KYUNGPOOK Math. J. 54(2014), 501-509
http://dx.doi.org/10.5666/KMJ.2014.54.3.501

The Order of Normal Form Generalized Hypersubstitutions of Type $\tau = (2)$

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Abstract. In 2000, K. Denecke and K. Mahdavi showed that there are many idempotent elements in $Hyp_{N_\varphi}(V)$ the set of normal form hypersubstitutions of type $\tau = (2)$ which are not idempotent elements in $Hyp(2)$ the set of all hypersubstitutions of type $\tau = (2)$. They considered in which varieties, idempotent elements of $Hyp(2)$ are idempotent elements of $Hyp_{N_\varphi}(V)$. In this paper, we study the similar problems on the set of all generalized hypersubstitutions of type $\tau = (2)$ and the set of all normal form generalized hypersubstitutions of type $\tau = (2)$ and determine the order of normal form generalized hypersubstitutions of type $\tau = (2)$.

1. Introduction

The order of generalized hypersubstitutions of type $\tau = (2)$ was studied by W. Puninagool and S. Leeratanavalee [6]. In this paper, we used the order of generalized hypersubstitutions of type $\tau = (2)$ as a tool to characterize the order of normal form generalized hypersubstitutions of type $\tau = (2)$.

A generalized hypersubstitution of type $\tau = (n_i)_{i \in I}$ is a mapping σ which maps each n_i-ary operation symbol to the set $W_\tau(X)$ of all terms of type τ built up by operation symbols from $\{f_i | i \in I\}$ where f_i is n_i-ary and variables from a countably infinite alphabet $X := \{x_1, x_2, x_3, ...\}$ which dose not necessarily preserve the arity. We denote the set of all generalized hypersubstitutions of type τ by $Hyp_G(\tau)$. To

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Received September 28, 2012; accepted April 15, 2013.
2010 Mathematics Subject Classification: 08A40, 20M07.
Key words and phrases: Order, normal form generalized hypersubstitution, idempotent element.

define a binary operation on $Hyp_G(\tau)$, we define at first the concept of *generalized superposition of terms* $S^m : W_\tau(X)^{m+1} \longrightarrow W_\tau(X)$ by the following steps:

(i) If $t = x_j, 1 \leq j \leq m$, then $S^m(x_j, t_1, ..., t_m) := t_j$.

(ii) If $t = x_j, m < j \in \mathbb{N}$, then $S^m(x_j, t_1, ..., t_m) := x_j$.

(iii) If $t = f_i(s_1, ..., s_{n_i})$, then
$S^m(t, t_1, ..., t_m) := f_i(S^m(s_1, t_1, ..., t_m), ..., S^m(s_{n_i}, t_1, ..., t_m))$.

We extend a generalized hypersubstitution σ to a mapping $\hat{\sigma} : W_\tau(X) \longrightarrow W_\tau(X)$ inductively defined as follows:

(i) $\hat{\sigma}[x] := x \in X$,

(ii) $\hat{\sigma}[f_i(t_1, ..., t_{n_i})] := S^{n_i}(\sigma(f_i), \hat{\sigma}[t_1], ..., \hat{\sigma}[t_{n_i}])$, for any n_i-ary operation symbol f_i, supposed that $\hat{\sigma}[t_j], 1 \leq j \leq n_i$ are already defined.

Then we define a binary operation $\circ_G$ on $Hyp_G(\tau)$ by $\sigma_1 \circ_G \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$ where $\circ$ denotes the usual composition of mappings and $\sigma_1, \sigma_2 \in Hyp_G(\tau)$. Let σ_{id} be the hypersubstitution which maps each n_i-ary operation symbol f_i to the term $f_i(x_1, ..., x_{n_i})$.

In [3], S. Leeratanavalee and K. Denecke proved that: For arbitrary terms $t, t_1, ..., t_n \in W_\tau(X)$ and for arbitrary generalized hypersubstitutions $\sigma, \sigma_1, \sigma_2$ we have

(i) $S^n(\sigma[t], \sigma[t_1], ..., \sigma[t_n]) = \hat{\sigma}[S^n(t, t_1, ..., t_n)]$,

(ii) $(\hat{\sigma}_1 \circ \sigma_2)\hat{} = \hat{\sigma}_1 \circ \hat{\sigma}_2$.

It turns out that $\underline{Hyp_G(\tau)} = (Hyp_G(\tau); \circ_G, \sigma_{id})$ is a monoid where σ_{id} is the identity element and the set of all hypersubstitutions of type τ forms a submonoid of $\underline{Hyp_G(\tau)}$.

