCTIONS

Jittisak Rakbud

Abstract. Let K be a compact Hausdorff space, $\mathscr{A}$ a commutative complex Banach algebra with identity and $\mathscr{C}(\mathscr{A})$ the set of characters of $\mathscr{A}$. In this note, we study the class of functions $f : K \to \mathscr{A}$ such that $\Omega_{\mathscr{A}} \circ f$ is continuous, where $\Omega_{\mathscr{A}}$ denotes the Gelfand representation of $\mathscr{A}$. The inclusion relations between this class, the class of continuous functions, the class of bounded functions and the class of weakly continuous functions relative to the weak topology $\sigma(\mathscr{A}, \mathscr{C}(\mathscr{A}))$, are discussed. We also provide some results on its completeness under the norm defined by $|||f||| = \|\Omega_{\mathscr{A}} \circ f\|_\infty$.

1. Introduction and preliminaries

A complex algebra $\mathscr{A}$ is called a *normed algebra* if $\mathscr{A}$ is in addition a normed space together with the property that $\|xy\| \leq \|x\| \|y\|$ for all $x, y \in \mathscr{A}$. A normed algebra is called a *normed algebra with identity* or a *unital normed algebra* if it has the identity of norm 1. A normed algebra becomes a *Banach algebra* if the norm is a complete norm. A class of Banach algebras which plays a key role in mathematical analysis is the class of Banach algebras of continuous complex-valued functions on compact Hausdorff spaces. We will generally discuss those algebras as follows. Let K be a compact Hausdorff space and $\mathcal{X}$ a Banach space with the dual $\mathcal{X}^*$. Let $C(K, \mathcal{X})$ be the set of continuous functions from K into $\mathcal{X}$. For the case where $\mathcal{X} = \mathbb{C}$, the set $C(K, \mathbb{C})$ will be denoted by just $C(K)$. By the compactness of K, we have for any $f \in C(K, \mathcal{X})$ that $\|f\|_\infty = \sup_{t \in K} \|f(t)\| < \infty$. It is well known that $C(K, \mathcal{X})$ equipped with the norm $\|\cdot\|_\infty$ is a Banach space (see [3], Example 1.7.2, page 49). If, in addition, $\mathcal{X}$ is a Banach algebra, then so is $C(K, \mathcal{X})$ under the usual multiplication. For any subset $\mathscr{F}$ of the dual $\mathcal{X}^*$ of $\mathcal{X}$ separating points of $\mathcal{X}$ in the sense that for each non-zero element $x \in \mathcal{X}$, there is a $\rho \in \mathscr{F}$ such that $\rho(x) \neq 0$, let $\sigma(\mathcal{X}, \mathscr{F})$ be the weak topology on $\mathcal{X}$ induced by $\mathscr{F}$, and let $C^{\mathscr{F}}(K, \mathcal{X})$ be the set of continuous functions from K into $(\mathcal{X}, \sigma(\mathcal{X}, \mathscr{F}))$, which

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are called *weakly continuous functions relative to the weak topology* $\sigma(\mathcal{X}, \mathscr{F})$. It is well known that (see [3], Proposition 1.3.2, page 29) $C^{\mathscr{F}}(K, \mathcal{X})$ is precisely the set of functions $f : K \to \mathcal{X}$ such that $\varphi f \in C(K)$ for all $\varphi \in \mathscr{F}$. Thus $C(K, \mathcal{X}) \subseteq C^{\mathscr{F}}(K, \mathcal{X})$. For the case of the weak topology on $\mathcal{X}$, i.e., $\mathscr{F} = \mathcal{X}^*$, we denote the set of weakly continuous functions from K into $\mathcal{X}$ by $C^w(K, \mathcal{X})$. By the closed graph theorem, the norm $\|\cdot\|_\infty$ can be defined and is a Banach norm on $C^w(K, \mathcal{X})$. However, $C^w(K, \mathcal{X})$ may not be closed under the usual multiplication when $\mathcal{X}$ is a Banach algebra, except for the case where K is finite.

For any normed algebra $\mathscr{A}$, the set of *characters*, i.e., non-zero multiplicative linear functionals, of $\mathscr{A}$ is denoted by $\mathscr{C}(\mathscr{A})$. A multiplicative linear operator from an algebra $\mathscr{A}$ into an algebra $\mathscr{B}$ is called a *homomorphism* from $\mathscr{A}$ into $\mathscr{B}$. A homomorphism from a normed algebra $\mathscr{A}$ into a normed algebra $\mathscr{B}$ is called an *isomorphism* if it is in addition a homeomorphism from $\mathscr{A}$ onto $\mathscr{B}$. A surjective homomorphism T from a normed algebra $\mathscr{A}$ into a normed algebra $\mathscr{B}$ is called an *isometric isomorphism* if it satisfies the property that $\|Tx\| = \|x\|$ for all $x \in \mathscr{A}$. Obviously, every isometric isomorphism is an isomorphism.

