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Rings Whose Simple Singular Modules are PS-Injective

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Abstract. Let R be a ring. A right R-module M is PS-injective if every R-homomorphism $f : aR \to M$ for every principally small right ideal aR can be extended to $R \to M$. We investigate, in this paper, rings whose simple singular modules are PS-injective. New characterizations of semiprimitive rings and semisimple Artinian rings are given.

1. Introduction

Throughout this paper, R is an associative ring with identity and all modules are unitary. The Jacobson radical of R is denoted by $J(R)$ and the right singular ideal is denoted by $Z(R_R)$. For $a \in R$, $l(a)$(resp. $r(a)$) denote the left (resp. right) annihilator of a in R. For the usual notations we refer the reader to [3], [7] and [10].

A right ideal I of R is called small if for every proper right ideal K of R, $K + I \neq R$. A right R-module M is right PS-injective if every R-homomorphism $f : aR \to M$ for every principally small right ideal aR can be extended to $R \to M$ (see [13]). The ring R is said to be right PS-injective if R_R is right PS-injective. This concept was introduced as a non-trivial generalization of right small injective rings and right P-injective rings. Given a right R-module M, we set $Z(M) = \{x \in M | xI = 0$ for some essential right ideal I of $R\}$. The module M is called singular module provided $Z(M) = M$. In what follows, we say that R satisfies (P) if every simple singular right R-module is PS-injective. Recall that:

(1) A ring R is semiprimitive if $J(R) = 0$.

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(2) A right ideal of R is reduced if it contains no nonzero nilpotent elements.
(3) A ring R is called an MERT if every essential maximal right ideal of R is an ideal.
(4) R is a left (right) Kasch ring if every maximal left (right) ideal is a left (right) annihilator of R.

Motivated by the well known result of Kaplansky (i.e., A commutative ring R is von Neumann regular if and only if every simple R-module is injective), many authors studied rings whose simple (singular) modules are injective (P-injective, GP-injective) (see [1], [2], [4-6], [9], [11], [12], [14], [15]). It was proven that: (1) R is strongly regular if and only if R is a left (or right) quasi-duo ring whose simple left R-modules are injective (or P-injective) (see [11]); (2) A ring R is strongly regular if and only if R is a left duo ring whose simple singular left R-modules are P-injective (see [14]); (3) A ring R is strongly regular if and only if R is a left duo ring whose simple singular left R-modules are YJ-injective if and only if R is a left quasi-duo ring whose simple left R-modules are YJ-injective (see [2]); (4) A ring R is strongly regular if and only if R is a weakly right duo ring whose simple singular right R-modules are right GP-injective (see [6]). The aim of present paper is to investigate rings whose simple singular right R-modules are PS-injective. We prove that a NI ring satisfying (P) are right nonsingular. Semiprimitive rings, nonsingular rings and semisimple Artinian rings are characterized in terms of PS-injectivity.

2. Main Results

We start with the following lemmas needed frequently in the sequel.

Lemma 2.1. Let R satisfy (P). Then for any $a \in J(R)$, there exists a right ideal L of R such that $(RaR + r(a)) \oplus L = R$.

Proof. For the right ideal $RaR+r(a)$ of R, there exists a right ideal L of R such that $(RaR + r(a)) \oplus L$ is an essential right ideal of R. Suppose $(RaR + r(a)) \oplus L \neq R$. Then it must be contained in a maximal right ideal M, whence M is essential. Define $f : aR \to R/M$ by $f(ax) = x + M$ for $x \in R$. It is easy to check that f is well-defined. Since R satisfies (P), R/M is PS-injective. Thus there exists $b \in R$ such that $1+M = f(a) = (b+M)a = ba+M$, and hence $1-ba \in M$. Note that $1-ba$ is invertible, contradicting with the maximality of M. Thus, $(RaR+r(a)) \oplus L = R$. □

Lemma 2.2. Let R satisfy (P). Then $J(R) \cap Z(R_R) = 0$.

Proof. Take any $0 \neq b \in J(R) \bigcap Z(R_R)$. By Lemma 2.1, there exists a right ideal L of R such that $(RbR + r(b)) \oplus L = R$. Since $b \in Z(R_R)$, $r(b)$ is an essential right ideal of R. Now $r(b) \cap L = 0$, so $L = 0$. This proves that $RbR + r(b) = R$, and hence $r(b) = R$ because RbR is a small ideal of R. This implies $b = 0$, a required contradiction. □

Recall that a ring R is a NI ring [8] if the set of nilpotent elements $N(R)$ in R is an ideal. A ring R is a NI ring if and only if the nilradical $Nil^*(R) = N(R)$.