For more details on generalized hypersubstitutions see [3]. In this paper, we consider the type $\tau = (2)$ with the binary operation symbol, say f. Let W_{x_1} denote the set of all words using only the letter x_1, and dually for W_{x_2}. For $s \in W_{(2)}(X)$, we denote :

$\sigma_s :=$ the generalized hypersubstitution which maps the binary operation f to the term s,

$leftmost(s) :=$ the first variable (from the left) that occurs in s,
$rightmost(s) :=$ the last variable (from the right) that occurs in s,
$W^G_{(2)}(\{x_1\}) := \{s \in W_{(2)}(X) | x_1 \in var(s), x_2 \notin var(s)\}$,
$W^G_{(2)}(\{x_2\}) := \{s \in W_{(2)}(X) | x_2 \in var(s), x_1 \notin var(s)\}$,
$W^G := \{t \in W_{(2)}(X) | t \notin X, x_1, x_2 \in var(t)\}$,
$G := \{\sigma_s \in Hyp_G(2) | s \in W_{(2)}(X) \backslash X, x_1, x_2 \notin var(s)\}$,

$P_G(2) := \{\sigma_{x_i} \in Hyp_G(2) | i \in \mathbb{N}, x_i \in X\}$,
$E^G_{x_1} := \{\sigma_{f(x_1,s)} \in Hyp_G(2) | s \in W_{(2)}(X), x_2 \notin var(s)\}$,
$E^G_{x_2} := \{\sigma_{f(s,x_2)} \in Hyp_G(2) | s \in W_{(2)}(X), x_1 \notin var(s)\}$,
$T_1 := \{\sigma_s \in Hyp_G(2) | s \in W^G_{(2)}(\{x_1\})$ and $leftmost(s) = x_m\}$ where $m > 2$,
$T_2 := \{\sigma_s \in Hyp_G(2) | s \in W^G_{(2)}(\{x_2\})$ and $rightmost(s) = x_m\}$ where $m > 2$.

In [6], W. Puninagool and S. Leeratanavalee proved that the following statements hold.

(i) Let σ_t be a generalized hypersubstitution of type $\tau = (2)$. Then σ_t is idempotent if and only if $\hat{\sigma}[t] = t$.

(ii) $P_G(2) \cup E^G_{x_1} \cup E^G_{x_2} \cup G \cup \{\sigma_{id}\}$ is the set of all idempotent elements in $Hyp_G(2)$.

(iii) $T_1 \cup T_2 \cup \{\sigma_{f(x_2,x_1)}\}$ is the set of all elements has order 2 in $Hyp_G(2)$.

(iv) If $\sigma \in Hyp_G(2) \setminus (P_G(2) \cup E^G_{x_1} \cup E^G_{x_2} \cup G \cup \{\sigma_{id}\} \cup T_1 \cup T_2 \cup \{\sigma_{f(x_2,x_1)}\})$, then $\sigma^n \neq \sigma^{n+1}$ for all $n \in \mathbb{N}$ with $n \geq 1$ (i.e. σ has infinite order).

(v) If $\sigma \in Hyp_G(2) \setminus (P_G(2) \cup E^G_{x_1} \cup E^G_{x_2} \cup G \cup \{\sigma_{id}\} \cup T_1 \cup T_2 \cup \{\sigma_{f(x_2,x_1)}\})$, then the length of the word $(\sigma \circ_h \sigma)(f)$ is greater than the length of $\sigma(t)$.

2. Normal Form Generalized Hypersubstitutions

The concept of normal form hypersubstitutions was introduced by J. Płonka in 1994 [5]. In [4], S. Leeratanavalee and K. Denecke generalized the concept of normal form hypersubstitutions to normal form generalized hypersubstitutions. We recall first the definition of V-generalized equivalent.

Definition 2.1. Let V be a variety of type τ. Two generalized hypersubstitutions σ_1 and σ_2 of type τ are called *V-generalized equivalent* if $\sigma_1(f_i) \approx \sigma_2(f_i)$ are identities in V for all $i \in I$. In this case we write $\sigma_1 \sim_{VG} \sigma_2$.

Clearly, the relation $\sim_{VG}$ is an equivalence relation on $Hyp_G(\tau)$ and has the following properties:

Proposition 2.2.([4]) Let V be a variety of type τ and let $\sigma_1, \sigma_2 \in Hyp_G(\tau)$. Then the following are equivalent.

(i) $\sigma_1 \sim_{VG} \sigma_2$.

(ii) For all $t \in W_\tau(X)$ the equation $\hat{\sigma_1}[t] \approx \hat{\sigma_2}[t]$ is an identity in V.

In general, the relation $\sim_{VG}$ is not a congruence relation on $Hyp_G(\tau)$. Let V be a variety of type τ and IdV be the set of identities satisfied in the variety V. If $s \approx t$ is an identity and for any $\sigma \in Hyp_G(\tau), \hat{\sigma}[s] \approx \hat{\sigma}[t] \in IdV$ then $s \approx t$ is called a *strong hyperidentity.* A variety V is called *strongly solid* if every identity in V is satisfied as a strong hyperidentity. For a strongly solid variety V the relation $\sim_{VG}$ is a congruence relation on $Hyp_G(\tau)$ and the factor monoid $\underline{Hyp_G(\tau)/_{\sim_{VG}}}$ exists.

