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TWO GENERAL HYPERGEOMETRIC TRANSFORMATION FORMULAS

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Abstract. A large number of summation and transformation formulas involving (generalized) hypergeometric functions have been developed by many authors. Here we aim at establishing two (presumably) new general hypergeometric transformations. The results are derived by manipulating the involved series in an elementary way with the aid of certain hypergeometric summation theorems obtained earlier by Rakha and Rathie. Relevant connections of certain special cases of our main results with several known identities are also pointed out.

1. Introduction and preliminaries

Throughout this paper, $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{Z}_0^-$ denote the sets of positive integers, real numbers, complex numbers, and nonpositive integers, respectively, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

In the theory of hypergeometric and generalized hypergeometric series, classical summation theorems such as those of Gauss, Kummer's first, second and third for the series ${}_2F_1$ with arguments 1, -1 and $\frac{1}{2}$, and Watson, Dixon, Whipple and Saalschütz for the series ${}_3F_2$ with argument 1 play a key role. For more details about the series ${}_2F_1$ and ${}_3F_2$, and their convergence conditions, we refer to the familiar books such as Bailey [1], Rainville [7], and Slater [9] (see also Srivastava and Choi [10, 11]).

Here, we are concerned with the following Gauss summation theorem and Kummer second summation theorem (see, *e.g.*, [1]), respectively:

$$(1.1) \qquad {}_2F_1\left[\begin{matrix} a,\, b; \\ c; \end{matrix}\; 1\right] = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)} \quad (\Re(c-a-b)>0)$$

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and

$$ {}_2F_1\left[\begin{matrix} a,\, b; \\ \frac{1}{2}(a+b+1); \end{matrix}\ \frac{1}{2}\right] = \frac{\Gamma\left(\frac{1}{2}\right)\,\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\,\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)}. \tag{1.2} $$

Recently a good deal of progress has been made in the direction of generalizing and extending the above-mentioned classical summation theorems. For this, we refer to the recent works by Lavoie *et al.* [6], Kim *et al.* [4, 5], Rakha and Rathie [8], and the references therein.

In particular, in 1996, Lavoie *et al.* [6] obtained a generalization of Kummer's second summation theorem (1.2) and presented explicit expressions of

$$ {}_2F_1\left[\begin{matrix} a,\, b; \\ \frac{1}{2}(a+b+i+1); \end{matrix}\ \frac{1}{2}\right] \quad (i = 0,\, \pm 1,\, \ldots,\, \pm 5)\,. \tag{1.3} $$

In 2011, using (1.3), Choi *et al.* [2] established certain generalization of the well known transformation formula due to Kummer:

$$ \begin{aligned} &{}_2F_1\left[\begin{matrix} a,\, b; \\ \frac{1}{2}(a+b+1); \end{matrix}\ \frac{1}{2}(1+z)\right] \\ &= \frac{\Gamma\left(\frac{1}{2}\right)\,\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\,\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)}\,{}_2F_1\left[\begin{matrix} \frac{1}{2}a,\, \frac{1}{2}b; \\ \frac{1}{2}; \end{matrix}\ z^2\right] \\ &\quad + \frac{2\,\Gamma\left(\frac{1}{2}\right)\,\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a\right)\,\Gamma\left(\frac{1}{2}b\right)}\,{}_2F_1\left[\begin{matrix} \frac{1}{2}a+\frac{1}{2},\, \frac{1}{2}b+\frac{1}{2}; \\ \frac{3}{2}; \end{matrix}\ z^2\right] \quad (|z|<1) \end{aligned} \tag{1.4} $$

and obtained eleven results closely related to (1.4).

Here, in this paper, we aim at presenting two (presumably) new and (potentially) useful general transformation formulas involving generalized hypergeometric functions by using the same technique given in [2]. The results are derived by manipulating the involved series in an elementary way with the aid of certain known hypergeometric summation theorems like the following identity (see, *e.g.*, [7, p. 49]):

$$ {}_2F_1\left[\begin{matrix} -\frac{1}{2}n,\, -\frac{1}{2}n+\frac{1}{2}; \\ c+\frac{1}{2}; \end{matrix}\ 1\right] = \frac{2^n\,(c)_n}{(2c)_n} \quad (\Re(c)>0;\ n\in\mathbb{N}_0)\,, \tag{1.5} $$

which is a special case of (1.1). Here $(\alpha)_n$ is the Pochhammer symbol defined ($\alpha \in \mathbb{C}$) by $(\alpha)_n := \alpha(\alpha+1)\cdots(\alpha+n-1)$ ($n \in \mathbb{N}$) and 1 ($n = 0$).

