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ASYMPTOTIC BEHAVIOR OF STRONG SOLUTIONS TO 2D g-NAVIER-STOKES EQUATIONS

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Abstract. Considered here is the first initial boundary value problem for the two-dimensional g-Navier-Stokes equations in bounded domains. We first study the long-time behavior of strong solutions to the problem in term of the existence of a global attractor and global stability of a unique stationary solution. Then we study the long-time finite dimensional approximation of the strong solutions.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary Γ. In this paper we consider the following two-dimensional (2D) g-Navier-Stokes equations:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla p & = f \text{ in } (0,\infty)\times\Omega, \\ \nabla\cdot(gu) & = 0 \text{ in } (0,\infty)\times\Omega, \\ u & = 0 \text{ on } (0,\infty)\times\Gamma, \\ u(x,0) & = u_0(x),\ x\in\Omega, \end{cases} \tag{1.1}$$

where $u = u(x,t) = (u_1,u_2)$ is the unknown velocity vector, $p = p(x,t)$ is the unknown pressure, $\nu > 0$ is the kinematic viscosity coefficient, u_0 is the initial velocity.

The 2D g-Navier-Stokes equations arise in a natural way when we study the standard 3D problem in the thin domain $\Omega_g = \Omega\times(0,g)$. We refer the reader to [9] for a derivation of the 2D g-Navier-Stokes equations from the 3D Navier-Stokes equations and a relationship between them. As mentioned in [9], good properties of the 2D g-Navier-Stokes equations initiate the study of the Navier-Stokes equations on the thin three-dimensional domain Ω_g. In the last few years, the existence and long-time behavior of weak solutions to 2D g-Navier-Stokes equations have been studied extensively in both autonomous and non-autonomous cases (see e.g. [1, 4, 5, 6, 7, 8, 10, 14]). However, to the

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best of our knowledge, little seems to be known about strong solutions of the 2D g-Navier-Stokes equations.

In a recent work [2], the authors proved the existence and finite-time numerical approximation of strong solutions to the 2D g-Navier-Stokes equations. In this paper, we continue studying the long-time behavior and the long-time finite dimensional approximation of the strong solutions. To do this, we assume that the function g satisfies the following assumption:

(**G**) $g \in W^{1,\infty}(\Omega)$ such that

$$0 < m_0 \leq g(x) \leq M_0 \text{ for all } x = (x_1, x_2) \in \Omega, \text{ and } |\nabla g|_\infty < m_0 \lambda_1^{1/2},$$

where $\lambda_1 > 0$ is the first eigenvalue of the g-Stokes operator in Ω (i.e., the operator A is defined in Section 2 below).

It is noticed that after studying the existence of solutions, as mentioned in [12, 13] for the Navier-Stokes equations, the long-time behavior and long-time approximation of the strong solutions are important questions because the problem of numerical computation of turbulent flows is directly connected with the computation of the solutions for large time. This is the main motivation of the present paper.

The plan of the paper is as follows. In Section 2, for convenience of the reader, we recall some auxiliary results on function spaces and inequalities for the nonlinear terms related to the g-Navier-Stokes equations. In Section 3, when the external force $f \in H_g$ is assumed to be time-independent, we show that the long-time behavior of strong solutions is determined by the existence of a compact global attractor in V_g for the continuous semigroup $S(t) : V_g \to V_g$ generated by the strong solutions to the problem. To do this, we construct a bounded absorbing set in the space $D(A)$, the domain of the operator A, and using the compactness of the embedding $D(A) \hookrightarrow V_g$. We also prove the existence, uniqueness and exponential stability of a stationary solution when the external force is time-independent and "small" when compared with the viscosity coefficient ν. Long-time finite dimensional approximation of strong solutions is studied in the last section. The results obtained here, in particular, generalize the corresponding results for the 2D Navier-Stokes equations in [11, 12, 13].

2. Preliminary results

2.1. Function spaces and inequalities for the nonlinear terms

Let $L^2(\Omega, g) = (L^2(\Omega))^2$ and $H_0^1(\Omega, g) = (H_0^1(\Omega))^2$ be endowed, respectively, with the inner products

$$(u, v)_g = \int_\Omega u \cdot v g dx, \; u, v \in L^2(\Omega, g),$$

and

$$((u,v))_g = \int_\Omega \sum_{j=1}^{2} \nabla u_j \cdot \nabla v_j g dx,\ u=(u_1,u_2), v=(v_1,v_2) \in H_0^1(\Omega,g),$$

and norms $|u|^2 = (u,u)_g$, $||u||^2 = ((u,u))_g$. Thanks to assumption (**G**), the norms $|\cdot|$ and $||\cdot||$ are equivalent to the usual ones in $(L^2(\Omega))^2$ and in $(H_0^1(\Omega))^2$.