Let $\mathscr{A}$ be a commutative Banach algebra with identity. It is well known that $\mathscr{C}(\mathscr{A}) \neq \emptyset$ (see [1], Theorem 2.35, page 41), and that every character of $\mathscr{A}$ is continuous and has the norm equal to 1 (see [1], Proposition 2.22, page 36). It is also known that $\mathscr{C}(\mathscr{A})$ equipped with the topology relative to the weak* topology on $\mathscr{A}^*$ is a compact Hausdorff space (see [1], Proposition 2.23, page 36). For each $x \in \mathscr{A}$, the *spectrum* of x denoted by $\sigma(x)$ is the set of complex numbers λ such that $x - \lambda$ is not invertible. It is well known that $\sigma(x)$ is a nonempty compact subset of $\mathbb{C}$ (see [1], Proposition 2.28 and Theorem 2.29, page 38), and that $\sigma(x) = \{\rho(x) : \rho \in \mathscr{C}(\mathscr{A})\}$ for all $x \in \mathscr{A}$ (see [1], Corollary 2.36, page 41). Thus, for each $x \in \mathscr{A}$, the real number $r(x) := \sup_{\lambda \in \sigma(x)} |\lambda|$, which is called the *spectral radius* of x, is well-defined. The *Gelfand transform* of each $x \in \mathscr{A}$ is the continuous complex-valued function $\widehat{x}$ on $\mathscr{C}(\mathscr{A})$, under the relative weak* topology, which is defined by $\varphi \mapsto \varphi(x)$. It is evident that for each $x \in \mathscr{A}$, $r(x) = \|\widehat{x}\|_\infty \leq \|x\|$. The *Gelfand representation* of $\mathscr{A}$ denoted by $\Omega_{\mathscr{A}}$ is the bounded homomorphism from $\mathscr{A}$ into $C(\mathscr{C}(\mathscr{A}))$ defined by $x \mapsto \widehat{x}$. Notice that if the Gelfand representation of $\mathscr{A}$ is injective, then $\mathscr{C}(\mathscr{A})$ separates points of $\mathscr{A}$. Whence, in this situation, we obtain that every continuous function $f : K \to \mathscr{A}$, where K is a compact Hausdorff space, is a weakly continuous function relative to the weak topology $\sigma(\mathscr{A}, \mathscr{C}(\mathscr{A}))$, that is, $C(K, \mathscr{A}) \subseteq C^{\mathscr{C}(\mathscr{A})}(K, \mathscr{A})$. By the bounded inverse theorem, we have that the Gelfand representation of a commutative Banach algebra with identity is an isomorphism if and only if it is bijective. The following is a sufficient condition for the Gelfand representation of a commutative Banach algebra with identity to be an isomorphism. It was provided in [2] by W. Fupinwong and S. Dhompongsa as a preliminary.

Theorem 1.1 ([2], Lemma 4.1, page 12)**.** *For any commutative Banach algebra $\mathscr{B}$ with identity, if* $\inf\{\|\widehat{x}\|_\infty : x \in \mathscr{B}, \|x\| = 1\} > 0$ *and $\mathscr{B}$ satisfies the following property*:

($\star$) *for each $x \in \mathscr{B}$, there exists a $y \in \mathscr{B}$ such that $\varphi(x) = \overline{\varphi(y)}$ for all $\varphi \in \mathscr{C}(\mathscr{B})$,*

then the Gelfand representation of $\mathscr{B}$ is an isomorphism.

For any commutative Banach algebra $\mathscr{B}$ with identity, it is clear that the condition that $\inf\{\|\widehat{x}\|_\infty : x \in \mathscr{B}, \|x\| = 1\} > 0$ implies the injectivity of the Gelfand representation of $\mathscr{B}$, which is equivalent to the semi-simplicity of $\mathscr{B}$. The converse of this statement is not true (see Remark 2.12). It is also clear that the condition ($\star$) is equivalent to the closedness of the subalgebra $\Omega_{\mathscr{B}}(\mathscr{B})$ of $C(\mathscr{C}(\mathscr{B}))$ under the complex conjugation. Notice that every commutative C^*-algebra with identity satisfies the two conditions of the above theorem. Moreover, its Gelfand representation is an isometric $*$-isomorphism (see [3], Theorem 4.4.3, page 270).

For any compact Hausdorff spaces X and Y, it is well known that $C(X, C(Y))$ is isometrically isomorphic to $C(Y, C(X))$ by the isomorphism defined by $f \mapsto \widetilde{f}$, where $\widetilde{f}$ is a function from Y into $C(X)$ such that $\widetilde{f}(y)$ is given by $x \mapsto f(y)(x)$ (see for more details [4], page 849). From this fact, it is not hard to see that for each function f from a compact Hausdorff space K into a commutative Banach algebra $\mathscr{A}$ with identity, the following two conditions are equivalent:

(C_1) $\Omega_{\mathscr{A}} \circ f$ is continuous.

(C_1') $\varphi \circ f$ is continuous for all $\varphi \in \mathscr{C}(\mathscr{A})$ and the function $\Psi_f^{(\mathscr{A},K)}$ from $\mathscr{C}(\mathscr{A})$, along with the topology relative to the weak* topology on $\mathscr{A}^*$, into $C(K)$ defined by $\Psi_f^{(\mathscr{A},K)}(\varphi) = \varphi \circ f$ for all $\varphi \in \mathscr{C}(\mathscr{A})$ is continuous.

And in these two situations, we have $\|\Omega_{\mathscr{A}} \circ f\|_\infty = \left\|\Psi_f^{(\mathscr{A},K)}\right\|_\infty$.

In this note, we deal mainly with the classes of continuous functions and functions satisfying the condition (C_1), from a compact Hausdorff space into a commutative Banach algebra with identity.

2. Results

In the entire contents of this section, let K and $\mathscr{A}$ be respectively a compact Hausdorff space and a commutative Banach algebra with identity which are arbitrarily fixed. In addition, the set $\mathscr{C}(\mathscr{A})$ of characters of $\mathscr{A}$ will be considered as a topological space equipped with the topology relative to the weak* topology on $\mathscr{A}^*$. Recall that $\Omega_{\mathscr{A}}$ denotes the Gelfand representation of $\mathscr{A}$.

Let $C_1(K, \mathscr{A})$ and $C_1^b(K, \mathscr{A})$ be the sets of functions and bounded functions from K into $\mathscr{A}$ respectively which satisfy the condition (C_1). The inclusion relations among the three sets $C(K, \mathscr{A})$, $C_1^b(K, \mathscr{A})$ and $C_1(K, \mathscr{A})$ are as follows.

Theorem 2.1. $C(K,\mathscr{A}) \subseteq C_1^b(K,\mathscr{A}) \subseteq C_1(K,\mathscr{A})$.

Proof. Since K is compact, it follows easily that $C(K,\mathscr{A}) \subseteq C_1^b(K,\mathscr{A})$. □

Note that by the equivalence of the two conditions (C_1) and (C_1') mentioned above, we obtain in addition for the case where the Gelfand representation of $\mathscr{A}$ is injective that $C_1(K,\mathscr{A}) \subseteq C^{\mathscr{C}(\mathscr{A})}(K,\mathscr{A})$. It is clear that a sufficient condition which implies $C(K,\mathscr{A}) = C_1^b(K,\mathscr{A}) = C_1(K,\mathscr{A})$ is that the Gelfand representation of $\mathscr{A}$ is an isomorphism. We will see later that in this situation the inclusion $C_1(K,\mathscr{A}) \subseteq C^{\mathscr{C}(\mathscr{A})}(K,\mathscr{A})$ can still be proper.

