

NEWFORMS OF LEVEL 4 AND OF TRIVIAL CHARACTER

YICHAO ZHANG

ABSTRACT. In this paper, we consider characters of $SL_2(\mathbb{Z})$ and then apply them to newforms of integral weight, level 4 and of trivial character. More precisely, we prove that all of them are actually level 1 forms of some nontrivial character. As a byproduct, we prove that they all are eigenfunctions of the Fricke involution with eigenvalue -1 .

Introduction

The Fricke involution W_N of level N , also known as the canonical involution, acts on the space of newforms of level N , integral weight k , and trivial character. Here k is necessarily even and positive. It is well-known that Hecke eigenforms behave well under the Fricke involution. More specifically, if f is a normalized Hecke eigenform of some level N , weight k and of trivial character, then we have $f|_k W_N = cg$ with $c \in \mathbb{C}^\times$ and g another normalized Hecke eigenform in the same space (see Lemma 1.1 below or Theorem 4.6.16 in [6]). The Fourier coefficients of g can be explicitly determined by that of f but the scalar c is left mysterious in general.

Question 1. Can we explicitly determine c with the information on f ?

Let $f = \sum_{n=1}^{\infty} a_n q^n$, with $q = e^{2\pi iz}$. If N is square-free, one can express c in terms of a_p with $p \mid N$; for example, one can compute it explicitly using Corollary 4.6.18 in [6] and a similar result as Lemma 2.1 in [10]. For example, if $N = p$ is a prime, then $g = f$ and $c = -p^{1-\frac{k}{2}} a_p \in \{\pm 1\}$. The determination of c in the case of non-square-free N is more subtle, due the vanishing of some Fourier coefficients a_p when $p^2 \mid N$. When $N = 4N_1$ with N_1 odd, Yang showed that the eigenvalue for the W_4 -operator is always -1 in his unpublished note [7]. We shall prove the same result in the simplest case, namely the case when $N = 4$, using a very different approach.

On the other hand, unlike the Lie group $SL_2(\mathbb{R})$, $SL_2(\mathbb{Z})$ has non-trivial finite dimensional representations. For example, each discriminant form of even

Received February 28, 2014; Revised August 28, 2014.

2010 *Mathematics Subject Classification.* Primary 11F11, 11F06.

Key words and phrases. Fricke involution, non-Dirichlet character.

signature gives a finite dimensional unitary representation of $\mathrm{SL}_2(\mathbb{Z})$, the well-known Weil representations. Weil representations are crucial in Borcherds's theory of singular theta lift; see [2] and also see [3], [10] and [9] for restating Borcherds's theory in terms of scalar-valued modular forms. In this paper, we consider one-dimensional representations, or characters, of $\mathrm{SL}_2(\mathbb{Z})$. Note that one-dimensional Weil representation is the trivial representation. Recall that the modular group

$$\mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/\{\pm I\} = \langle S, T : S^2 = (ST)^3 = I \rangle,$$

where S, T denote the corresponding images in $\mathrm{PSL}_2(\mathbb{Z})$ of the pair of standard generators of $\mathrm{SL}_2(\mathbb{Z})$. We can use instead the pair of generators S and ST , and it follows that $\mathrm{PSL}_2(\mathbb{Z})$, hence $\mathrm{SL}_2(\mathbb{Z})$, possess some non-trivial characters χ . From the defining relations, we can find all such characters easily. For example, an explicit example of such a χ of (maximal) order 12 can be found in [4]. If we restrict our attention to the modular group $\mathrm{PSL}_2(\mathbb{Z})$, such χ has order at most 6. Note in particular that these characters are not induced by Dirichlet characters on the level, and we call them *non-Dirichlet* characters to allow potential consideration of congruence subgroups. Denote by $\mathcal{S}(\mathrm{SL}_2(\mathbb{Z}), k, \chi)$ the space of modular forms of weight k and type χ for $\mathrm{SL}_2(\mathbb{Z})$ (see the next section for the precise definition). A natural question then arises:

Question 2. Can the space $\mathcal{S}(\mathrm{SL}_2(\mathbb{Z}), k, \chi)$ be non-zero for some weight k and some non-trivial character χ ?

In this paper, we give a positive answer to this question by considering a real non-trivial character χ .