Let

$$\mathcal{V} = \{u \in (C_0^\infty(\Omega))^2 : \nabla \cdot (gu) = 0\}.$$

Denote by H_g the closure of $\mathcal{V}$ in $L^2(\Omega,g)$, and by V_g the closure of $\mathcal{V}$ in $H_0^1(\Omega,g)$. It follows that $V_g \subset H_g \equiv H_g' \subset V_g'$, where the injections are dense and continuous. We will use $||\cdot||_*$ for the norm in V_g', and $\langle \cdot,\cdot \rangle$ for duality pairing between V_g and V_g'.

We now define the trilinear form b by

$$b(u,v,w) = \sum_{i,j=1}^{2} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j g dx,$$

whenever the integrals make sense. It is easy to check that if $u,v,w \in V_g$, then

$$b(u,v,w) = -b(u,w,v).$$

Hence

$$b(u,v,v) = 0,\ \forall u,v \in V_g.$$

Set $A : V_g \to V_g'$ by $\langle Au, v\rangle = ((u,v))_g$, $B : V_g \times V_g \to V_g'$ by $\langle B(u,v), w\rangle = b(u,v,w)$, and put $Bu = B(u,u)$. Denote $D(A) = \{u \in V_g : Au \in H_g\}$, then $D(A) = H^2(\Omega,g) \cap V_g$ and $Au = -P_g \Delta u, \forall u \in D(A)$, where P_g is the ortho-projector from $L^2(\Omega,g)$ onto H_g.

Lemma 2.1 ([1])**.** *If $n=2$, then*

$$|b(u,v,w)| \leq \begin{cases} c_1|u|^{1/2}\|u\|^{1/2}\|v\||w|^{1/2}\|w\|^{1/2},\ \forall u,v,w \in V_g, \\ c_2|u|^{1/2}\|u\|^{1/2}\|v\|^{1/2}|Av|^{1/2}|w|,\ \forall u \in V_g, v \in D(A), w \in H_g, \\ c_3|u|^{1/2}|Au|^{1/2}\|v\||w|,\ \forall u \in D(A), v \in V_g, w \in H_g, \\ c_4|u|\|v\||w|^{1/2}|Aw|^{1/2},\ \forall u \in H_g, v \in V_g, w \in D(A), \end{cases}$$

where $c_i, i=1,\dots,4$, are appropriate constants.

Lemma 2.2 ([2])**.** *Let $u \in L^2(0,T;D(A)) \cap L^\infty(0,T;V_g)$. Then the function Bu defined by*

$$(Bu(t),v)_g = b(u(t),u(t),v),\ \forall v \in H_g,\ \text{a.e. } t \in [0,T],$$

belongs to $L^4(0,T;H_g)$, therefore also belongs to $L^2(0,T;H_g)$.

Lemma 2.3 ([3])**.** *Let $u \in L^2(0,T;V_g)$. Then the function Cu defined by*

$$(Cu(t),v)_g = ((\frac{\nabla g}{g} \cdot \nabla)u, v)_g = b(\frac{\nabla g}{g}, u, v),\ \forall v \in V_g,$$

belongs to $L^2(0,T;H_g)$*, and hence also belongs to* $L^2(0,T;V_g')$*. Moreover,*

$$|Cu(t)| \le \frac{|\nabla g|_\infty}{m_0} \cdot ||u(t)|| \quad \textit{for a.e. } t \in (0,T),$$

and

$$||Cu(t)||_* \le \frac{|\nabla g|_\infty}{m_0 \lambda_1^{1/2}} \cdot ||u(t)|| \quad \textit{for a.e. } t \in (0,T).$$

Since

$$-\frac{1}{g}(\nabla \cdot g\nabla)u = -\Delta u - (\frac{\nabla g}{g}.\nabla)u,$$

we have

$$(-\Delta u, v)_g = ((u,v))_g + ((\frac{\nabla g}{g}.\nabla)u, v)_g = (Au, v)_g + ((\frac{\nabla g}{g} \cdot \nabla)u, v)_g, \forall u, v \in V_g.$$

2.2. Existence of strong solutions

We recall the result on the existence and uniqueness of a strong solution to problem (1.1) in [2] which will be used later.

Definition 2.1. Given $f \in L^2(0,T;H_g)$ and $u_0 \in V_g$, a strong solution on the $(0,T)$ of problem (1.1) is a function $u \in L^2(0,T;D(A)) \cap L^\infty(0,T;V_g)$ with $u(0) = u_0$, and such that

$$\frac{d}{dt}(u(t),v)_g + \nu((u(t),v))_g + \nu(Cu(t),v)_g + b(u(t),u(t),v) = (f(t),v)_g \tag{2.1}$$

for all $v \in V_g$, and for a.e. $t \in (0,T)$.