In the arbitrary case we form also $Hyp_G(2)/_{\sim_{VG}}$ and consider a choice function

$$\varphi : Hyp_G(2)/_{\sim_{VG}} \to Hyp_G(2), \text{ with } \varphi([\sigma_{id}]_{\sim_{VG}}) = \sigma_{id}$$

which selects from each equivalence class exactly one element. Then we obtain the set $Hyp_{GN_\varphi}(V) := \varphi(Hyp_G(2)/_{\sim_{VG}})$ of all normal form generalized hypersubstitutions with respect to $\sim_{VG}$ and φ.

On the set $Hyp_{GN_\varphi}(V)$ we define a binary operation

$$\circ_{GN} : Hyp_{GN_\varphi}(V) \times Hyp_{GN_\varphi}(V) \to Hyp_{GN_\varphi}(V)$$

by $\sigma_1 \circ_{GN} \sigma_2 := \varphi(\sigma_1 \circ_G \sigma_2)$. This mapping is well-defined, but in general not associative.

For example, we consider the variety $V = Mod\{(xy)z \approx x(yz), xyuv \approx xuyv, x^3 \approx x\}$. Let f be our binary operation symbol and x_1x_2 abbreviates $f(x_1, x_2)$. So we can construct the set $W_{(2)}(X)/IdV$. These are some elements in $W_{(2)}(X)/IdV$: $[x_1]_{IdV}, [x_2]_{IdV}, [x_m]_{IdV}, [x_1x_m]_{IdV}, [x_mx_1]_{IdV}, [x_2x_m]_{IdV}, [x_mx_2]_{IdV}, [x_1x_1]_{IdV}, [x_2x_2]_{IdV}, [x_1x_2]_{IdV}, [x_2x_1]_{IdV}, [x_mx_k]_{IdV}, [x_mx_kx_n]_{IdV}, [x_mx_1x_n]_{IdV}, [x_mx_2x_n]_{IdV}, [x_mx_nx_1]_{IdV}, [x_mx_1x_1]_{IdV}, [x_mx_2x_1]_{IdV}, [x_mx_mx_2]_{IdV}, [x_mx_1x_2]_{IdV}$, where $m, k, n > 2$.

So we get some corresponding elements in $Hyp_{GN_\varphi}(V)$: $\sigma_{x_1}, \sigma_{x_2}, \sigma_{x_m}, \sigma_{x_1x_m}, \sigma_{x_mx_1}, \sigma_{x_2x_m}, \sigma_{x_mx_2}, \sigma_{x_1x_1}, \sigma_{x_2x_2}, \sigma_{x_1x_2}, \sigma_{x_2x_1}, \sigma_{x_mx_k}, \sigma_{x_mx_kx_n}, \sigma_{x_mx_1x_n}, \sigma_{x_mx_2x_n}, \sigma_{x_mx_nx_1}, \sigma_{x_mx_1x_1}, \sigma_{x_mx_2x_1}, \sigma_{x_mx_mx_2}, \sigma_{x_mx_1x_2}, \sigma_{x_mx_2x_2}, \sigma_{x_1x_nx_m}, \sigma_{x_1x_1x_m}, \sigma_{x_1x_2x_m}, \sigma_{x_1x_mx_1}, \sigma_{x_1x_2x_1}, \sigma_{x_1x_mx_2}, \sigma_{x_1x_1x_2}, \sigma_{x_1x_2x_2}, \sigma_{x_2x_mx_n}$, where $m, k, n > 2$. Since $\sigma_{x_mx_1}, \sigma_{x_1x_2x_2}, \sigma_{x_1x_1} \in Hyp_{GN_\varphi(V)}$ we consider

$$(\sigma_{x_mx_1} \circ_{GN} \sigma_{x_1x_2x_2}) \circ_{GN} \sigma_{x_1x_1} = \sigma_{x_mx_1} \circ_{GN} \sigma_{x_1x_1} = \sigma_{x_mx_1},$$

$$\sigma_{x_mx_1} \circ_{GN} (\sigma_{x_1x_2x_2} \circ_{GN} \sigma_{x_1x_1}) = \sigma_{x_mx_1} \circ_{GN} \sigma_{x_1} = \sigma_{x_1}.$$

So $(Hyp_{GN_\varphi}(V); \circ_{GN}, \sigma_{id})$ is not a monoid.

We call this structure a groupoid of normal form generalized hypersubstitutions. Next, we consider, how to characterize the idempotent elements of $Hyp_{GN_\varphi}(V)$ where V is a variety of semigroups.

Proposition 2.3. Let V be a variety of semigroups and let

$$\varphi : Hyp_G(2)/_{\sim VG} \to Hyp_G(2).$$

be a choice function. Then

(i) $\sigma \in Hyp_{GN_\varphi}(V)$ is an idempotent element iff $\sigma \circ_G \sigma \sim_{VG} \sigma$.