It turns out that Questions 1 and 2 are related. They can be both answered by an isomorphism between the space of newforms of level 4 and trivial character and that of modular forms of level 1 and type χ (Theorem 1.2). Here χ is the unique non-trivial real character of $\mathrm{PSL}_2(\mathbb{Z})$ and it satisfies $\chi(S) = \chi(T) = -1$. In other words, the level 4 newforms are actually level 1 forms, that is, they are “oldforms” when above non-Dirichlet characters are taken into account. This is analogous to the results in [10] and [9], where by considering more general *types* (Weil representations), we proved that forms in a subspace of the space of modular forms of some level N and character $\left(\frac{N}{\cdot}\right)$ are actually level 1 (vector-valued) modular forms.

The isomorphism given in Theorem 1.2 answers Question 2 directly and the first non-zero space appears when $k = 6$. It also answers Question 1; under the isomorphism, we show in Theorem 1.3 that the eigenvalue of the Fricke involution is given by $\chi(S) = -1$.

In Section 1, we set up the notations and state the main results. In Section 2, we prove some results of modular forms between different levels (or rather, different congruence groups), and also consider possible real characters of $\mathrm{PSL}_2(\mathbb{Z})$. In the last section, we give the proof of Theorem 1.2 and Theorem 1.3 and end the paper with an example.

Acknowledgments. The author would like to thank Yifan Yang for sending me his unpublished note on newforms of level $4N$, and Wen-Ching Winnie Li for pointing out Yang's result to me. The author is also grateful to Henry H. Kim and Kyu-Hwan Lee for their support. Thanks also go to the reviewer, who has made very useful comments on the proofs.

1. Statements of main theorems

In this section, we fix the notations and state the main results. For unexplained notations and terminology, we refer the readers to any standard textbook on modular forms, for example [6].

For a positive integer N , we have congruence subgroups $\Gamma_0(N)$ and $\Gamma(N)$ defined as follows:

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{N} \right\},$$

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\},$$

where $*$ means no restriction on the corresponding entry. We shall be only interested in the case when $N = 2$ or 4 .

Let k be an even positive integer and \mathbb{H} the upper half plane. Recall that for a real matrix M of positive determinant and a function f on \mathbb{H} , the weight- k slash operator of M is defined by

$$(f|_k M)(\tau) = \det(M)^{\frac{k}{2}} (c\tau + d)^{-k} f(M\tau), \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $M\tau = \frac{a\tau + b}{c\tau + d}$. Let Γ be any congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and χ be any character (homomorphisms into \mathbb{C}^\times of finite order) of Γ . Denote by $\mathcal{S}(\Gamma, k, \chi)$ the space of cusp forms for Γ of weight k and of character χ , namely holomorphic functions f on \mathbb{H} such that $f|_k M = \chi(M)f$ for all $M \in \Gamma$ and f vanishes at cusps. If $\Gamma = \Gamma_0(N)$ and χ is a Dirichlet character modulo N , we denote $\mathcal{S}^{\mathrm{new}}(\Gamma_0(N), k, \chi)$ the subspace of newforms for $\Gamma_0(N)$ of weight k and of character χ . Recall that the Fricke involution, defined by the weight k slash operator of

$$W_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix},$$

acts on the space $\mathcal{S}^{\mathrm{new}}(\Gamma_0(N), k, 1)$. We have also the Hecke operators T_p , for $p \nmid N$ a rational prime, and U_p for $p \mid N$, and we write down the action of U_p explicitly as follows

$$f|_k U_p = p^{\frac{k}{2}-1} \sum_{j \pmod{p}} f \Big|_k \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix}.$$

Note that there are different normalizations in the literature on the Hecke operators. For ease of notations, we denote

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, T_{1/2} = \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix}, V_N = \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix}.$$

Assume that f has Fourier expansion $\sum_n a_n q^n$ at ∞ ; here $q = e^{2\pi i\tau}$. Recall that a non-zero f is called a Hecke eigenform or a primitive form if f is a common eigenfunction for all the Hecke operators T_p . If so, $a_1 \neq 0$ and f is called normalized if $a_1 = 1$. We recall the following well-known result:

Lemma 1.1. *Let f be a normalized Hecke eigenform in $\mathcal{S}^{\text{new}}(\Gamma_0(N), k, 1)$ and let $f = \sum_n a_n q^n$ be its Fourier expansion at ∞ . Then*

- (1) $f|_k T_p = a_p f$ for each $p \nmid N$.
- (2) f is an eigenfunction for U_p and $f|_k U_p = a_p f$ for each $p \mid N$.
- (3) $f|_k W_N = cg$, where $c \in \mathbb{C}^\times$, $g = \sum_n b_n q^n$ is a normalized Hecke eigenform such that $b_p = a_p$ if $p \nmid N$ and $b_p = \overline{a_p}$ if $p \mid N$.