Theorem 2.1 ([2]). *Suppose that* $f \in L^2_{loc}(0,\infty;H_g)$ *and* $u_0 \in V_g$ *are given. Then for any* $T > 0$*, there exists a unique strong solution* u *of problem* (1.1) *on* $(0,T)$*. Moreover, the map* $u_0 \mapsto u(t)$ *is continuous on* V_g *for all* $t \in [0,T]$*, that is, the strong solution depends continuously on the initial data.*

3. Long-time behavior of strong solutions

In this section, we assume that $f \in H_g$ is independent of time t. Then, by Theorem 2.1, we can define a (nonlinear) continuous semigroup $S(t) : V_g \to V_g$ by

$$S(t)u_0 = u(t),\ t \ge 0,\ u_0 \in V_g,$$

where $u(t)$ is the unique strong solution of problem (1.1) with the initial datum $u(0) = u_0$. We will prove that this semigroup possesses a compact connected global attractor $\mathcal{A}$ in V_g (we refer the reader to [13] about the general theory of global attractors), and when the external force f is "small" enough, the attractor has a very simple form $\mathcal{A} = \{u^*\}$, where u^* is the unique strong stationary solution of problem (1.1).

3.1. Existence of a global attractor in V_g

Proposition 3.1. *If $f \in H_g$, then there exist a time $t_0 = t_0(|u_0|)$, a ρ_{H_g} and an I_{V_g} such that*

$$|u(t)| \leq \rho_{H_g}, \tag{3.1}$$

and

$$\int_t^{t+1} ||u(s)||^2 ds \leq I_{V_g}, \ \forall t \geq t_0. \tag{3.2}$$

Proof. In (2.1) taking $v = u(t)$ and arguing exactly as in the proof of Lemma 3.1 in [2], we have

$$\frac{d}{dt}|u|^2 + 2\nu(\gamma_0 - \epsilon)\|u\|^2 \leq \frac{|f|^2}{2\nu\epsilon\lambda_1}, \tag{3.3}$$

and then using the inequality $||u||^2 \geq \lambda_1 |u|^2$, we obtain

$$\frac{d}{dt}|u|^2 + 2\nu\lambda_1(\gamma_0 - \epsilon)|u|^2 \leq \frac{|f|^2}{2\nu\epsilon\lambda_1}.$$

By Gronwall's lemma, we get

$$|u(t)|^2 \leq |u_0|^2 e^{-2\nu\lambda_1(\gamma_0-\epsilon)t} + \frac{|f|^2}{4\nu^2\lambda_1^2\epsilon(\gamma_0 - \epsilon)},$$

and so there is a time $t_0 = t_0(|u_0|)$ such that for all $t \geq t_0$,

$$|u(t)|^2 \leq \frac{|f|^2}{2\nu^2\lambda_1^2\epsilon(\gamma_0 - \epsilon)} \leq \frac{2|f|^2}{\nu^2\lambda_1^2\gamma_0^2} = \rho_H^2. \tag{3.4}$$

The estimate (3.2) follows by integrating (3.3) from t to $t+1$ and using (3.4). □

We now prove the existence of a bounded absorbing set in V_g for the semigroup $S(t)$.

Proposition 3.2. *If $f \in H_g$, then there exist a time $t_1 = t_1(t_0)$, a ρ_{V_g} and an I_A such that*

$$||u(t)|| \leq \rho_{V_g}, \tag{3.5}$$

and

$$\int_t^{t+1} |Au(s)|^2 ds \leq I_A, \quad \forall t \geq t_1. \tag{3.6}$$

Proof. From (2.1), replacing v by $Au(t)$ and repeating arguments in the proof of Lemma 3.2 in [2], we get

$$\begin{aligned} &\frac{d}{dt}||u(t)||^2 + \nu\Big(1 - \frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\Big)|Au(t)|^2 \\ &\leq \frac{2}{\nu}|f|^2 + 2c_3'|u(t)|^2||u(t)||^4 + \frac{\nu|\nabla g|_\infty}{2m_0\lambda_1^{1/2}}||u(t)||^2, \end{aligned} \tag{3.7}$$

and hence by Gronwall's lemma,

$$||u(t)||^2 \leq ||u(s)||^2 \exp\big(\int_s^t \big(2c_3'|u(\tau)|^2|||u(\tau)||^2 + \frac{\nu|\nabla g|_\infty}{2m_0\lambda_1^{1/2}}\big)d\tau\big)$$
$$+ \frac{2}{\nu}|f|^2 \int_s^t \exp\big(\int_s^t \big(2c_3'|u(\tau)|^2||u(\tau)||^2 + \frac{\nu|\nabla g|_\infty}{2m_0\lambda_1^{1/2}}\big)d\tau\big)dr.$$

Using (3.1) and (3.2), we get

$$||u(t)||^2 \leq C_1||u(s)||^2 + C_2\frac{2}{\nu}|f|^2.$$

Now integrating between $s = t-1$ and $s = t$, we have

$$||u(t)||^2 \leq C_1 \int_{t-1}^t ||u(s)||^2 ds + C_2\frac{2}{\nu}|f|^2.$$

Using (3.2) once again, we obtain $||u(t)||^2 \leq \rho_{V_g}$. Integrating (3.7) from t to $t+1$, we obtain (3.6). □

We can now show the existence of a bounded absorbing set in $D(A)$ for the semigroup $S(t)$, which implies the existence of a global attractor in V_g.