(ii) $\sigma_{x_2x_1} \circ_{GN} \sigma_{x_2x_1} = \sigma_{x_1x_2}$ if $\sigma_{x_2x_1} \in Hyp_{GN_\varphi}(V)$,
$\sigma_{x_mx_1} \circ_{GN} \sigma_{x_mx_1} = \sigma_{x_mx_m}$ if $\sigma_{x_mx_1} \in Hyp_{GN_\varphi}(V)$ where $m > 2$,
$\sigma_{x_2x_m} \circ_{GN} \sigma_{x_2x_m} = \sigma_{x_mx_m}$ if $\sigma_{x_2x_m} \in Hyp_{GN_\varphi}(V)$ where $m > 2$.

Proof. (i) If σ is an idempotent of $Hyp_{GN_\varphi}(V)$, then $\sigma \circ_{GN} \sigma = \sigma \sim_{VG} (\sigma \circ_G \sigma)$. Conversely, we assume that $\sigma \sim_{VG} (\sigma \circ_G \sigma)$. Because of $\sigma \in Hyp_{GN_\varphi}(V)$, so $\sigma \circ_{GN} \sigma = \sigma$.
(ii) Since $(\sigma_{x_2x_1} \circ_{GN} \sigma_{x_2x_1}) \sim_{VG} (\sigma_{x_2x_1} \circ_G \sigma_{x_2x_1}) = \sigma_{x_1x_2} \in Hyp_{GN_\varphi}(V)$. Thus $\sigma_{x_2x_1} \circ_{GN} \sigma_{x_2x_1} = \sigma_{x_1x_2}$, $(\sigma_{x_mx_1} \circ_{GN} \sigma_{x_mx_1}) \sim_{VG} (\sigma_{x_mx_1} \circ_G \sigma_{x_mx_1}) = \sigma_{x_mx_m} \in Hyp_{GN_\varphi}(V)$. Thus $\sigma_{x_mx_1} \circ_{GN} \sigma_{x_mx_1} = \sigma_{x_mx_m}$ and $(\sigma_{x_2x_m} \circ_{GN} \sigma_{x_2x_m}) \sim_{VG} (\sigma_{x_2x_m} \circ_G \sigma_{x_2x_m}) = \sigma_{x_mx_m} \in Hyp_{GN_\varphi}(V)$. Thus $\sigma_{x_2x_m} \circ_{GN} \sigma_{x_2x_m} = \sigma_{x_mx_m}$. □

3. Idempotents in $Hyp_{GN_\varphi(V)}$

In general, if σ is an idempotent of $Hyp_G(2)$ and $\sigma \in Hyp_{GN_\varphi}(V)$, then it is also an idempotent in $Hyp_{GN_\varphi}(V)$ for any variety V of semigroups and any choice function φ. But if σ is an idempotent in $Hyp_{GN_\varphi}(V)$, then it is not necessarily be idempotent in $Hyp_G(2)$. As an example, let $V = Mod\{(xy)z \approx x(yz), xyuv \approx xuyv, x^3 \approx x\}$. We consider

$$\sigma_{x_mx_1x_2} \circ_{GN} \sigma_{x_mx_1x_2} = \sigma_{x_mx_1x_2},$$

$$\sigma_{x_mx_1x_2} \circ_G \sigma_{x_mx_1x_2} = \sigma_{x_mx_mx_mx_1x_2}.$$

We get $\sigma_{x_mx_1x_2}$ is idempotents in $Hyp_{GN_\varphi}(V)$ which is not idempotents in $Hyp_G(2)$.