Theorem 2.2. *If the Gelfand representation of $\mathscr{A}$ is an isomorphism, then $C(K,\mathscr{A}) = C_1^b(K,\mathscr{A}) = C_1(K,\mathscr{A})$.*

Another condition which also implies the three sets $C(K,\mathscr{A})$, $C_1^b(K,\mathscr{A})$ and $C_1(K,\mathscr{A})$ to be equal is that $\inf\{\|\widehat{x}\|_\infty : x \in \mathscr{A}, \|x\| = 1\} > 0$. To prove this, we need the following lemma.

Lemma 2.3. *Suppose that $\inf\{\|\widehat{x}\|_\infty : x \in \mathscr{A}, \|x\| = 1\} > 0$ and $0 < \delta < \inf\{\|\widehat{x}\|_\infty : x \in \mathscr{A}, \|x\| = 1\}$. Then for any $a \in \mathscr{A}$, $\|\widehat{a}\|_\infty < \delta^2$ implies $\|a\| < \delta$.*

Proof. Suppose to the contrary that $\|\widehat{a}\|_\infty < \delta^2$, but $\|a\| \geq \delta$. Then $\frac{1}{\|a\|} \leq \frac{1}{\delta}$. Thus

$$0 < \delta < \inf\{\|\widehat{x}\|_\infty : x \in \mathscr{A}, \|x\| = 1\} \leq \left\| \widehat{\left(\frac{a}{\|a\|}\right)} \right\|_\infty < \delta,$$

which is a contradiction. So we obtain that $\|\widehat{a}\|_\infty < \delta^2$ implies $\|a\| < \delta$ as required. □

Theorem 2.4. *If $\inf\{\|\widehat{x}\|_\infty : x \in \mathscr{A}, \|x\| = 1\} > 0$, then $C(K,\mathscr{A}) = C_1^b(K,\mathscr{A}) = C_1(K,\mathscr{A})$.*

Proof. Suppose that $\inf\{\|\widehat{x}\|_\infty : x \in \mathscr{A}, \|x\| = 1\} > 0$. Since $C(K,\mathscr{A}) \subseteq C_1^b(K,\mathscr{A}) \subseteq C_1(K,\mathscr{A})$, it suffices to show only that $C(K,\mathscr{A}) = C_1(K,\mathscr{A})$. Let $f \in C_1(K,\mathscr{A})$. To get that $f \in C(K,\mathscr{A})$, let $s \in K$ and $\epsilon > 0$. Then by the continuity of $\Omega_{\mathscr{A}} \circ f$, there is an open neighborhood V of s such that

$$\left\| \widehat{f(s)} - \widehat{f(t)} \right\|_\infty < \beta^2 \text{ for all } t \in V,$$

where $\beta = \dfrac{\min\{\epsilon, \inf\{\|\widehat{x}\|_\infty : x \in \mathscr{A}, \|x\| = 1\}\}}{2}$. Thus, by Lemma 2.3,

$$\|f(s) - f(t)\| < \beta \leq \frac{\epsilon}{2} < \epsilon \text{ for all } t \in V.$$

This yields the continuity of f. □

The following example shows that there are a commutative Banach algebra $\mathscr{B}$ with identity and a compact Hausdorff space E such that the Gelfand representation of $\mathscr{B}$ is injective but not surjective, and the inclusions $C(E,\mathscr{B}) \subseteq C_1^b(E,\mathscr{B}) \subseteq C_1(E,\mathscr{B}) \subseteq C^{\mathscr{C}(\mathscr{B})}(E,\mathscr{B})$ are all proper.

Example 2.5. Consider the unitization $(l^2)_e$ of the Hilbert space l^2, i.e., $(l^2)_e := \mathbb{C} \oplus l^2$ along with the norm $\|(\lambda, \{x_k\}_{k=1}^\infty)\| := |\lambda| + \|\{x_k\}_{k=1}^\infty\|_2$ and the multiplication defined by $(\lambda, \{x_k\}_{k=1}^\infty)(\gamma, \{y_k\}_{k=1}^\infty) = (\lambda\gamma, \lambda\{y_k\}_{k=1}^\infty + \gamma\{x_k\}_{k=1}^\infty + \{x_k y_k\}_{k=1}^\infty)$ for all $(\lambda, \{x_k\}_{k=1}^\infty)$, $(\gamma, \{y_k\}_{k=1}^\infty) \in \mathbb{C} \oplus l^2$. Here, for any $\lambda \in \mathbb{C}$ and $\{x_k\}_{k=1}^\infty \in l^2$, we write $(\lambda, 0) = \lambda$ and $(0, \{x_k\}_{k=1}^\infty) = \{x_k\}_{k=1}^\infty$. Hence every $(\lambda, \{x_k\}_{k=1}^\infty) \in \mathbb{C} \oplus l^2 = (l^2)_e$ can be written in a form of addition as follows: $(\lambda, \{x_k\}_{k=1}^\infty) = \lambda + \{x_k\}_{k=1}^\infty$. For each integer $n \geq 1$, let $\varphi_n : (l^2)_e \to \mathbb{C}$ be defined by $\varphi_n(\lambda + \{x_k\}_{k=1}^\infty) = \lambda + x_n$ for all $\lambda + \{x_k\}_{k=1}^\infty \in (l^2)_e$, and let $\varphi_0 : (l^2)_e \to \mathbb{C}$ be defined by $\varphi_0(\lambda + \{x_k\}_{k=1}^\infty) = \lambda$ for all $\lambda + \{x_k\}_{k=1}^\infty \in (l^2)_e$. Then $\varphi_n \in \mathscr{C}\left((l^2)_e\right)$ for all $n \geq 0$. First of all, we need that $\mathscr{C}\left((l^2)_e\right) = \{\varphi_n : n = 0, 1, 2, \ldots\}$, and that the Gelfand representation of $(l^2)_e$ is injective but not surjective. We do not know whether these results are well known. For completeness and self-containedness of the contents in this example, we will prove them again. Let $\varphi \in \mathscr{C}\left((l^2)_e\right)$. If $\varphi = \varphi_0$, then we are done. Suppose that $\varphi \neq \varphi_0$. Let $\{e_n\}_{n=1}^\infty$ be the standard orthonormal basis for l^2, and let $\xi_n = \varphi(e_n)$ for all n. Since, for each n, we have $\xi_n = \varphi(e_n) = \varphi(e_n^2) = \varphi(e_n)^2 = (\xi_n)^2$, it follows that ξ_n is either 0 or 1. Furthermore, if $n \neq k$, then $\xi_n + \xi_k$ is either 0 or 1 as well, due to the fact that $\xi_n + \xi_k = \varphi(e_n + e_k) = \varphi((e_n + e_k)^2) = \varphi(e_n + e_k)^2 = (\xi_n + \xi_k)^2$. From these, we obtain that if $\xi_n \neq 0$ for some n, then ξ_k must be 0 for all $k \neq n$. Since $\varphi \neq \varphi_0$, we have that the restriction of φ on l^2 is not zero, which implies that there is a sequence $\{y_n\}_{n=1}^\infty \in l^2$ such that