Proposition 3.3. *If $f \in H_g$, then there exist a time $t_2 = t_2(t_1)$ and a ρ_A such that*

$$|Au(t)| \leq \rho_A, \quad \forall\, t \geq t_2.$$

Proof. Observe first that if $u \in D(A)$, then $B(u,u) \in H_g$, with

$$|B(u,u)| \leq k|u|^{1/2}||u|||Au|^{1/2}.$$

On the other hand, since

$$\frac{du}{dt} + \nu Au + \nu Cu + B(u,u) = f, \tag{3.8}$$

we get

$$|u_t| \leq \nu|Au| + \nu|Cu| + k|u|^{1/2}||u|||Au|^{1/2} + |f|.$$

Using Lemma 2.3 and Young's inequality, we have

$$|u_t| \leq \nu\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}||u|| + k_1|Au| + k_2|u|||u||^2 + |f|.$$

Using (3.1) and (3.5), we get for all $t \geq t_1$,

$$|u_t| \leq c|Au| + \nu\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\rho_{V_g} + c\rho_{H_g}\rho_{V_g}^2 + |f|.$$

Integrating from t to $t+1$, using (3.6), we have

$$\int_t^{t+1} |u_t|^2 ds \leq C_t \ \text{ for all } t \geq t_1.$$

We now differentiate (3.8) with respect to t and take the inner product with u_t to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|u_t|^2 + \nu||u_t||^2 &\le \nu|(Cu_t, u_t)| + |b(u_t, u, u_t)| \\ &\le \nu\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}||u_t||^2 + k||u||||u_t||||u_t|| \\ &\le \nu\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}||u_t||^2 + \epsilon\nu||u_t||^2 + \frac{k^2}{4\nu\epsilon}||u||^2|u_t|^2.\end{aligned}$$

Hence

$$\frac{d}{dt}|u_t|^2 + 2\nu(\gamma_0 - \epsilon)||u_t||^2 \le \frac{k^2}{2\nu\epsilon}||u||^2|u_t|^2.$$

It follows that for t large enough,

$$\frac{d}{dt}|u_t|^2 \le \frac{k^2\rho_{V_g}^2}{2\nu\epsilon}|u_t|^2.$$

We integrate this inequality between s and $t+1$ with $t < s < t+1$ to get

$$|u_t(t+1)|^2 \le |u_t(s)|^2 + \frac{k^2\rho_{V_g}^2}{2\nu\epsilon}\int_s^{t+1}|u_t(s)|^2 ds,$$

and then again between t and $t+1$ so that

$$|u_t(t+1)|^2 \le (1 + \frac{k^2\rho_{V_g}^2}{2\nu\epsilon})\int_t^{t+1}|u_t(s)|^2 ds \le C_t(1 + \frac{k^2\rho_{V_g}^2}{2\nu\epsilon}). \tag{3.9}$$

From (3.8), we have

$$\begin{aligned}\nu|Au| &\le |u_t| + \nu|Cu| + |B(u,u)| + |f| \\ &\le |u_t| + \nu\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}||u|| + k|u|^{1/2}||u|||Au|^{1/2} + |f|.\end{aligned}$$

Using Young's inequality, we get

$$\begin{aligned}\frac{\nu}{2}|Au| &\le |u_t| + \nu\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}||u|| + \frac{k^2}{2\nu}|u|||u||^2 + |f| \\ &\le |u_t| + \nu\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\rho_{V_g} + \frac{k^2}{2\nu}\rho_{H_g}\rho_{V_g}^2 + |f|,\end{aligned}$$

and so we have that $|Au(t+1)|$ is bounded, using (3.9). This completes the proof. □

Because the compactness of the embedding $D(A) \hookrightarrow V_g$ and the connectedness of V_g, from Theorem 1.1 in [13, Chapter 1] we immediately get the following result.