Certain special cases of our main formulas are seen to yield several known summation identities. For our purpose, we require the following known summation theorem, a generalization of (1.2), due to Rakha and Rathie [8]:

$$
\begin{aligned}
&{}_2F_1\left[\begin{matrix} a,\, b; \\ \frac{1}{2}(a+b+i+1); \end{matrix}\ \frac{1}{2}\right] \\
&= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}i+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a-\frac{1}{2}b-\frac{1}{2}i+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}b\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}i+\frac{1}{2}\right)} \\
&\quad \times \sum_{r=0}^{i}\binom{i}{r}(-1)^r\,\frac{\Gamma\left(\frac{1}{2}b+\frac{1}{2}r\right)}{\Gamma\left(\frac{1}{2}a-\frac{1}{2}i+\frac{1}{2}r+\frac{1}{2}\right)} \qquad (i\in\mathbb{N}_0)
\end{aligned}
\tag{1.6}
$$

and

$$
\begin{aligned}
&{}_2F_1\left[\begin{matrix} a,\, b; \\ \frac{1}{2}(a+b-i+1); \end{matrix}\ \frac{1}{2}\right] \\
&= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b-\frac{1}{2}i+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}b\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)} \times \sum_{r=0}^{i}\binom{i}{r}\frac{\Gamma\left(\frac{1}{2}b+\frac{1}{2}i\right)}{\Gamma\left(\frac{1}{2}a-\frac{1}{2}i+\frac{1}{2}r+\frac{1}{2}\right)} \qquad (i\in\mathbb{N}_0)\,.
\end{aligned}
\tag{1.7}
$$

It is noted that the special case of (1.6) or (1.7) when $i=0$ is immediately seen to yield the Kummer second theorem (1.2). It is also remarked in passing that Choi *et al.* [3] presented certain interesting hypergeometric identities by using the Beta integral formula.

2. Main transformation formulas

Here we establish the following two general transformation formulas:

$$
\begin{aligned}
&{}_{G+3}F_{H+2}\left[\begin{matrix} a,\, b,\, c,\, g_1,\, \ldots,\, g_G; \\ \frac{1}{2}(a+b+i+1),\, 2c,\, h_1,\, \ldots,\, h_H; \end{matrix}\ y\right] \\
&= \sum_{m=0}^{\infty}\frac{(a)_{2m}\,(b)_{2m}\,(g_1)_{2m}\cdots(g_G)_{2m}}{\left(\frac{a+b+i+1}{2}\right)_{2m}(h_1)_{2m}\cdots(h_H)_{2m}\left(c+\frac{1}{2}\right)_m 2^{4m}\,m!} \\
&\quad \times {}_{G+2}F_{H+1}\left[\begin{matrix} a+2m,\, b+2m,\, g_1+2m,\, \ldots,\, g_G+2m; \\ \frac{1}{2}(a+b+1+4m+i),\, h_1+2m,\, \ldots,\, h_H+2m; \end{matrix}\ \frac{1}{2}y\right] \\
&\quad (i\in\mathbb{N}_0)
\end{aligned}
\tag{2.1}
$$

and

$$
\begin{aligned}
&{}_{G+3}F_{H+2}\left[\begin{matrix} a,\, b,\, c,\, g_1,\, \ldots,\, g_G; \\ \frac{1}{2}(a+b-i+1),\, 2c,\, h_1,\, \ldots,\, h_H; \end{matrix}\, y\right] \\
&= \sum_{m=0}^{\infty} \frac{(a)_{2m}\,(b)_{2m}\,(g_1)_{2m}\cdots(g_G)_{2m}}{\left(\frac{a+b-i+1}{2}\right)_{2m}\,(h_1)_{2m}\cdots(h_H)_{2m}\,\left(c+\frac{1}{2}\right)_m\,2^{4m}\,m!} \\
&\quad\times {}_{G+2}F_{H+1}\left[\begin{matrix} a+2m,\, b+2m,\, g_1+2m,\, \ldots,\, g_G+2m; \\ \frac{1}{2}(a+b+1+4m-i),\, h_1+2m,\, \ldots,\, h_H+2m; \end{matrix}\, \frac{1}{2}y\right] \\
&(i \in \mathbb{N}_0)\,,
\end{aligned} \tag{2.2}
$$

where ${}_pF_q$ $(p,\, q \in \mathbb{N}_0)$ denote the familiar generalized hypergeometric series (see, *e.g.*, [10, Section 1.4] and [11, Section 1.5]).