$$\sum_{n=1}^\infty y_n \xi_n = \varphi\left(\sum_{n=1}^\infty y_n e_n\right) = \varphi(\{y_n\}_{n=1}^\infty) \neq 0.$$

Thus there exists a positive integer n such that $\xi_n \neq 0$. This yields that $\xi_n = 1$ and $\xi_k = 0$ for all $k \neq n$. Hence, for any $\lambda + \{x_k\}_{k=1}^\infty \in (l^2)_e$, we have

$$\begin{aligned}\varphi(\lambda + \{x_k\}_{k=1}^\infty) &= \lambda + \varphi(\{x_k\}_{k=1}^\infty) = \lambda + \varphi\left(\sum_{k=1}^\infty e_k x_k\right) \\ &= \lambda + \sum_{k=1}^\infty x_k \xi_k = \lambda + x_n \\ &= \varphi_n(\lambda + \{x_k\}_{k=1}^\infty).\end{aligned}$$

That is, $\varphi = \varphi_n$. Therefore, we obtain $\mathscr{C}\left((l^2)_e\right) = \{\varphi_n : n = 0, 1, 2, \ldots\}$ as required. Notice that φ_0 is the only limit point of $\mathscr{C}\left((l^2)_e\right)$. Indeed, for each $n \geq 1$, the set

$$V_n = \left\{\varphi \in \mathscr{C}\left((l^2)_e\right) : |\varphi(e_n) - \varphi_n(e_n)| < 1\right\}$$

is an open neighborhood of φ_n in $\mathscr{C}\left((l^2)_e\right)$ which doesn't contain all other elements of $\mathscr{C}\left((l^2)_e\right)$, and for any $\Theta = \lambda + \{x_k\}_{k=1}^\infty \in (l^2)_e$, $|\varphi_n(\Theta) - \varphi_0(\Theta)| = |x_n| \to 0$. Next, we will show that the Gelfand representation of $(l^2)_e$ is injective but not surjective. To see that the Gelfand representation of $(l^2)_e$ is injective, let $\Theta = \lambda + \{x_k\}_{k=1}^{\infty} \in (l^2)_e$, and suppose that $\widehat{\Theta} = 0$. Then $\varphi_n(\Theta) = 0$ for all $n \geq 0$. Since $\varphi_0(\Theta) = 0$, we have $\lambda = 0$. Thus $\Theta = \{x_k\}_{k=1}^{\infty}$. Since for every $n \geq 1$, $\varphi_n(\{x_k\}_{k=1}^{\infty}) = \varphi_n(\Theta) = 0$, it follows that $x_n = 0$ for all n. Thus $\Theta = 0$. To get that the Gelfand representation of $(l^2)_e$ is not surjective, let $f : \mathscr{C}\left((l^2)_e\right) \to \mathbb{C}$ be defined by $f(\varphi_n) = \frac{1}{\sqrt{n}}$ for all $n \geq 1$ and $f(\varphi_0) = 0$. We claim that f is continuous and $f \neq \widehat{\Theta}$ for all $\Theta \in (l^2)_e$. By the note above, it is clear that f is continuous at φ_n for all $n \geq 1$. To see that f is continuous at φ_0, let $\epsilon > 0$ be given. Then there is a positive integer N such that $\frac{1}{n} < \epsilon^2$ for all $n \geq N$. Let $U = \{\varphi_n : n \geq N\} \cup \{\varphi_0\}$. Then, by the fact that $\mathscr{C}\left((l^2)_e\right)$ is a Hausdorff space, we get that U is an open neighborhood of φ_0 in $\mathscr{C}\left((l^2)_e\right)$. Since $|f(\varphi_n) - f(\varphi_0)| = |f(\varphi_n)| = \frac{1}{\sqrt{n}} < \epsilon$ for all $n \geq N$, it follows that f is continuous at φ_0. Hence f is continuous. If $\Theta = \lambda + \{x_k\}_{k=1}^{\infty} \in (l^2)_e$ and $\widehat{\Theta} = f$, then $\lambda = \varphi_0(\Theta) = \widehat{\Theta}(\varphi_0) = f(\varphi_0) = 0$ and $x_n = \varphi_n(\Theta) = \widehat{\Theta}(\varphi_n) = f(\varphi_n) = \frac{1}{\sqrt{n}}$ for all $n \geq 1$. It follows that $\Theta = \left\{\frac{1}{\sqrt{k}}\right\}_{k=1}^{\infty}$, which is impossible since $\left\{\frac{1}{\sqrt{k}}\right\}_{k=1}^{\infty}$ doesn't belong to l^2. Hence $\widehat{\Theta} \neq f$ for all $\Theta \in (l^2)_e$. Therefore, the Gelfand representation of $(l^2)_e$ is not surjective.

Let

$$E = \left\{\frac{1}{n} : n = 1, 2, 3, \dots\right\} \cup \{0\}$$

be equipped with the topology relative to the usual topology on the set of real numbers. Then E is a compact Hausdorff space. We now turn our attention to confirm that the inclusions

$$C(E, (l^2)_e) \subseteq C_1^b(E, (l^2)_e) \subseteq C_1(E, (l^2)_e) \subseteq C^{\mathscr{C}((l^2)_e)}(E, (l^2)_e)$$

are all proper.