Theorem 3.4. *The semigroup $S(t)$ generated by problem* (1.1) *possesses a compact connected global attractor $\mathcal{A}$ in the space V_g.*

3.2. Existence and exponential stability of stationary solutions

A strong stationary solution to problem (1.1) is an element $u^* \in D(A)$ such that

$$\nu((u^*, v))_g + \nu(Cu^*, v)_g + b(u^*, u^*, v) = (f, v)_g, \ \forall v \in V_g. \tag{3.10}$$

Theorem 3.5. *If $f \in H_g$, then*

(a) *Problem* (1.1) *admits at least one strong stationary solution u^*. Moreover, any such strong stationary solution satisfies the estimate*

$$\nu(1 - \frac{|\nabla g|_\infty}{m_0 \lambda_1^{1/2}})||u^*|| \le \frac{1}{\lambda_1^{1/2}}|f|. \tag{3.11}$$

(b) *If the following condition holds*

$$\Big[\nu(1 - \frac{|\nabla g|_\infty}{m_0 \lambda_1^{1/2}})\Big]^2 > \frac{c_1|f|}{\lambda_1}, \tag{3.12}$$

where c_1 is the constant in Lemma 2.1, then the strong stationary solution of (1.1) *is unique.*

Proof. **(i) Existence.** The estimate (3.11) can be obtained taking into account that in particular any stationary solution u^*, if it exists, should verify

$$\nu((u^*, u^*))_g + \nu(Cu^*, u^*)_g = (f, u^*)_g,$$

and therefore

$$\nu||u^*||^2 \le \frac{1}{\lambda_1^{1/2}}|f|||u^*|| + \frac{\nu|\nabla g|_\infty}{m_0 \lambda_1^{1/2}}||u^*||^2.$$

or

$$\nu(1 - \frac{|\nabla g|_\infty}{m_0 \lambda_1^{1/2}})||u^*|| \le \frac{1}{\lambda_1^{1/2}}|f|.$$

For the existence, let $v_1, v_2, \dots,$ be the basis of V_g consisting of eigenfunctions of the operator A. For each $m \ge 1$, let us denote $V_m = \text{span}\{v_1, \dots, v_m\}$ and we would like to define an approximate strong stationary solutions u^m of (1.1) by

$$u^m = \sum_{i=1}^m \gamma_{mi} v_i,$$

such that

$$\nu((u^m, v))_g + \nu(Cu^m, v)_g + b(u^m, u^m, v) = (f, v)_g, \ \forall v \in V_g. \tag{3.13}$$

To prove the existence of u^m, we define operators $R_m : V_m \to V_m$ by

$$((R_m u, v)) = \nu\langle Au, v\rangle_g + \nu(Cu, v)_g + b(u, u, v) - (f, v)_g \ \forall u, v \in V_m.$$

For all $u \in V_m$,

$$\begin{aligned} ((R_m u, u)) &= \nu\langle Au, u\rangle_g + \nu(Cu, u)_g - (f, u)_g \\ &\ge \nu||u||^2 - \frac{1}{\lambda_1^{1/2}}|f|||u|| - \frac{\nu|\nabla g|_\infty}{m_0 \lambda_1^{1/2}}||u||^2 \end{aligned}$$

$$= \nu(1 - \frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}})||u||^2 - \frac{1}{\lambda_1^{1/2}}|f|||u||.$$

Thus, if we take

$$\beta = \frac{|f|}{\lambda_1^{1/2}\nu(1 - \frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}})},$$

we obtain $((R_m u, u)) \geq 0$ for all $u \in V_m$ such that $||u|| = \beta$. Consequently, by a corollary of the Brouwer fixed point theorem, for each $m \geq 1$ there exists $u_m \in V_m$ such that $R_m(u_m) = 0$, with $\|u_m\| \leq \beta$. Taking $v = Au^m$ in (3.13) we get

$$\begin{aligned}\nu|Au^m|^2 &= (f, Au^m)_g - \nu(Cu^m, Au^m)_g - b(u^m, u^m, Au^m)\\ &\leq |f||Au^m| + \frac{\nu|\nabla g|_\infty}{m_0}|u^m||Au^m| + c_3|u^m|^{1/2}\|u^m\||Au^m|^{3/2}\\ &\leq \frac{1}{2\epsilon}|f|^2 + \epsilon|Au^m|^2 + \frac{\nu|\nabla g|_\infty}{m_0\lambda_1^{1/2}}|Au^m|^2 + \frac{\nu|\nabla g|_\infty}{4m_0\lambda_1^{1/2}}\|u^m\|^2 + c_3'\|u^m\|^3.\end{aligned}$$

Hence using (3.11) we deduce that

$$\nu(1 - \frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}} - \epsilon)|Au^m|^2 \leq C(|f|, \nu, \lambda_1, |\nabla g|_\infty), \tag{3.14}$$

where $\epsilon > 0$ is chosen such that $\gamma_0 - \epsilon > 0$. Hence we deduce that the sequence $\{u^m\}$ is bounded in $D(A)$, and consequently, by the compact injection of $D(A)$ in V_g, we can extract a subsequence $\{u^{m'}\} \subset \{u^m\}$ that converges weakly in $D(A)$ and strongly in V_g to an element $u^* \in D(A)$. It is now standard to take limits in (3.13) and to obtain that u^* is a strong stationary solution of (1.1).