All idempotent elements of $Hyp_{GN_\varphi}(V)$ are $\{\sigma_{x_1}, \sigma_{x_2}, \sigma_{x_1x_1}, \sigma_{x_1x_2}, \sigma_{x_2x_2}, \sigma_{x_1x_1x_2}, \sigma_{x_1x_2x_2}, \sigma_{x_1x_2x_2x_1}, \sigma_{x_1x_1x_2x_1}, \sigma_{x_1x_1x_2x_2}, \sigma_{x_2x_1x_2x_2}, \sigma_{x_2x_1x_1x_2}, \sigma_{x_1x_1x_2x_2x_1}, \sigma_{x_2x_1x_1x_2x_2}\} \cup$ $\{\sigma_t | t \in [s]$ where $[s] \in \{[x_{k_1}^{i_1}x_{k_2}^{i_2}...x_{k_j}^{i_j}], [x_1x_{k_1}^{i_1}x_{k_2}^{i2}...x_{k_j}^{i_j}], [x_{k_1}^{i_1}x_{k_2}^{i_2}...x_{k_{2j-1}}^{i_{2j-1}}x_2]$, $[x_{k_1}^{i_1}x_{k_2}^{i_2}...x_{k_{2j-1}}^{i_{2j-1}}x_1x_2], [x_{k_1}^{i_1}x_{k_2}^{i_2}...x_{k_j}^{i_j}x_2x_2], [x_1x_1x_{k_1}^{i_1}x_{k_2}^{i_2}...x_{k_j}^{i_j}], [x_1x_2x_{k_1}^{i_1}x_{k_2}^{i_2}...x_{k_{2j-1}}^{i_{2j-1}}]$, $[x_1x_{k_1}^{i_1}x_{k_2}^{i_2}...x_{k_j}^{i_j}x_1], [x_1x_{k_1}^{i_1}x_{k_2}^{i_2}...x_{k_{2j-1}}^{i_{2j-1}}x_2], [x_2x_{k_1}^{i_1}x_{k_2}^{i_2}...x_{k_j}^{i_j}x_2], [x_{k_1}^{i_1}x_{k_2}^{i_2}...x_{k_{2j-1}}^{i_{2j-1}}x_1x_1x_2]$, $[x_1x_2x_2x_{k_1}^{i_1}x_{k_2}^{i_2}...x_{k_j}^{i_j}], [x_1x_1x_{k_1}^{i_1}x_{k_2}^{i_2}...x_{k_j}^{i_j}x_1], [x_1x_2x_{k_1}^{i_1}x_{k_2}^{i_2}...x_{k_j}^{i_j}x_2], [x_1x_1x_{k_1}^{i_1}x_{k_2}^{i_2}...x_{k_{2j-1}}^{i_{2j-1}}x_2]$, $[x_2x_2x_{k_1}^{i_1}x_{k_2}^{i_2}...x_{k_{2j-1}}^{i_{2j-1}}x_2], [x_{k_1}^{i_1}x_{k_2}^{i_2}...x_{k_j}^{i_j}x_1x_1x_2x_2], [x_1x_1x_2x_2x_{k_1}^{i_1}x_{k_2}^{i_2}...x_{k_j}^{i_j}]$, $[x_1x_2x_2x_{k_1}^{i_1}x_{k_2}^{i_2}...x_{k_j}^{i_j}x_1], [x_1x_1x_2x_{k_1}^{i_1}x_{k_2}^{i_2}...x_{k_{2j-1}}^{i_{2j-1}}x_1], [x_1x_1x_2x_{k_1}^{i_1}x_{k_2}^{i_2}...x_{k_j}^{i_j}x_2]$, $[x_2x_1x_2x_{k_1}^{i_1}x_{k_2}^{i_2}...x_{k_{2j-1}}^{i_{2j-1}}x_2], [x_2x_1x_1x_{k_1}^{i_1}x_{k_2}^{i_2}...x_{k_j}^{i_j}x_2], [x_mx_1x_1x_2x_2x_{k_1}^{i_1}x_{k_2}^{i_2}...x_{k_j}^{i_j}]$, $[x_1x_1x_2x_2x_{k_1}^{i_1}x_{k_2}^{i_2}...x_{k_j}^{i_j}x_1], [x_2x_1x_1x_2x_{k_1}^{i_1}x_{k_2}^{i_2}...x_{k_j}^{i_j}x_2], [x_mx_1x_2x_{k_1}^{i_1}x_{k_2}^{i_2}...x_{k_{2j}}^{i_{2j}}]\}$ where $j \geq 1, k_j \geq 3, i_j \leq 2$; $k_m \neq k_n$ for $m \neq n$ and $m \geq 3\}$.

Now we consider which varieties at most the idempotents of $Hyp_G(2)$ are idempotent of $Hyp_{GN_\varphi}(V)$.

Theorem 3.1. *For a variety V of semigroups the following are equivalent:*

(i) $Mod\{(xy)z \approx x(yz), xy \approx yx\} \subseteq V$.

(ii) $\{\sigma | \sigma \in Hyp_{GN_\varphi}(V)$ and $\sigma \circ_{GN} \sigma = \sigma\} = \{\sigma | \sigma \in Hyp_G(2)$ *and* $\sigma \circ_G \sigma = \sigma\} \cap Hyp_{GN_\varphi}(V)$ *for each choice function* φ.

Proof. Let $x_i \in W_{(2)}(X)$ where $i > 2$ and $l(t)$ denote the length of t where $t \in W_{(2)}(X)$. Let φ be an arbitrary choice function.

(i) $\Rightarrow$ (ii)

Let $Mod\{(xy)z \approx x(yz), xy \approx yx\} \subseteq V$. It is clear that $\{\sigma|\sigma \in Hyp_G(2)$ and $\sigma \circ_G \sigma = \sigma\} \cap Hyp_{GN_\varphi}(V) \subseteq \{\sigma|\sigma \in Hyp_{GN_\varphi}(V)$ and $\sigma \circ_{GN} \sigma = \sigma\}$ for each choice function φ.