We will begin with proving that $C(E, (l^2)_e) \subsetneq C_1^b(E, (l^2)_e)$. To show this, let $f : E \to (l^2)_e$ be defined by

$$f(x) = \begin{cases} \Bigg\{\underbrace{\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}}}_{n \text{ terms}}, 0, 0, \dots\Bigg\} & \text{if } x = \frac{1}{n} \text{ for some } n, \\ 0 & \text{if } x = 0. \end{cases}$$

For each $n \geq 1$, we have

$$\left\|f\left(\frac{1}{n}\right) - f(0)\right\| = \left\|f\left(\frac{1}{n}\right)\right\| = \left\|\Bigg\{\underbrace{\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}}}_{n \text{ terms}}, 0, 0, \dots\Bigg\}\right\|_2 = 1.$$

From this, we get that f is bounded, and that $f\left(\frac{1}{n}\right) \nrightarrow f(0)$ in $(l^2)_e$, which yields the discontinuity of f. So $f \notin C(E, (l^2)_e)$. To show that $\Omega_{(l^2)_e} \circ f$ is continuous, from the fact that 0 is the only limit point of E in E itself, it suffices to prove just that $\Omega_{(l^2)_e} \circ f$ is continuous at 0. Let $\epsilon > 0$ be given. Then there is a positive integer N such that $\frac{1}{k} < \frac{\epsilon}{4}^2$ for all $k \geq N$. Let $W = \left\{\frac{1}{k} : k \geq N\right\} \cup \{0\}$. Then W is an open neighborhood of 0 in E. Notice that for every $n \geq 1$, we have that the function $\varphi_n \circ f : E \to \mathbb{C}$ is determined by

$$\varphi_n \circ f\left(\frac{1}{k}\right) = \varphi_n\left(f\left(\frac{1}{k}\right)\right) = \begin{cases} 0 & \text{if } k < n, \\ \dfrac{1}{\sqrt{k}} & \text{if } k \geq n, \end{cases}$$

and $\varphi_0 \circ f = 0$. Let $n \geq 1$ and $k \geq N$. If $n > k$, then

$$\left|\varphi_n\left(f\left(\frac{1}{k}\right)\right)\right| = 0 < \frac{\epsilon}{2}.$$

If $n \leq k$, then

$$\left|\varphi_n\left(f\left(\frac{1}{k}\right)\right)\right| = \frac{1}{\sqrt{k}} < \frac{\epsilon}{2}.$$

It follows that

$$\begin{aligned} \left\|\Omega_{(l^2)_e} \circ f(x) - \Omega_{(l^2)_e} \circ f(0)\right\|_\infty &= \left\|\widehat{f(x)} - \widehat{f(0)}\right\|_\infty = \left\|\widehat{f(x)}\right\|_\infty \\ &= \sup_{n\geq 0} \left|\varphi_n\left(f\left(x\right)\right)\right| \\ &< \epsilon \quad \text{for all } x \in W. \end{aligned}$$

Whence the function $\Omega_{(l^2)_e} \circ f$ is continuous. Previously, it has already been shown that f is bounded. Therefore, $f \in C_1^b((l^2)_e, E)$.

Next, we will show that $C_1^b(E, (l^2)_e) \subsetneq C_1(E, (l^2)_e)$. Let $g : E \to (l^2)_e$ be defined by

$$g(x) = \begin{cases} \left\{\underbrace{\frac{1}{n^{1/3}}, \frac{1}{n^{1/3}}, \ldots, \frac{1}{n^{1/3}}}_{n \text{ terms}}, 0, 0, \ldots\right\} & \text{if } x = \frac{1}{n} \text{ for some } n, \\ 0 & \text{if } x = 0. \end{cases}$$

Then for each $n \geq 1$, we have

$$\left\|g\left(\frac{1}{n}\right)\right\| = \left\|\left\{\underbrace{\frac{1}{n^{1/3}}, \frac{1}{n^{1/3}}, \ldots, \frac{1}{n^{1/3}}}_{n \text{ terms}}, 0, 0, \ldots\right\}\right\|_2 = n^{1/6}.$$

Thus g is unbounded. By an argument similar to that for proving the continuity of $\Omega_{(l^2)_e} \circ f$, the continuity of $\Omega_{(l^2)_e} \circ g$ is obtained. Hence we get $C_1^b(E, (l^2)_e) \subsetneq C_1(E, (l^2)_e)$ as required.

Finally, we will prove that $C_1(E,(l^2)_e) \subsetneq C^{\mathscr{C}((l^2)_e)}(E,(l^2)_e)$. Let $h : E \to (l^2)_e$ be defined by

$$h(x) = \begin{cases} \Big\{\underbrace{0,0,\dots,0}_{n \text{ terms}},1,0,0,\dots\Big\} & \text{if } x = \frac{1}{n} \text{ for some } n, \\ 0 & \text{if } x = 0. \end{cases}$$

Then for each $n \geq 1$, the function $\varphi_n \circ h : E \to \mathbb{C}$ is determined by

$$\varphi_n \circ h\left(\frac{1}{k}\right) = \varphi_n\left(h\left(\frac{1}{k}\right)\right) = \begin{cases} 0 & \text{if } k+1 \neq n, \\ 1 & \text{if } k+1 = n, \end{cases}$$

and $\varphi_0 \circ h = 0$. This yields that $\varphi_n \circ h$ is continuous for all $n \geq 0$. So $h \in C^{\mathscr{C}((l^2)_e)}(E,(l^2)_e)$. It is apparent for every $x \in E$ with $x \neq 0$ that

$$\left\|\Omega_{(l^2)_e} \circ h(x) - \Omega_{(l^2)_e} \circ h(0)\right\|_\infty = \left\|\widehat{h(x)}\right\|_\infty = \sup_{n\geq 0}|\varphi_n(h(x))| = 1.$$

This implies that $\Omega_{(l^2)_e} \circ h$ is not continuous. Consequently, $h \notin C_1(E,(l^2)_e)$.