Obviously, 2-primal rings (i.e., $P(R) = N(R)$, where $P(R)$ is the prime radical of R.) are NI rings.

Proposition 2.3. If R is a NI ring and satisfies (P), then R is right nonsingular.

Proof. Suppose that $Z(R_R) \neq 0$. Then $Z(R_R)$ contains nonzero nilpotent elements. To see this, let $0 \neq x \in Z(R_R)$, so $r(x)$ is an essential right ideal of R. Thus $r(x) \cap xR \neq 0$, and hence there exists $r \in R$ such that $xr \neq 0$ and $x^2 r = 0$. So we have $(xrx)^2 = 0$, whence $xrx = 0$. It implies $(xr)^2 = 0$, and hence $xr = 0$, a contradiction.

Now take $0 \neq b \in Z(R_R)$ with $b^2 = 0$, so $b \in J(R)$ since R is a NI ring. Then $b \in J(R) \cap Z(R_R) = 0$ by Lemma 2.2. This is a contradiction. □

It is known that a ring R is semiprimitive if and only if every right simple R-module is PS-injective (cf. [13, Proposition 2.18]). But a ring satisfying (P) need not be semiprimitive by the following example.

Example 2.4. let $R = \begin{pmatrix} F & 0 \\ F & F \end{pmatrix}$, where F is a field. Then $0 \neq \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in J(R)$. Note that $T = \begin{pmatrix} F & 0 \\ F & 0 \end{pmatrix}$ is the unique essential maximal right ideal of R. It is easy to show that every simple singular right R-module is PS-injective.

Now we consider when a ring R satisfying (P) is semiprimitive.

Proposition 2.5. If R satisfies (P) and every complement right ideal is an ideal, then R is semiprimitive.

Proof. We first prove that $J(R)$ contains no nonzero nilpotent elements. Let $a \in J(R)$ with $a^2 = 0$. So there exists a right ideal L of R such that $r(a) \oplus L$ is right essential. By hypothesis, L is an ideal. Then $aL \subseteq L \cap r(a) = 0$, so $L \subseteq r(a)$, and hence $r(a)$ is an essential right ideal of R. Then $a \in Z(R_R)$. So $a \in J(R) \cap Z(R_R) = 0$ by Lemma 2.2.

Now let $b \in J(R)$. By Lemma 2.1, there exists a right ideal L of R such that $((RbR + r(b)) \oplus L = R$. Thus $RbR + r(b) = eR$ with $e^2 = e \in R$. So $b^2 = beb = b^2 ab$ for some $a \in R$, and hence $b^2(1 - ab) = 0$, which implies $b^2 = 0$ because $1 - ab$ is invertible. Thus $b = 0$ by the preceding result. □

A ring is called a right duo ring if every right ideal is an ideal.

Corollary 2.6. If R is a right duo ring and satisfies (P), then it is semiprimitive.

Recall that a ring R is right weakly continuous [10] if R is semiregular and $J(R) = Z(R_R)$. Examples of this rings include mininjective semiregular rings R in which $soc(R_R) \subseteq^{ess} R_R$, P-injective semiregular rings, right continuous rings, and the endomorphism rings of free continuous right modules.

Proposition 2.7. Let R be a right weakly continuous ring. If R satisfies (P), then it is semiprimitive.

Proof. Note that $J(R) = Z(R_R)$ since R is right weakly continuous. Thus, the result follows by Lemma 2.2. □

A ring R is called idempotent reflexive if $eRa = 0$ implies $aRe = 0$ for any a and $e^2 = e \in R$. Abelian rings and semiprime rings are idempotent reflexive. Now we have the following results.

Theorem 2.8. *The following are equivalent for a ring R.*

(1) *R is semiprimitive.*

(2) *R is a semiprime ring satisfying* (P).

(3) *R is an idempotent reflexive ring satisfying* (P).

(4) *R is a right PS-injective ring satisfying* (P).