(ii) Uniqueness. Suppose that u^* and v^* are two strong stationary solutions of (1.1). Then

$$\nu\langle Au^* - Av^*, v\rangle_g + b(u^*, u^*, v) - b(v^*, v^*, v) + \nu(Cu^* - Cv^*, v)_g = 0$$

for all $v \in V_g$. Taking $v = u^* - v^*$, we have

$$\nu\langle Au^* - Av^*, u^* - v^*\rangle_g = b(u^* - v^*, v^*, u^* - v^*) - \nu(Cu^* - Cv^*, u^* - v^*)_g.$$

Hence

$$\nu||u^* - v^*||^2 \leq c_1\lambda_1^{-1/2}||u^* - v^*||^2||v^*|| + \frac{\nu|\nabla g|_\infty}{m_0\lambda_1^{1/2}}||u^* - v^*||^2,$$

or

$$\nu(1 - \frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}})||u^* - v^*||^2 \leq c_1\lambda_1^{-1/2}||u^* - v^*||^2||v^*||. \tag{3.15}$$

From (3.11) and (3.15) we have

$$\Big[\nu(1 - \frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}})\Big]^2||u^* - v^*||^2 \leq c_1\lambda_1^{-1}|f|||u^* - v^*||^2, \tag{3.16}$$

and the uniqueness follows from (3.12) and (3.16). □

Theorem 3.6. *If $f \in H_g$ and condition* (3.12) *is satisfied, then for any solution* $u(\cdot)$ *of problem* (1.1) *we have*

$$|u(t) - u^*| \to 0 \quad \textit{as } t \to \infty.$$

Proof. Denote $w(t) = u(t) - u^*$, one has

$$\frac{d}{dt}(w(t), v)_g + \nu((w(t), v))_g + \nu(Cw(t), v)_g + b(u(t), u(t), v) - b(u^*, u^*, v) = 0, \ \forall v \in V_g.$$

Replacing v by $w(t)$ and noting that

$$b(u(t), u(t), w(t)) - b(u^*, u^*, w(t)) = b(w(t), u^*, w(t)),$$

we get

$$\frac{d}{dt}(w(t), w(t))_g + \nu((w(t), w(t)))_g + \nu(Cw(t), w(t))_g + b(w(t), u^*, w(t)) = 0.$$

Introducing an exponential term $e^{\lambda t}$ with a positive value λ to be fixed later on, by Lemmas 2.1 and 2.3, we have

$$\begin{aligned}
&\frac{d}{dt}(e^{\lambda t}|w(t)|^2) \\
&= e^{\lambda t}\Big[\lambda|w(t)|^2 - 2\nu\|w(t)\|^2 - 2\nu(Cw(t), w(t))_g - 2b(w(t), u^*, w(t))\Big] \\
&\le e^{\lambda t}\Big[\lambda|w(t)|^2 - 2\nu\|w(t)\|^2 + \frac{2\nu|\nabla g|_\infty}{m_0\lambda_1^{1/2}}||w(t)||^2 + 2c_1\|u^*\||w(t)|\|w(t)\|\Big] \\
&\le e^{\lambda t}\Big[\frac{\lambda}{\lambda_1} - 2\nu + \frac{2\nu|\nabla g|_\infty}{m_0\lambda_1^{1/2}} + \frac{2c_1}{\lambda_1^{1/2}}\|u^*\|\Big]\|w(t)\|^2 \\
&\le e^{\lambda t}\Big[\frac{\lambda}{\lambda_1} - 2\nu + \frac{2\nu|\nabla g|_\infty}{m_0\lambda_1^{1/2}} + \frac{2c_1|f|}{\lambda_1\nu(1 - \frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}})}\Big]\|w(t)\|^2,
\end{aligned}$$

where we have used the estimate (3.11) for the stationary solution u^*.