If the Gelfand representation $\Omega_{\mathscr{A}}$ of $\mathscr{A}$ is an isomorphism, then by Theorem 2.2 and the injectivity of $\Omega_{\mathscr{A}}$, we have that

$$C(K,\mathscr{A}) = C_1^b(K,\mathscr{A}) = C_1(K,\mathscr{A}) \subseteq C^{\mathscr{C}(\mathscr{A})}(K,\mathscr{A}).$$

The following example shows that in this situation the two sets $C_1(K,\mathscr{A})$ and $C^{\mathscr{C}(\mathscr{A})}(K,\mathscr{A})$ may not be equal.

Example 2.6. In this example, we consider the commutative C^*-algebra with identity $C[0,1]$ of continuous complex valued functions on $[0,1]$ and the Alexandroff one-point compactification of $[1,\infty)$ which is denoted by $[1,\infty]$. Let $f : [1,\infty] \to C[0,1]$ be defined by $f(r) = f_r$ for all $r \in [1,\infty]$, where for any $r \in [1,\infty)$, $f_r : [0,1] \to \mathbb{R}$ is defined by $f_r(t) = \frac{rt}{1+r^2t^2}$ for all $t \in [0,1]$, and $f_\infty = 0$. For each $t \in [0,1]$, we have $\delta_t \circ f(r) = \delta_t(f_r) = f_r(t) = \frac{rt}{1+r^2t^2}$ for all $r \in [1,\infty)$, where $\delta_t \in \mathscr{C}(C[0,1])$ which is the point evaluation at $t \in [0,1]$. Since $\lim_{r\to\infty} f_r(t) = 0 = f_\infty(t)$ for all $t \in [0,1]$, it follows that $\delta_t \circ f$ is continuous on $[1,\infty]$ for all $t \in [0,1]$. It is easy to check that $\|f_r\|_\infty = \frac{1}{2}$ for all $r \in [1,\infty)$. From this, we obtain that $\|f_n - f_\infty\|_\infty = \|f_n\|_\infty = \frac{1}{2}$ for all positive integer n. It follows that $\|f_n - f_\infty\|_\infty \nrightarrow 0$. Thus f is not continuous.

It is easy to see that $C_1(K,\mathscr{A})$ is an algebra containing both $C(K,\mathscr{A})$ and $C_1^b(K,\mathscr{A})$ as subalgebras. We next investigate the completeness of these three algebras under the norm $|||\cdot|||$ on $C_1(K,\mathscr{A})$ defined naturally by

$$|||f||| := \|\Omega_{\mathscr{A}} \circ f\|_\infty = \sup_{t\in K}\left\|\widehat{f(t)}\right\|_\infty.$$

Notice that $|||f||| \leq \|f\|_\infty$ for all $f \in C_1^b(K,\mathscr{A})$.

Proposition 2.7. *The vector space $C_1(K,\mathscr{A})$ equipped with the norm $|||\cdot|||$ is a normed space if and only if the Gelfand representation of $\mathscr{A}$ is injective. In this case, the normed space $(C_1(K,\mathscr{A}),|||\cdot|||)$ is furthermore a normed algebra.*

Proof. Suppose that the Gelfand representation of $\mathscr{A}$ is injective. We will show that the function $|||\cdot|||$ is precisely a norm on $C_1(K,\mathscr{A})$. It is clear that $\|\lambda f\| = |\lambda|\,|||f|||$ and $\|f+g\| \le |||f||| + \|g\|$ for all $f,g \in C_1(K,\mathscr{A})$ and $\lambda \in \mathbb{C}$, and that $|||0||| = 0$. We need to show that $|||f||| = 0$ implies $f = 0$ for all $f \in C_1(K,\mathscr{A})$. Let $f \in C_1(K,\mathscr{A})$, and assume that $|||f||| = 0$. Then $\left\|\widehat{f(t)}\right\|_\infty = 0$ for all $t \in K$. This gives us that $\widehat{f(t)} = 0$ for all $t \in K$. So, by the injectivity of the Gelfand representation of $\mathscr{A}$, we have for each $t \in K$ that $f(t) = 0$, which yields $f = 0$. Thus the vector space $C_1(K,\mathscr{A})$ equipped with the norm $|||\cdot|||$ is a normed space. It is obvious that $fg \in C_1(K,\mathscr{A})$ and $|||fg||| \le |||f|||\,|||g|||$ for all $f,g \in C_1(K,\mathscr{A})$. Hence $(C_1(K,\mathscr{A}),|||\cdot|||)$ is in addition a normed algebra. Conversely, suppose that $(C_1(K,\mathscr{A}),|||\cdot|||)$ is a normed space. To prove that the Gelfand representation of $\mathscr{A}$ is injective, let $a \in \mathscr{A}$, and suppose that $\widehat{a} = 0$. Let $f : K \to \mathscr{A}$ be defined by $f(t) = a$ for all $t \in K$. It is clear that f is continuous. Since $|||f||| = \|\widehat{a}\|_\infty = 0$, it follows by the assumption that $f = 0$. Therefore, by the definition of f, we have $a = 0$. □

Lemma 2.8. *If the Gelfand representation of $\mathscr{A}$ is an isomorphism, then the two norms $\|\cdot\|_\infty$ and $|||\cdot|||$ on $C(K,\mathscr{A})$ $(= C_1^b(K,\mathscr{A}) = C_1(K,\mathscr{A}))$ are equivalent.*

Proof. Since the Gelfand representation of $\mathscr{A}$ is an isomorphism, there is a $c > 0$ such that $\|x\| \le c\,\|\widehat{x}\|_\infty$ for all $x \in \mathscr{A}$, and we obtain by Proposition 2.7 that $|||\cdot|||$ is a norm on $C(K,\mathscr{A})$. Hence, for any $f \in C(K,\mathscr{A})$,

$$\|f\|_\infty = \sup_{t\in K}\|f(t)\| \le c\left(\sup_{t\in K}\left\|\widehat{f(t)}\right\|_\infty\right) = c\,|||f|||\,.$$

So, by the fact that $|||f||| \le \|f\|_\infty$ for all $f \in C(K,\mathscr{A})$, we now complete the proof. □