Conversely, let $\sigma_w \in \{\sigma|\sigma \in Hyp_{GN_\varphi}(V)$ and $\sigma \circ_{GN} \sigma = \sigma\}$ and σ_w is not idempotent in $Hyp_G(2)$. Since $yx \approx xy, x_i s \approx x_1 s$ and $tx_i \approx tx_2$ (where $s \in W^G_{(2)}(\{x_1\})$, $t \in W^G_{(2)}(\{x_2\})$) are identity in V, so we choose $xy, x_1 s, tx_2$ are respresentatives of its classes in $Hyp_{GN_\varphi}(V)$. Then $yx, x_i s, tx_i \notin Hyp_{GN_\varphi}(V)$. Since σ_w is not idempotent in $Hyp_G(2)$ and $\sigma_w \neq \sigma_{yx}, \sigma_{x_i s}, \sigma_{tx_i}$, so σ_w has infinite order.

Since σ_w has infinite order, so $l(\sigma_w) \neq l(\sigma_w \circ_G \sigma_w)$. We get $\sigma_w \approx (\sigma_w \circ_G \sigma_w) \notin IdMod\{(xy)z \approx x(yz), xy \approx yx\}$. But σ_w is idempotent on $Hyp_{GN_\varphi}(V)$, so $l(\sigma_w) = l(\sigma_w \circ_{GN} \sigma_w) = l(\varphi(\sigma_w \circ_G \sigma_w))$, i.e. , $\sigma_w \approx (\sigma_w \circ_G \sigma_w) \in IdV$, a contradiction. So $IdV \subseteq IdMod\{(xy)z \approx x(yz), xy \approx yx\}$.

(ii) $\Rightarrow$ (i)

Let $\{\sigma|\sigma \in Hyp_{GN_\varphi}(V)$ and $\sigma \circ_{GN} \sigma = \sigma\} = \{\sigma|\sigma \in Hyp_G(2)$ and $\sigma \circ_G \sigma = \sigma\} \cap Hyp_{GN_\varphi}(V)$ for each choice function φ.

Assume that $Mod\{(xy)z \approx x(yz), xy \approx yx\} \nsubseteq V$. Then there exists $x^k \approx x^n \in IdV$ with $1 \leq k \leq n \in \mathbb{N}$. Next, we will construct an idempotent element of $Hyp_{GN_\varphi}(V)$ which is not in $P_G(2) \cup E^G_{x_1} \cup E^G_{x_2} \cup G \cup \{\sigma_{id}\}$. We consider into six cases:

Case 1 : We set $m = n - k$ and $w = f(f(x_1, x_1), u)$ where $u \in W_{x_1}$. Clearly, $\sigma_w \notin P_G(2) \cup E^G_{x_1} \cup E^G_{x_2} \cup G \cup \{\sigma_{id}\}$. It is easy to see that the length of σ_w is $3km$ and the length of $(\sigma_w \circ_G \sigma_w)$ is $(3km)^2$. In fact, from $x^k \approx x^n \in IdV$ it follows that $x^a \approx x^{a+bm} \in IdV$ for all $a \geq k$ and $b \geq 1$ where $a, b \in \mathbb{N}$. Then we have $x^{3km} \approx x^{3km+(9k^2m-3k)m} = x^{3km+9k^2m^2-3km} = x^{9k^2m^2} = x^{(3km)^2}$. Hence $\sigma_w(f) \approx x^{3km} \approx x^{(3km)^2} \approx (\sigma_w \circ_G \sigma_w)(f)$.

Case 2 : We set $m = n - k$ and $w = f(f(f(...f(x_1, x_i), ...), x_i), x_i)$. Clearly, $\sigma_w \notin P_G(2) \cup E^G_{x_1} \cup E^G_{x_2} \cup G \cup \{\sigma_{id}\}$. It is easy to see that the length of σ_w is $km + 1$ and the length of $(\sigma_w \circ_G \sigma_w)$ is $(km)^2 + 1$. In fact, from $x^k \approx x^n \in IdV$ it follows that $x^a \approx x^{a+bm} \in IdV$ for all $a \geq k$ and $b \geq 1$ where $a, b \in \mathbb{N}$. Then we have $x^{km} \approx x^{km+(k^2m-k)m} = x^{km+k^2m^2-km} = x^{k^2m^2} = x^{(km)^2}$. Hence $\sigma_w(f) \approx x^{km} \approx x^{(km)^2} \approx (\sigma_w \circ_G \sigma_w)(f)$.