This lemma is part of the theory of newforms, also known as Atkin-Lehner-Li theory ([1] and [5]). Statements (1) and (2) are standard, and one may see Chapter 4 of [6]. To work out a proof of (3), keeping in mind that $\chi = 1$, we first apply Lemma 1.1 of [10], which slightly generalizes Theorem 4.6.16(3) of [6], to write the canonical involution W_N as a product of local operators. We then apply Theorem 4.6.16(4) of [6] to conclude the proof.

Let χ be the non-trivial real character on $\text{PSL}_2(\mathbb{Z})$ such that $\chi(S) = \chi(T) = -1$. Actually this is the unique one (see Lemma 2.1 below).

Now we state our main theorems.

Theorem 1.2. *The map*

$$\begin{aligned} \mathcal{S}(SL_2(\mathbb{Z}), k, \chi) &\rightarrow \mathcal{S}^{\text{new}}(\Gamma_0(4), k, 1) \\ f(\tau) &\mapsto g(\tau) = f(2\tau) \end{aligned}$$

defines an isomorphism.

As a byproduct, we obtain:

Theorem 1.3. *For any $g \in \mathcal{S}^{\text{new}}(\Gamma_0(4), k, 1)$, we have $g|_k W_4 = -g$.*

For any $g = \sum_n a_n q^n \in \mathcal{S}^{\text{new}}(\Gamma_0(4), k, 1)$, we have its L-function

$$L(s, g) = \sum_n a_n n^{-s},$$

and its completed L-function $\Lambda(s, g) = \pi^{-s} \Gamma(s) L(s, g)$. It is well-known that $\Lambda(s, g)$ can be analytically continued to the whole s -plane and satisfies the following functional equation

$$\Lambda(s, g) = i^k \Lambda(k - s, g|_k W_4).$$

With Theorem 1.3, this functional equation can be made more precise:

Corollary 1.4. *For any $g \in \mathcal{S}^{\text{new}}(\Gamma_0(4), k, 1)$, we have $\Lambda(s, g) = -i^k \Lambda(k - s, g)$.*

2. Modular forms between different levels

As before, we denote also by S, T their images in $\text{PSL}_2(\mathbb{Z})$ respectively. It is well-known that $\text{PSL}_2(\mathbb{Z})$ is generated by S, T ; more precisely,

$$\text{PSL}_2(\mathbb{Z}) = \langle S, T : S^2 = (ST)^3 = I \rangle.$$

We consider characters χ of $\text{PSL}_2(\mathbb{Z})$; that is, homomorphisms $\chi: \text{PSL}_2(\mathbb{Z}) \rightarrow \mathbb{C}^\times$ with finite images. In case of real characters, we have the following elementary lemma.

Lemma 2.1. *Let χ be a character of $\text{PSL}_2(\mathbb{Z})$. Then χ is a real character if and only if $\chi(T)$ is real, in which case either χ is trivial or χ is the unique character such that $\chi(S) = \chi(T) = -1$.*

Proof. It is clear that $\chi(S) \in \{\pm 1\}$ and $\chi(T)$ is a sixth root of unity since $S^2 = (ST)^3 = I$, and the first statement follows.

If χ is real, then $\chi(S), \chi(T) \in \{\pm 1\}$. Because of the defining relation $(ST)^3 = I$, we must have $\chi(S) = \chi(T)$. Now that $S^2 = (ST)^3 = I$ are the only definition relations, the conditions $\chi(S) = \chi(T) = -1$ indeed give a character. □

Proposition 2.2. *Let χ be a real character of $\text{PSL}_2(\mathbb{Z})$. We have $f \in \mathcal{S}(SL_2(\mathbb{Z}), k, \chi)$ if and only if $f \in \mathcal{S}(\Gamma(2), k, 1)$ and $f|_k S = \chi(S)f, f|_k T = \chi(T)f$.*

Proof. We first recall the well-known fact that $\Gamma_0(4)$, considered in $\text{PSL}_2(\mathbb{Z})$, is generated by T^2 and ST^4S (for a proof, see page 26 in [8]). Since $\Gamma(2)$ and $\Gamma_0(4)$ are conjugate via V_2 , we see that $\Gamma(2)$ is generated by T^2 and ST^2S . So clearly $\chi(\Gamma(2)) = \{1\}$, and the proposition follows. □