If condition (3.12) holds, then we can choose $\lambda > 0$ such that

$$\frac{\lambda}{\lambda_1} - 2\nu + \frac{2\nu|\nabla g|_\infty}{m_0\lambda_1^{1/2}} + \frac{2c_1|f|}{\lambda_1\nu(1 - \frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}})} < 0.$$

Hence, we deduce that $|w(t)|^2 \le e^{-\lambda t}|w(0)|^2$, and this completes the proof. □

4. Long-time finite dimensional approximation

Let f_1 and f_2 be two continuous bounded functions from $[0, \infty)$ into H_g, let u_0 and v_0 be given in V_g, and let u and v denote the corresponding strong solutions to the following problems

$$\begin{cases} \dfrac{du}{dt} + \nu Au + \nu Cu + Bu &= f_1(t), \\ u(0) &= u_0, \end{cases} \tag{4.1}$$

and

$$\begin{cases} \dfrac{dv}{dt} + \nu Av + \nu Cv + Bv &= f_2(t), \\ v(0) &= v_0. \end{cases} \tag{4.2}$$

Let us consider a finite-dimensional subspace E of V_g. We denote by $P(E)$ the orthogonal projector in H_g onto E, and $Q(E) = I - P(E)$. We now show that there exists $\rho(E)$, $0 \le \rho(E) < 1$, such that

$$|((\phi, \psi))_g| \le \rho(E)||\phi||||\psi||, \ \ \forall \phi \in E, \forall \psi \in V_g, \ P(E)\psi = 0.$$

Indeed, if this is not true, there exist two sequences $\phi_j \in E$, $\psi_j \in V_g$, $j \ge 1$, $P(E)\psi_j = 0$ such that

$$\|\phi_j\|\|\psi_j\| \ge |((\phi_j, \psi_j))_g| \ge (1 - \frac{1}{j})\|\phi_j\|\|\psi_j\|.$$

Setting $\phi'_j = \phi_j/\|\phi_j\|, \psi'_j = \psi_j/\|\psi_j\|$, we have

$$1 \ge |((\phi'_j, \psi'_j))_g| \ge (1 - \frac{1}{j}).$$

We can extract a subsequence, still denoted by j, such that ϕ'_j converges strongly in E to $\phi \in E$, $\|\phi\| = 1$ (since E is finite dimension), and ψ'_j converges weakly in V_g to $\psi \in V_g$, $\|\psi\| \le 1$, $P(E)\psi = 0$. Hence, passing to the limit, we have

$$|((\phi, \psi))_g| = 1, \ \|\phi\| = 1, \ \|\psi\| \le 1,$$

so that $\|\psi\| = 1$, $\psi = k\phi \neq 0$, in contradiction with $P(E)\psi = 0$.

We now associate with E the two numbers $\lambda(E)$, $\mu(E)$,

$$\lambda(E) = \inf\{||\phi||^2, \ \phi \in V_g, \ P(E)\phi = 0, \ |\phi| = 1\},$$
$$\mu(E) = \sup\{||\psi||^2, \ \psi \in E, \ |\psi| = 1\},$$

so that

$$|\phi| \le \lambda(E)^{-1/2}||\phi||, \ \ \forall \phi \in V_g, \ P(E)\phi = 0, \ \text{and} \ ||\psi|| \le \mu(E)^{1/2}|\psi|, \ \ \forall \psi \in E.$$

We now prove the following result.

Theorem 4.1. *Assume that u and v are two strong solutions of* (4.1) *and* (4.2)*, respectively. Let E be a finite-dimensional subspace of V_g such that*

$$\lambda(E) > (\frac{c_1\rho_A}{\nu\gamma_0})^2, \tag{4.3}$$

where c_1 is the constant in Lemma 2.1, ρ_A is the constant in Proposition 3.3, $\gamma_0 = 1 - \frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}} > 0$. Then, if

$$|P(E)(u(t) - v(t))| \to 0 \ \ as\, t \to \infty,$$

and

$$|(I - P(E))(f_1(t) - f_2(t))| \to 0 \ \ as\, t \to \infty,$$

we have

$$|u(t) - v(t)| \to 0 \quad as\, t \to \infty.$$

Proof. For brevity, we will write P, Q, ρ, λ, μ instead of $P(E)$, and so forth.

We consider the two solutions u, v of (4.1) and (4.2), and set

$$w = u - v, \;\; p = Pw, \; q = Qw, \; e = f_1 - f_2.$$

We apply the operator Q to the difference between (4.1) and (4.2) to obtain

$$\frac{dq}{dt} + \nu QAw + \nu QCw + QB(v, w) + QB(w, u) = Qe.$$

We then take the scalar product in H_g with q,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|q|^2 + \nu||q||^2 = (Qe, q)_g - \nu((p, q))_g - \nu(Cq, q)_g \\ - \nu(Cp, q)_g - (B(v, p), q)_g - (B(p, u), q)_g - (B(q, u), q)_g.\end{aligned}$$

Using Lemmas 2.1 and 2.3, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|q|^2 + \nu||q||^2 \le \frac{1}{\lambda_1^{1/2}}|Qe|||q|| + \nu\rho\mu^{1/2}|p|||q|| + \nu\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}||q||^2 \\ + \nu\frac{|\nabla g|_\infty}{m_0}|p|||q|| + c_1(|Au| + |Av|)|p|||q|| + c_1|Au||q|||q||.\end{aligned}$$