Proof. In order to establish the first general transformation formula (2.1), for convenience and simplicity, let us denote the left-hand side of (2.1) by $\mathcal{S}$. Expressing ${}_{G+3}F_{H+2}$ as a series, we have

$$
\mathcal{S} = \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n\,(g_1)_n\cdots(g_G)_n}{\left(\frac{1}{2}(a+b+i+1)\right)_n\,(h_1)_n\cdots(h_H)_n}\,\frac{y^n}{2^n\,n!}\left\{\frac{2^n\,(c)_n}{(2c)_n}\right\}.
$$

Using the known result (1.5), we obtain

$$
\mathcal{S} = \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n\,(g_1)_n\cdots(g_G)_n}{\left(\frac{1}{2}(a+b+i+1)\right)_n\,(h_1)_n\cdots(h_H)_n}\,\frac{y^n}{2^n\,n!}\,{}_2F_1\left[\begin{matrix} -\frac{1}{2}n,\, -\frac{1}{2}n+\frac{1}{2}; \\ c+\frac{1}{2}; \end{matrix}\, 1\right].
$$

Expressing ${}_2F_1$ as a series, after some simplification, we get

$$
\mathcal{S} = \sum_{n=0}^{\infty}\sum_{m=0}^{[\frac{n}{2}]} \frac{(a)_n\,(b)_n\,(g_1)_n\cdots(g_G)_n}{\left(\frac{1}{2}(a+b+i+1)\right)_n\,(h_1)_n\cdots(h_H)_n}\,\frac{y^n}{2^n\,n!}\,\frac{\left(-\frac{n}{2}\right)_m\,\left(-\frac{n}{2}+\frac{1}{2}\right)_m}{\left(c+\frac{1}{2}\right)_m\,m!}.
$$

Using an easily-derivable identity:

$$
(\alpha)_{2n} = 2^{2n}\left(\frac{1}{2}\alpha\right)_n\left(\frac{1}{2}\alpha+\frac{1}{2}\right)_n,
$$

we have

$$
\mathcal{S} = \sum_{n=0}^{\infty}\sum_{m=0}^{[\frac{n}{2}]} \frac{(a)_n\,(b)_n\,(g_1)_n\cdots(g_G)_n}{\left(\frac{1}{2}(a+b+i+1)\right)_n\,(h_1)_n\cdots(h_H)_n}\,\frac{y^n}{2^n\,n!}\,\frac{(-n)_{2m}}{2^{2m}\,\left(c+\frac{1}{2}\right)_m\,m!}.
$$

Again, using a known identity:

$$
(-n)_{2m} = \frac{n!}{(n-2m)!},
$$

we obtain

$$\mathcal{S}=\sum_{n=0}^{\infty}\sum_{m=0}^{[\frac{n}{2}]}\frac{(a)_n(b)_n(g_1)_n\cdots(g_G)_n}{\left(\frac{1}{2}(a+b+i+1)\right)_n\,(h_1)_n\cdots(h_H)_n}\frac{y^n}{(n-2m)!m!2^{n+2m}\left(c+\frac{1}{2}\right)_m}.$$

Now, using the following series manipulation identity (see, *e.g.*, [7, p. 57, Lemma 11]):

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{n}{2}]}A(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n+2k),$$

after a little simplification, we get

$$\begin{aligned}\mathcal{S}=&\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(a)_{n+2m}\,(b)_{n+2m}\,(g_1)_{n+2m}\cdots(g_G)_{n+2m}}{\left(\frac{1}{2}(a+b+i+1)\right)_{n+2m}\,(h_1)_{n+2m}\cdots(h_H)_{n+2m}}\\&\times\frac{y^{n+2m}}{n!\,m!\,2^{n+4m}\,\left(c+\frac{1}{2}\right)_m}.\end{aligned}$$