Case 3 : From $x^k \approx x^n \in IdV$ implies $x_1^n x_i^r \approx x_1^k x_i^s \in IdV$. We set $m = n - k$, $t = r - s$ and $w = f(f(x_1, x_1), u)$ where $u \in W^G_{(2)}(\{x_1\})$. Clearly, $\sigma_w \notin P_G(2) \cup E^G_{x_1} \cup E^G_{x_2} \cup G \cup \{\sigma_{id}\}$. It is easy to see that the length of σ_w is $2km + st$ and the length of $(\sigma_w \circ_G \sigma_w)$ is $(2km)^2 + st(2km + 1)$. In fact, from $x_1^n x_i^r \approx x_1^k x_i^s$ it follows that $x_1^a x_i^c \approx x_1^{a+bm} x_i^{c+dt} \in IdV$ for all $a \geq k$, $b \geq 1$ $c \geq s$ and $d \geq 1$ where $a, b, c, d \in \mathbb{N}$. The we have $x_1^{2km} \approx x_1^{2km+(4k^2m-2k)m} = x_1^{2km+4k^2m^2-2km} = x_1^{4k^2m^2} = x_1^{(2km)^2}$ and $x_i^{st} \approx x_i^{st+(2kms)t} = x_i^{st+2kmst} = x_i^{st(2km+1)}$. Hence $\sigma_w(f) \approx x_1^{2km} x_i^{st} \approx$

$x_1^{(2km)^2} x_i^{st(2km+1)} \approx (\sigma_w \circ_G \sigma_w)(f)$.
Case 4 : We set $m = n - k$ and $w = f(u, f(x_2, x_2))$ where $u \in W_{x_2}$.
Case 5 : We set $m = n - k$ and $w = f(x_i, f(x_i, ...f(x_i, x_2)...))$.
Case 6 : From $x^k \approx x^n \in IdV$ implies $x_i^r x_2^n \approx x_i^s x_2^k \in IdV$. We set $m = n - k$, $t = r - s$ and $w = f(u, f(x_2, x_2))$ where $u \in W_{(2)}^G(\{x_2\})$.
The proof of Case 4, 5, 6 is similar to Case 1, 2, 3 respectively.

From all cases, we have $(\sigma_w \circ_G \sigma_w) \sim_{VG} \sigma_w$. And from (ii), $(\sigma_w \circ_{GN} \sigma_w) \sim_{VG} (\sigma_w \circ_G \sigma_w)$. So $(\sigma_w \circ_{GN} \sigma_w) \sim_{VG} (\sigma_w \circ_G \sigma_w) \sim_{VG} \sigma_w$ it follows that $\sigma_w \circ_{GN} \sigma_w = \sigma_w$. Therefore σ_w is idempotent on $Hyp_{GN_\varphi}(V)$, a contradiction. □

4. Elements of Infinite Order

In this section, we will characterize the set of all elements in $Hyp_{GN_\varphi}(V)$ which have infinite order where $V = Mod\{(xy)z \approx x(yz), xy \approx yx\}$. Let $O(\sigma)$ denote the order of the generalized hypersubstitution $\sigma \in Hyp_{GN_\varphi}(V)$.

Theorem 4.1. *Let V be a variety of semigroups and $\langle\sigma\rangle_{\circ_{GN}}$ be the cyclic subsemigroup generated by σ. Then following are equivalent:*

(i) $Mod\{(xy)z \approx x(yz), xy \approx yx\} \subseteq V$.

(ii) $\{\sigma | \sigma \in Hyp_{GN_\varphi}(V)$ and the order of σ is infinite$\} = Hyp_{GN_\varphi}(V) \backslash (A_1 \cup A_2 \cup A_3 \cup A_4)$ *where*
$A_1 = P_G(2) \cup E_{x_1}^G \cup E_{x_2}^G \cup G \cup \{\sigma_{id}\} \cup \{\sigma_{f(x_2,x_1)}\}$
$A_2 = \{\sigma \mid \sigma \in Hyp_{GN_\varphi}(V) \cap (T_1 \cup \{\sigma_v \mid v \in W_{(2)}^G(\{x_1\})$ *where leftmost*(v) $= x_1\} \backslash \sigma_{x_1} \cup \sigma_{f(x_1,s)}$ *where* $s \in W_{(2)}^G(\{x_1\})$ *and* $\langle\sigma\rangle_{\circ_{GN}} \cap \{\sigma_{x_1 u} \mid u \in W_2(X)\} \neq \emptyset)\}$
$A_3 = \{\sigma \mid \sigma \in Hyp_{GN_\varphi}(V) \cap (T_2 \cup \{\sigma_v \mid v \in W_{(2)}^G(\{x_2\})$ *where rightmost*(v) $= x_2\} \backslash \sigma_{x_2} \cup \sigma_{f(s,x_2)}$ *where* $s \in W_{(2)}^G(\{x_2\})$ *and* $\langle\sigma\rangle_{\circ_{GN}} \cap \{\sigma_{u x_2} \mid u \in W_2(X)\} \neq \emptyset)\}$.

Proof. Let $x_i \in W_{(2)}(X)$ where $i > 2$ and $l(t)$ denote the length of t where $t \in W_{(2)}(X)$. Let φ be an arbitrary choice function.