By Proposition 3.3, there exists a number $T > 0$ such that

$$|Au(t)| \le \rho_A, \; |Av(t)| \le \rho_A \;\text{ for all } t \ge T.$$

Hence if $t \ge T$, using Cauchy's inequality we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|q|^2 + \nu||q||^2 &\le \frac{1}{\lambda_1^{1/2}}|Qe|||q|| + \nu\rho\mu^{1/2}|p|||q|| + \nu\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}||q||^2 \\ &\quad + \nu\frac{|\nabla g|_\infty}{m_0}|p|||q|| + 2c_1\rho_A|p|||q|| + c_1\rho_A\lambda(E)^{-1/2}||q||^2 \\ &\le \frac{\nu\epsilon}{4}||q||^2 + \frac{1}{\nu\lambda_1\epsilon}|Qe|^2 + \frac{\nu\epsilon}{4}||q||^2 + \frac{\nu\rho^2\mu}{\epsilon}|p|^2 \\ &\quad + \nu\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}||q||^2 + \frac{\nu\epsilon}{4}||q||^2 + \frac{\nu|\nabla g|_\infty^2}{m_0^2\epsilon}|p|^2 + \frac{\nu\epsilon}{4}||q||^2 \\ &\quad + \frac{4c_1\rho_A}{\nu\epsilon}|p|^2 + c_1\rho_A\lambda(E)^{-1/2}||q||^2.\end{aligned}$$

Then, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|q|^2 + (\nu(\gamma_0 - \epsilon) - c_1\rho_A\lambda(E)^{-1/2})||q||^2 \\ &\le \frac{1}{\nu\lambda_1\epsilon}|Qe|^2 + (\frac{\nu\rho^2\mu}{\epsilon} + \frac{\nu|\nabla g|_\infty^2}{m_0^2\epsilon} + \frac{4c_1\rho_A}{\nu\epsilon})|p|^2.\end{aligned}$$

Choosing $\epsilon > 0$ such that $\nu(\gamma_0 - \epsilon) - c_1\rho_A\lambda(E)^{-1/2} > 0$, we deduce that

$$\frac{d}{dt}|q|^2 + \nu_1\lambda|q|^2 \le c_1'|Qe|^2 + c_2'|p|^2,$$

where $\nu_1 = 2(\nu(\gamma_0 - \epsilon) - c_1\rho_A\lambda(E)^{-1/2})$, $c_1' = \frac{2}{\nu\lambda_1\epsilon}$ and $c_2' = 2(\frac{\nu\rho^2\mu}{\epsilon} + \frac{\nu|\nabla g|_\infty^2}{m_0^2\epsilon} + \frac{4c_1\rho_A}{\nu\epsilon})$. Whence for $t \ge t_0 \ge T$,

$$|q(t)|^2 \le |q(t_0)|^2 e^{-\nu_1\lambda(t-t_0)} + \int_{t_0}^t [c_1'|e(\tau)|^2 + c_2'|p(\tau)|^2] e^{-\nu_1\lambda(t-\tau)} d\tau. \tag{4.4}$$

Given $\delta > 0$, there exists M (which we can assume $\ge T$) such that for $t \ge M$,

$$|P(u(t) - v(t))|^2 \le \delta, \quad |(I-P)(f_1(t) - f_2(t))|^2 \le \delta.$$

Therefore, for $t \ge t_0 + M$, (4.4) implies

$$\begin{aligned}|q(t)|^2 \le c(u,v)e^{-\nu_1\lambda(t-t_0)} &+ \delta(c_1' + c_2')\int_{t-M}^t e^{-\nu_1\lambda(t-\tau)} d\tau \\ &+ [c_1'c(f_1,f_2) + c_2'c(u,v)]\int_{t_0}^{t-M} e^{-\nu_1\lambda(t-\tau)} d\tau,\end{aligned}$$

where

$$c(u,v) = \sup_{t\ge t_0}|u(t) - v(t)|, \quad c(f_1,f_2) = \sup_{t\ge t_0}|f_1(t) - f_2(t)|.$$

As $t \to \infty$, then

$$\limsup_{t\to\infty}|q(t)|^2 \le \delta(c_1' + c_2')\frac{1 - e^{-\nu_1\lambda M}}{\nu_1\lambda} + [c_1'c(f_1,f_2) + c_2'c(u,v)]\frac{e^{-\nu_1\lambda M}}{\nu_1\lambda}.$$

Letting $\delta \to 0$ and then $M \to \infty$, we get the desired result. □

Remark 4.1. Theorem 4.1 shows that if condition (4.3) is satisfied, then the behavior for $t \to +\infty$ of $u(t)$ is completely determined by that of $P(E)u(t)$.

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