Theorem 2.9. *If $\inf\{\|\widehat{x}\|_\infty : x \in \mathscr{A}, \|x\| = 1\} > 0$ or the Gelfand representation of $\mathscr{A}$ is an isomorphism, then $C(K,\mathscr{A})$ $(= C_1^b(K,\mathscr{A}) = C_1(K,\mathscr{A}))$ endowed with the norm $|||\cdot|||$ is a Banach space.*

Proof. If the Gelfand representation of $\mathscr{A}$ is an isomorphism, then by Proposition 2.7, $(C(K,\mathscr{A}),|||\cdot|||)$ is a normed space. And, by Lemma 2.8 and the completeness of $(C(K,\mathscr{A}),\|\cdot\|_\infty)$, we obtain that $(C(K,\mathscr{A}),|||\cdot|||)$ is a Banach space. Next, suppose that $\inf\{\|\widehat{x}\|_\infty : x \in \mathscr{A}, \|x\| = 1\} > 0$. Then the Gelfand representation of $\mathscr{A}$ is injective. Thus, by Proposition 2.7, $(C(K,\mathscr{A}),|||\cdot|||)$ is a normed space. To see that it is a Banach space, let $\{f_n\}_{n=1}^\infty$ be a Cauchy sequence in $(C(K,\mathscr{A}),|||\cdot|||)$. First, we will prove the following statement: for every $\epsilon > 0$, there is a positive integer N such that $\|f_n(t) - f_m(t)\| < \epsilon$ for all

$n, m \geq N$ and $t \in K$. To see this, let $\epsilon > 0$ be given, and put

$$\beta = \frac{\min\{\epsilon, \inf\{\|\widehat{x}\|_\infty : x \in \mathscr{A}, \|x\| = 1\}\}}{2}.$$

Then there is a positive integer N such that $|||f_n - f_m||| < \beta^2$ for all $n, m \geq N$. Since $0 < \beta < \inf\{\|\widehat{x}\|_\infty : x \in \mathscr{A}, \|x\| = 1\}$, we get for each $t \in K$ and $n, m \geq N$ by Lemma 2.3 that $\|f_n(t) - f_m(t)\| < \beta < \frac{\epsilon}{2} < \epsilon$. The statement is now completely proved. From this result, we have that $\{f_n(t)\}_{n=1}^\infty$ is a Cauchy sequence in $\mathscr{A}$ for all $t \in K$. Hence, by the completeness of $\mathscr{A}$, we obtain for each $t \in K$ that there is an $f(t)$ in $\mathscr{A}$ such that $f_n(t) \to f(t)$. Let $f : K \to \mathscr{A}$ be defined by $t \mapsto f(t)$. We will prove that $f \in C(K, \mathscr{A})$, and that $f_n \to f$ in $(C(K, \mathscr{A}), |||\cdot|||)$. To prove these, let $\epsilon > 0$ be given. Then by the statement provided and proved above again, there is a positive integer N such that for every $t \in K$,

$$(*) \qquad \|f_n(t) - f_m(t)\| < \frac{\epsilon}{6} \text{ for all } n, m \geq N.$$

Since $f_n(t) \to f(t)$ for all $t \in K$, it follows for each $t \in K$ by taking the limits as $m \to \infty$ on both sides of the inequality $(*)$ that

$$(\dagger) \qquad \|f_n(t) - f(t)\| \leq \frac{\epsilon}{6} \text{ for all } n \geq N.$$

To see that $f \in C(K, \mathscr{A})$, let $s \in K$. Since $f_N \in C(K, \mathscr{A})$, there is an open neighborhood V of s such that

$$(\ddagger) \qquad \|f_N(s) - f_N(t)\| < \frac{\epsilon}{6} \text{ for all } t \in V.$$

Thus, by $(\dagger)$ and $(\ddagger)$,

$$\begin{aligned} \|f(s) - f(t)\| &\leq \|f_N(s) - f(s)\| + \|f_N(s) - f_N(t)\| + \|f_N(t) - f(t)\| \\ &< \frac{\epsilon}{6} + \frac{\epsilon}{6} + \frac{\epsilon}{6} < \epsilon \text{ for all } t \in V. \end{aligned}$$

Accordingly, the continuity of f is obtained. By $(\dagger)$ again, we get

$$|||f_n - f||| \leq \|f_n - f\|_\infty \leq \frac{\epsilon}{6} < \epsilon \text{ for all } n \geq N.$$

Therefore, $f_n \to f$ in $(C(K, \mathscr{A}), |||\cdot|||)$. □

Proposition 2.10. *If the Gelfand representation of* $\mathscr{A}$ *is an isomorphism, then the embedding* $f \mapsto \Omega_{\mathscr{A}} \circ f$ *is an isometric isomorphism from the Banach algebra* $(C(K, \mathscr{A}), |||\cdot|||)$ *onto the Banach algebra* $(C(K, C(\mathscr{C}(\mathscr{A}))), \|\cdot\|_\infty)$.

Proof. Let $g \in C(K, C(\mathscr{C}(\mathscr{A})))$. Then by the surjectivity of the Gelfand representation $\Omega_{\mathscr{A}}$ of $\mathscr{A}$, we have that for each $t \in K$, there is an element $h(t) \in \mathscr{A}$ such that $g(t) = \widehat{h(t)}$. Next, we define a function $h : K \to \mathscr{A}$ by $t \mapsto h(t)$. It is clear that $g = \Omega_{\mathscr{A}} \circ h$. Since the Gelfand representation of $\mathscr{A}$ is an isomorphism, the function h is continuous. Hence the map $f \mapsto \Omega_{\mathscr{A}} \circ f$ from $(C(K, \mathscr{A}), |||\cdot|||)$ into $(C(K, C(\mathscr{C}(\mathscr{A}))), \|\cdot\|_\infty)$ is onto. □

Theorem 2.11. *If $\mathscr{A}$ satisfies the property* $(\star)$ *stated in Theorem* 1.1*, then the following are equivalent.*

(1) *The Gelfand representation of $\mathscr{A}$ is an isomorphism.*

(2) *The embedding $f \mapsto \Omega_{\mathscr{A}} \circ f$ is an isometric isomorphism from the normed algebra $(C(K,\mathscr{A}), |||\cdot|||)$ onto the Banach algebra $(C(K, C(\mathscr{C}(\mathscr{A}))), \|\cdot\|_\infty)$.*