Proof. (1)⇒(2), (2)⇒(3) and (1)⇒(4) are trivial. (3)⇒(1). For any $a \in J(R)$, by lemma 2.1, there exists a right ideal L of R such that $(RaR + r(a)) \oplus L = R$. Let $L = eR$, where $e^2 = e \in R$. Then $eRaR = LRaR \subseteq RaR \cap L = 0$, and hence $eRa = 0$. Thus, $aRe = 0$ since R is an idempotent reflexive ring. So $L \subseteq ReR \subseteq r(a)$. This implies $L = 0$. Then we have $RaR + r(a) = R$, and hence $r(a) = R$ since RaR is a small ideal of R. Therefore, $a = 0$. (4)⇒(1). By [13, Theorem 2.6], $J(R) \subseteq Z(R_R)$ since R is right PS-injective. Then $J(R) = J(R) \cap Z(R_R) = 0$ by Lemma 2.2. □

Remark 2.9. A left PS-injective ring satisfying (P) need not be semiprimitive. For example, let $R = \begin{pmatrix} K & 0 \\ K & A \end{pmatrix}$, where $K = \mathbb{Z}_2$ and

$$A = \{(a_1, a_2, \cdots, a_n, a, a, \cdots) |\ a, a_1, a_2, \cdots \in K, n \in \mathbb{N}\}.$$

If $k \in K$ and $(a_1, a_2, \cdots, a_n, a, a, \cdots) \in A$, let $k \cdot (a_1, a_2, \cdots, a_n, a, a, \cdots) = ka$. Then $R = \begin{pmatrix} K & 0 \\ K & \mathbb{Z}_2^{(N)} \end{pmatrix}$ is the unique maximal essential right ideal of R, where

$$\mathbb{Z}_2^{(N)} = \{(a_1, a_2, \cdots, a_n, 0, 0, \cdots) |\ a, a_1, a_2, \cdots \in \mathbb{Z}_2, n \in \mathbb{N}\}.$$

Analogous to the proof of [12,Example 2.13], we can show that R is a MERT, left PS-injective ring and satisfies (P). But it is not semiprimitive because $J(R) = \begin{pmatrix} 0 & 0 \\ \mathbb{Z}_2 & 0 \end{pmatrix} \neq 0$.

Lemma 2.10.([2, Lemma3.8]) A ring R is semisimple Artinian if and only if R has no an essential maximal left(right) ideal.

Theorem 2.11. *The following are equivalent for a ring R.*

(1) *R is a semisimple Artinian ring.*

(2) *R is a right Kasch ring satisfying* (P).

Proof. (1) ⇒ (2) is clear.
(2) ⇒ (1). Suppose that $M \neq 0$ is an essential maximal right ideal of R. Since R is a right Kasch ring, $M = r(a)$ for some $0 \neq a \in R$. Then $a \in Z(R_R)$. Note that $aR \cong R/M$ is simple, and hence $aR \subseteq soc(R_R)$. Thus $(aR)^2 \subseteq aRsoc(R_R) = asoc(R_R) \subseteq aM = 0$ since $soc(R_R)$ is the intersection of all essential right ideals of R, whence $aR \subseteq J(R)$. Then $aR \subseteq Z(R_R) \cap J(R) = 0$ by Lemma 2.2, and hence $a = 0$, a contradiction. Therefore, R has no an essential maximal right ideal, and hence (1) follows by Lemma 2.10. □

Proposition 2.12. If R is an MERT, left Kasch, NI ring such that essential maximal right ideals are PS-injective, then it is semisimple Artinian.

Proof. Assume that R is not semisimple Artinian, then it has an essential maximal right ideal M by lemma 2.10. Thus M is an ideal since R is MERT. Let M_1 be a maximal left ideal of R containing M. Then, in view of [10, Proposition 1.44], $M_1 = l(u)$ for some $0 \neq u \in R$ since R is a left Kasch ring. Now M is an essential right ideal of R, and hence $M \cap uR \neq 0$. Thus, there exists $r \in R$ such that $ur \neq 0$ and $ur \in M$, whence $uru = 0$ because $M \subseteq M_1$, which yields $(ur)^2 = 0$, and hence $ur \in J(R)$ since R is a NI ring. Then the inclusion map $urR \to M$ extends to $R \to M$ by the PS-injectivity of M. Therefore $ur = c \cdot (ur)$ for some $c \in M \subseteq M_1$, whence $1 - c \in l(ur)$. But $l(ur) = l(u) = M_1$. Thus $1 - c \in M_1$, which yields $1 \in M_1$, a contradiction. Then the result follows. □

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