Corollary 2.3. *Let χ be a real character of $\text{PSL}_2(\mathbb{Z})$. We have $f \in \mathcal{S}(SL_2(\mathbb{Z}), k, \chi)$ if and only if $g(\tau) = f(2\tau) \in \mathcal{S}(\Gamma_0(4), k, 1)$ and $g|_k W_4 = \chi(S)g, g|_k T_{1/2} = \chi(T)g$.*

Proof. Since $\Gamma(2)$ and $\Gamma_0(4)$ are conjugate, $f(\tau) \mapsto f(2\tau)$ defines an isomorphism between the spaces of cusp forms on $\Gamma(2)$ and on $\Gamma_0(4)$. Under such conjugation, when considered as operators, S corresponds to W_4 and T corresponds to $T_{1/2}$, so the corollary follows from the previous proposition. □

3. Proof of Theorems 1.2 and 1.3

We begin with the following lemma.

Lemma 3.1. *If $g \in \mathcal{S}^{\text{new}}(\Gamma_0(4), k, 1)$, then $g|_k U_2 = 0$. Consequently, $g|_k T_{1/2} = -g$, and if g is a Hecke eigenform, then $g|_k W_4 = cg$ for $c \in \{\pm 1\}$.*

Proof. Since $\mathcal{S}^{\text{new}}(\Gamma_0(4), k, 1)$ contains a basis of Hecke eigenforms (see Theorem 4.6.13 of [6]), we may assume that g is a Hecke eigenform. We then have $g|_k U_2 = a_2 g$, and we only need to prove that $a_2 = 0$, which in turn follows from Theorem 4.6.17 in [6].

Note that $g|_k U_2 = 0$ implies that only odd powers in q can appear in the Fourier expansion of g at ∞ , and we must have $g(\tau + \frac{1}{2}) = -g(\tau)$. The last statement follows from Theorem 4.6.16 in [6], since $a_2 = 0$ and hence real. \square

Proof of Theorem 1.3. By Lemma 2.1 and Lemma 3.1, the involution W_4 on $\mathcal{S}^{\text{new}}(\Gamma_0(4), k, 1)$ is diagonalizable with eigenvalues being either $+1$ or -1 . Therefore we may decompose the space into the direct sum of the plus space and the minus space according to the sign of the eigenvalues.

Suppose the plus space is not zero, and let g be one of the non-zero forms therein, that is, $g|_k W_4 = g$. Now consider $f(\tau) = g(\frac{\tau}{2})$, and we know that $f \in \mathcal{S}(\Gamma(2), k, 1)$. From the transformation behavior of g (see Lemma 3.1) and the isomorphism between $\Gamma_0(4)$ and $\Gamma(2)$, we see that

$$f|_k S = f, \quad \text{and} \quad f|_k T = -f.$$

Since S, T generate $\text{PSL}_2(\mathbb{Z})$, the one-dimensional subspace $\mathbb{C}f$ is invariant under the action of $\text{PSL}_2(\mathbb{Z})$. Hence we have a character χ of $\text{PSL}_2(\mathbb{Z})$ such that $f|_k M = \chi(M)f$ for any $M \in \text{PSL}_2(\mathbb{Z})$. Since $\chi(S), \chi(T) \in \{\pm 1\}$, χ is a real character. But that $\chi(S) = 1$ and $\chi(T) = -1$ contradicts Lemma 2.1. This completes the proof of Theorem 1.3. \square

From now on, let χ be the unique character of $\text{PSL}_2(\mathbb{Z})$ such that $\chi(S) = \chi(T) = -1$. With little more effort, we may give a proof of Theorem 1.2.

Proof of Theorem 1.2. If $g \in \mathcal{S}^{\text{new}}(\Gamma_0(4), k, 1)$, then by Theorem 1.3 and Lemma 3.1,

$$g|_k W_4 = -g, \quad g|_k T_{1/2} = -g.$$

Hence by Corollary 2.3, we see that $f(\tau) = g(\frac{\tau}{2}) \in \mathcal{S}(\text{SL}_2(\mathbb{Z}), k, \chi)$.