(3) *The set $C(K,\mathscr{A})$ equipped with the norm $|||\cdot|||$ is a Banach space.*

(4) *$C(K,\mathscr{A}) = C_1^b(K,\mathscr{A}) = C_1(K,\mathscr{A})$ and the norms $\|\cdot\|_\infty$ and $|||\cdot|||$ on $C(K,\mathscr{A})$ are equivalent.*

Replacing $C(K,\mathscr{A})$ appearing in the conditions (2) *and* (3) *by either $C_1^b(K, \mathscr{A})$ or $C_1(K,\mathscr{A})$, the conditions* (1)-(4) *are also equivalent.*

Proof. From Proposition 2.10, the implication (1) $\Rightarrow$ (2) is true, and it clear that (2) $\Rightarrow$ (3) is true as well. Now, we will prove that (3) $\Rightarrow$ (1) is true. Suppose that $C(K,\mathscr{A})$ equipped with the norm $|||\cdot|||$ is a Banach space. Then by Proposition 2.7, the Gelfand representation of $\mathscr{A}$ is injective. To see that it is surjective, we will show first that the image $\widehat{\mathscr{A}}$ of $\mathscr{A}$ under the Gelfand representation of $\mathscr{A}$, which is a subalgebra of $C(\mathscr{C}(\mathscr{A}))$, possesses the following properties:

(a) $\widehat{\mathscr{A}}$ separates the points of $\mathscr{C}(\mathscr{A})$ in the sense that for any τ_1 and τ_2 in $\mathscr{C}(\mathscr{A})$ with $\tau_1 \neq \tau_2$, there is an element a in $\mathscr{A}$ such that $\widehat{a}(\tau_1) \neq \widehat{a}(\tau_2)$;

(b) $\widehat{\mathscr{A}}$ does not annihilate any points of $\mathscr{C}(\mathscr{A})$;

(c) $\overline{\widehat{a}} \in \widehat{\mathscr{A}}$ for all $a \in \mathscr{A}$.

Since for any τ_1 and τ_2 in $\mathscr{C}(\mathscr{A})$ with $\tau_1 \neq \tau_2$, there is an element a in $\mathscr{A}$ such that $\widehat{a}(\tau_1) = \tau_1(a) \neq \tau_2(a) = \widehat{a}(\tau_2)$, it follows that $\widehat{\mathscr{A}}$ separates the points of $\mathscr{C}(\mathscr{A})$. So the property (a) is satisfied. It is obvious that the property (b) holds since for each $\tau \in \mathscr{C}(\mathscr{A})$, we have $\tau \neq 0$, which implies that there is an a in $\mathscr{A}$ such that $\widehat{a}(\tau) = \tau(a) \neq 0$. Satisfying the property $(\star)$ of $\mathscr{A}$ implies immediately that the property (c) holds. Thus, by the Stone-Weierstrass approximation theorem, we have $\overline{\widehat{\mathscr{A}}} = C(\mathscr{C}(\mathscr{A}))$. From this result, the surjectivity of the Gelfand representation of $\mathscr{A}$ will be obtained once we can show that $\widehat{\mathscr{A}}$ is closed in $C(\mathscr{C}(\mathscr{A}))$. To get this, we will prove that $\widehat{\mathscr{A}}$ is complete. Let $\{\widehat{a_n}\}_{n=1}^\infty$ be a Cauchy sequence in $\widehat{\mathscr{A}}$. For each n, let $f_n : K \to \mathscr{A}$ be defined by $f_n(t) = a_n$ for all $t \in K$. Then f_n is continuous and $|||f_n||| = \|\widehat{a_n}\|_\infty$ for all n. Moreover, $|||f_n - f_m||| = \|\widehat{a_n} - \widehat{a_n}\|_\infty$ for all n, m. This implies that $\{f_n\}_{n=1}^\infty$ is a Cauchy sequence in $(C(K,\mathscr{A}), |||\cdot|||)$. It follows by the completeness of $(C(K,\mathscr{A}), |||\cdot|||)$ that there is an $f \in C(K,\mathscr{A})$ such that $|||f_n - f||| \to 0$. From this, we have that $\widehat{a_n} \to \widehat{f(t)}$ for each fixed $t \in K$. Therefore, $\widehat{\mathscr{A}}$ is complete. Notice that by the uniqueness of the limit of the sequence $\{\widehat{a_n}\}_{n=1}^\infty$, we have $\widehat{f(t)} = \widehat{f(s)}$ for all $s, t \in K$. Hence, by the injectivity of the Gelfand representation of $\mathscr{A}$, we have $f(t) = f(s)$ for all $s, t \in K$. This yields that there is a unique a in $\mathscr{A}$ such that $\widehat{a_n} \to \widehat{a}$.

We now have the circle (1) ⇒ (2) ⇒ (3) ⇒ (1). To complete to the proof of the theorem, we will show that the implications (1) ⇒ (4) ⇒ (3) hold. By Theorem 2.2 and Lemma 2.8, the implication (1) ⇒ (4) is immediately obtained, and finally the completeness of $(C(K, \mathscr{A}), \|\cdot\|_\infty)$ implies that the implication (4) ⇒ (3) is true. □

We end this paper with the following observations.

Remark 2.12. (1) As proved in Example 2.5, the Gelfand representation of $(l^2)_e$ is injective but not surjective, and we can easily see that $(l^2)_e$ possesses the property ($\star$). Hence, by Theorem 1.1, we have $\inf \left\{ \|\widehat{x}\|_\infty : x \in (l^2)_e, \|x\| = 1 \right\} = 0$.

(2) By Theorem 2.11, we have for each compact Hausdorff space K that the three sets $C(K, (l^2)_e)$, $C_1^b(K, (l^2)_e)$ and $C_1(K, (l^2)_e)$ equipped with the norm $|\!|\!|\cdot|\!|\!|$ are incomplete normed algebras.

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Department of Mathematics
Faculty of Science
Silpakorn University
Nakhon Pathom, 73000, Thailand
E-mail address: jitti@su.ac.th