Then, using a known identity:

$$(\alpha)_{n+2m}=(\alpha)_{2m}\,(\alpha+2m)_n$$

in each of the involved terms, after some simplification, we have

$$\begin{aligned}\mathcal{S}=&\sum_{m=0}^{\infty}\frac{(a)_{2m}\,(b)_{2m}\,(g_1)_{2m}\cdots(g_G)_{2m}}{\left(\frac{1}{2}(a+b+i+1)\right)_{2m}\,(h_1)_{2m}\cdots(h_H)_{2m}}\frac{y^{2m}}{2^{4m}\,\left(c+\frac{1}{2}\right)_m}\\&\times\sum_{n=0}^{\infty}\frac{(a+2m)_n\,(b+2m)_n\,(g_1+2m)_n\cdots(g_G+2m)_n}{\left(\frac{1}{2}(a+b+i+1)+2m\right)_n\,(h_1+2m)_n\cdots(h_H+2m)_n}\frac{y^n}{2^n\,n!}.\end{aligned}$$

Finally, summing up the inner series, we are easily led to the right-hand side of (2.1).

Next, the same argument as above will easily establish the second general formula (2.2). □

3. Special cases

Here we present some of the known results that can be deduced from our main transformation formulas (2.1) and (2.2).

(i) In (2.1), if we set $i=0$, $G=H=0$ and $y=1$, we obtain

$$\begin{aligned}&{}_3F_2\left[\begin{matrix}a,\,b,\,c;\\ \frac{1}{2}(a+b+1),\,2c;\end{matrix}\;1\right]\\&=\sum_{m=0}^{\infty}\frac{(a)_{2m}\,(b)_{2m}}{\left(\frac{a+b+1}{2}\right)_{2m}\,2^{4m}\,m!}\,{}_2F_1\left[\begin{matrix}a+2m,\,b+2m;\\ \frac{1}{2}(a+b+4m+1);\end{matrix}\;\frac{1}{2}\right].\end{aligned}\tag{3.1}$$

Evaluating ${}_2F_1$ appearing on the right-hand side of (3.1) by using the Kummer's second summation theorem (1.2) and an easily-derivable identity:

$$(\alpha)_{2m} = 2^{2m} \left(\frac{1}{2}a\right)_m \left(\frac{1}{2}a + \frac{1}{2}\right)_m ,$$

and, after a little simplification, summing up the series and finally applying the Gauss summation theorem (1.1), we get the following classical Watson's summation theorem (see, *e.g.*, [1]):

$$\begin{aligned}
&{}_3F_2 \left[\begin{matrix} a,\, b,\, c; \\ \frac{1}{2} + \frac{1}{2}a + \frac{1}{2}b,\, 2c; \end{matrix} \; 1 \right] \\
&= \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(c + \frac{1}{2}\right) \Gamma\left(\frac{1}{2} + \frac{1}{2}a + \frac{1}{2}b\right) \Gamma\left(\frac{1}{2} - \frac{1}{2}a - \frac{1}{2}b + c\right)}{\Gamma\left(\frac{1}{2} + \frac{1}{2}a\right) \Gamma\left(\frac{1}{2} + \frac{1}{2}b\right) \Gamma\left(\frac{1}{2} - \frac{1}{2}a + c\right) \Gamma\left(\frac{1}{2} - \frac{1}{2}b + c\right)}
\end{aligned} \tag{3.2}$$

provided $\Re(2c - a - b) > -1$.

(ii) Similarly as in (i), taking $G = H = 0$ and $y = 1$ in (2.1), and $G = H = 0$ and $y = 1$ in (2.2), and using the results (1.6) and (1.7), respectively, we obtain the following general summation formulas:

$$\begin{aligned}
&{}_3F_2 \left[\begin{matrix} a,\, b,\, c; \\ \frac{1}{2}(a + b + i + 1),\, 2c; \end{matrix} \; 1 \right] \\
&= \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}i + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}a - \frac{1}{2}b - \frac{1}{2}i + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}b\right) \Gamma\left(\frac{1}{2}b + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}a - \frac{1}{2}b + \frac{1}{2}i + \frac{1}{2}\right)} \\
&\quad \times \sum_{r=0}^{i} (-1)^r \binom{i}{r} \frac{\Gamma\left(\frac{1}{2}b + \frac{1}{2}r\right)}{\Gamma\left(\frac{1}{2}a - \frac{1}{2}i + \frac{1}{2}r + \frac{1}{2}\right)} \\
&\quad \times {}_3F_2 \left[\begin{matrix} \frac{1}{2}a,\, \frac{1}{2}a + \frac{1}{2},\, \frac{1}{2}b + \frac{1}{2}r; \\ c + \frac{1}{2},\, \frac{1}{2}a + \frac{1}{2}r - \frac{1}{2}i + \frac{1}{2}; \end{matrix} \; 1 \right] \quad (i \in \mathbb{N}_0)
\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}
&{}_3F_2 \left[\begin{matrix} a,\, b,\, c; \\ \frac{1}{2}(a + b - i + 1),\, 2c; \end{matrix} \; 1 \right] \\
&= \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}i + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}b\right) \Gamma\left(\frac{1}{2}b + \frac{1}{2}\right)} \sum_{r=0}^{i} \binom{i}{r} \frac{\Gamma\left(\frac{1}{2}b + \frac{1}{2}r\right)}{\Gamma\left(\frac{1}{2}a - \frac{1}{2}i + \frac{1}{2}r + \frac{1}{2}\right)} \\
&\quad \times {}_3F_2 \left[\begin{matrix} \frac{1}{2}a,\, \frac{1}{2}a + \frac{1}{2},\, \frac{1}{2}b + \frac{1}{2}r; \\ c + \frac{1}{2},\, \frac{1}{2}a + \frac{1}{2}r - \frac{1}{2}i + \frac{1}{2}; \end{matrix} \; 1 \right] \quad (i \in \mathbb{N}_0)\,.
\end{aligned} \tag{3.4}$$

It is noted that (3.3) and (3.4) are known results obtained by Rakha and Rathie [8] who used a different method. Further, the cases $i = 0, 1, 2, 3, 4, 5$ of the results (3.3) and (3.4) were also obtained by Kim and Rathie [4].

(iii) In (2.1), if we take $i = 2$, $G = 1 = H$, $g_1 = d + 1$, $h_1 = d$ and $y = 1$, we have

$$
\begin{aligned}
&{}_4F_3\left[\begin{matrix} a,\, b,\, c,\, d+1; \\ \frac{1}{2}(a+b+3),\, 2c,\, d; \end{matrix}\; 1\right] \\
&= \sum_{m=0}^{\infty} \frac{(a)_{2m}\,(b)_{2m}\,(d+1)_{2m}}{\left(\frac{1}{2}(a+b+3)\right)_{2m}\,(d)_{2m}\,\left(c+\frac{1}{2}\right)_m\,2^{4m}\,m!} \\
&\quad \times {}_3F_2\left[\begin{matrix} a+2m,\, b+2m,\, d+2m+1; \\ \frac{1}{2}(a+b+3+4m),\, d+2m; \end{matrix}\; \frac{1}{2}\right].
\end{aligned} \tag{3.5}
$$

Now, we observe that the ${}_3F_2$ appearing on the right-hand side of (3.5) can be evaluated with the help of a known result [5, p. 15, Eq. (5.2)], then, after much arrangement, separating it into three terms and summing up all the series and, finally, evaluating each separated series by using Gauss's summation theorem (1.1), we are led to another known result due to Kim *et al.* [5]:

(3.6)

$$
\begin{aligned}
&{}_4F_3\left[\begin{matrix} a,\, b,\, c,\, d+1; \\ \frac{1}{2}(a+b+3),\, 2c,\, d; \end{matrix}\; 1\right] \\
&= \frac{2^{a+b-2}\,\Gamma\left(c+\frac{1}{2}\right)\,\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{3}{2}\right)\,\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b-\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\,\Gamma(a)\,\Gamma(b)} \\
&\quad \times \left\{\frac{\alpha\,\Gamma\left(\frac{1}{2}a\right)\,\Gamma\left(\frac{1}{2}b\right)}{\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}\right)\,\Gamma\left(c-\frac{1}{2}b+\frac{1}{2}\right)} + \frac{\beta\,\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\,\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(c-\frac{1}{2}a+1\right)\,\Gamma\left(c-\frac{1}{2}b+1\right)}\right\}
\end{aligned}
$$

provided $\Re(2c - a - b) > -1$, $d \in \mathbb{C} \setminus \mathbb{Z}_0^-$, and α and β are given by

$$\alpha = a(2c-a) + b(2c-b) - 2c + 1 - \frac{ab}{d}(4c - a - b - 1)$$

and

$$\beta = 8\left\{\frac{1}{2d}(a+b+1) - 1\right\}.$$

It is noted that Kim *et al.* [5] obtained (3.6) by using a different method. Further, if we take $d = \frac{1}{2}(a+b+1)$ in (3.6), we recover Watson's theorem (3.2). Similarly many other explicit results may be obtained from our main formulas (2.1) and (2.2).

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