Conversely, if $f \in \mathcal{S}(\text{SL}_2(\mathbb{Z}), k, \chi)$, then by Corollary 2.3, $g(\tau) = f(2\tau) \in \mathcal{S}(\Gamma_0(4), k, 1)$ and it satisfies

$$g|_k W_4 = -g, \quad g|_k T_{1/2} = -g.$$

We need to prove that g lies in the space of newforms. Assume that $g = g_0 + g_1$ with g_0 a newform and g_1 an oldform, so we only have to prove that $g_1 = 0$. By Corollary 2.3, g_0 also satisfies above transformation rule under W_4 and $T_{1/2}$, so does g_1 , namely

$$g_1|_k W_4 = -g_1, \quad g_1|_k T_{1/2} = -g_1.$$

By Lemma 4.6.9 of [6],

$$g_1(\tau) = \sum_{m=1,2} \sum_{l|\frac{\tau}{m}} h_{m,l}(l\tau), \quad h_{m,l} \in \mathcal{S}^{\text{new}}(\Gamma_0(m), k, 1).$$

But the transformation rule under $T_{1/2}$ says that the q -expansion of g_1 contains only odd powers of q , which forces

$$g_1(\tau) = \sum_{m=1,2} h_{m,1}(\tau), \quad h_{m,1} \in \mathcal{S}^{\text{new}}(\Gamma_0(m), k, 1).$$

In particular, $g_1 \in \mathcal{S}(\Gamma_0(2), k, 1)$. But $ST^2S \in \Gamma_0(2)$, so $g_1|_k ST^2 = g_1|_k S$. Now $g_1|_k W_4 = -g_1$ implies that $g_1|_k S = -g_1|_k V_4^{-1}$, and therefore

$$g_1|_k V_4^{-1} T^2 V_4 = -g_1|_k ST^2 V_4 = -g_1|_k S V_4 = -g_1|_k W_4 = g_1.$$

But $V_4^{-1} T^2 V_4 = T_{1/2}$, so

$$g_1 = g_1|_k V_4^{-1} T^2 V_4 = g_1|_k T_{1/2} = -g_1,$$

and $g_1 = 0$.

Since the maps are clearly inverse to each other, Theorem 1.2 follows. \square

We end this note with the following example.

Example 3.2. We set $k = 6$ and we know that the space $\mathcal{S}^{\text{new}}(\Gamma_0(4), 6, 1)$ is one-dimensional and generated by the Hecke eigenform

$$g = q - 12q^3 + 54q^5 - 88q^7 - 99q^9 + 540q^{11} - 418q^{13} + O(q^{15}).$$

Therefore, by Theorem 1.2, we see that the $\mathcal{S}(\text{SL}_2(\mathbb{Z}), 6, \chi)$ is one-dimensional and generated by

$$f = q^{1/2} - 12q^{3/2} + 54q^{5/2} - 88q^{7/2} - 99q^{9/2} + 540q^{11/2} - 418q^{13/2} + O(q^{15/2}).$$

References

- [1] A. O. L. Atkin and J. Lehner, *Hecke operators on $\Gamma_0(m)$* , Math. Ann. **185** (1970), no. 2, 134–160.
- [2] R. E. Borcherds, *Automorphic forms with singularities on Grassmannians*, Invent. Math. **132** (1998), no. 3, 491–562.
- [3] J. H. Bruinier and M. Bundschuh, *On Borcherds products associated with lattices of prime discriminant*, Ramanujan J. **7** (2003), no. 1-3, 49–61.
- [4] Keith Conrad, *SL₂(\mathbb{Z})*, [http://www.math.uconn.edu/~kconrad/blurbs/grouptheory/SL\(2, \$\mathbb{Z}\$ \).pdf](http://www.math.uconn.edu/~kconrad/blurbs/grouptheory/SL(2,Z).pdf).
- [5] W.-C. W. Li, *Newforms and functional equations*, Math. Ann. **212** (1975), no. 4, 285–315.
- [6] T. Miyake, *Modular Forms*, Springer Verlag, 2006.
- [7] Y. Yang, *Newforms of level $4N$* , Unpublished, 2010.
- [8] D. Zagier, *Elliptic modular forms and their applications*, The 1-2-3 of modular forms, 1–103, Universitext, Springer, 2008.
- [9] Y. Zhang, *Zagier duality and integrality of Fourier coefficients for weakly holomorphic modular forms*, preprint, arXiv: 1308.1037, 2013.
- [10] ———, *An isomorphism between scalar-valued modular forms and modular forms for Weil representations*, preprint, arXiv:1307.4390, 2014.

DEPARTMENT OF MATHEMATICS
 UNIVERSITY OF CONNECTICUT
 STORRS, CT 06269, USA
 E-mail address: yichao.zhang@uconn.edu