(i)⇒(ii)

Let $Mod\{(xy)z \approx x(yz), xy \approx yx\} \subseteq V$. We will show that $\{\sigma | \sigma \in Hyp_{GN_\varphi}(V)$ and the order of σ is infinite$\} = Hyp_{GN_\varphi}(V) \backslash (A_1 \cup A_2 \cup A_3)$. Let σ_w has infinite order on $Hyp_{GN_\varphi}(V)$. Since A_1 is set of all idempotent on $Hyp_{GN_\varphi}(V)$, i.e., all elements of A_1 has order 1. So $\sigma_w \notin A_1$. Assume that $\sigma_w \in A_2$ ($\sigma_w \in A_3$), then there exists a word $u \in W_2(X)$ ($s \in W_2(X)$) such that $\sigma_{x_1 u} \in \langle\sigma\rangle_{\circ_{GN}}$ ($\sigma_{s x_2} \in \langle\sigma\rangle_{\circ_{GN}}$). We get $\sigma_{x_1 u} = \sigma_w^m$ for each $m \in \mathbb{N}$ ($\sigma_{s x_2} = \sigma_w^n$ for each $n \in \mathbb{N}$) and $O(\sigma_{x_1 u}) = 1$ ($O(\sigma_{s x_2}) = 1$), so $O(\sigma_w^m) = 1$ ($O(\sigma_w^n) = 1$), i.e., σ_w^m is idempotent on $Hyp_{GN_\varphi}(V)$, contradicts to $O(\sigma_w^m) = \infty$. Thus $\sigma_w \notin (A_1 \cup A_2 \cup A_3)$. Hence $\sigma_w \in Hyp_{GN_\varphi}(V) \backslash (A_1 \cup A_2 \cup A_3)$.

(ii)⇒(i)

Let $\{\sigma|\sigma \in Hyp_{GN_\varphi}(V)$ and the order of σ is infinite$\} = Hyp_{GN_\varphi}(V)\backslash(A_1 \cup A_2 \cup A_3 \cup A_4)$.

Assume that $Mod\{(xy)z \approx x(yz), xy \approx yx\} \nsubseteq V$. Then there exists $x^k \approx x^n \in IdV$ with $1 \leq k \leq n \in \mathbb{N}$. We consider into two cases:

Case 1 : We set $m = n - k$ and $w = f(f(...f(x_1, x_2), ..., x_2), x_2)$. Clearly, $\sigma_w \notin (A_1 \cup A_2 \cup A_3)$. It is easy to see that the length of σ_w is $km + 1$ and the length of $(\sigma_w \circ_G \sigma_w)$ is $(km)^2 + 1$. In fact, from $x^k \approx x^n \in IdV$ it follows that $x^a \approx x^{a+bm} \in IdV$ for all $a \geq k$ and $b \geq 1$ where $a, b \in \mathbb{N}$. Then we have $x^{km} \approx x^{km+(k^2m-k)m} = x^{km+k^2m^2-km} = x^{(km)^2}$. Hence $\sigma_w(f) \approx x_1x_2^{km} \approx x_1x_2^{(km)^2} \approx (\sigma_w \circ_G \sigma_w)(f)$.

Case 2 : We set $m = n - k$ and $w = f(f(...f(x_1, f(x_2, x_i)), ..., x_i), x_i)$. Clearly, $\sigma_w \notin (A_1 \cup A_2 \cup A_3)$. It is easy to see that the length of σ_w is km and the length of $(\sigma_w \circ_G \sigma_w)$ is $km(km + 2)$. In fact, from $x^k \approx x^n \in IdV$ it follows that $x^a \approx x^{a+bm} \in IdV$ for all $a \geq k$ and $b \geq 1$ where $a, b \in \mathbb{N}$. Then we have $x^{km} \approx x^{km+(k^2m+k)m} = x^{km+k^2m^2+km} = x^{(km)^2+2km} = x^{km(km+2)}$. Hence $\sigma_w(f) \approx x_1x_2x_m^{km} \approx x_1x_2x_m^{km(km+2)} \approx (\sigma_w \circ_G \sigma_w)(f)$.

From all cases, we have $(\sigma_w \circ_G \sigma_w) \sim_{VG} \sigma_w$. And from (ii), $(\sigma_w \circ_{GN} \sigma_w) \sim_{VG} (\sigma_w \circ_G \sigma_w)$. So $(\sigma_w \circ_{GN} \sigma_w) \sim_{VG} (\sigma_w \circ_G \sigma_w) \sim_{VG} \sigma_w$ it follows $\sigma_w \circ_{GN} \sigma_w = \sigma_w$. Therefore σ_w is idempotent on $Hyp_{GN_\varphi}(V)$, a contradiction. □

Acknowledgements. The corresponding author was supported by Chiang Mai University, Chiang Mai 50200, Thailand.

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