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Certain Class of Analytic Functions Defined by Ruscheweyh Derivative with Varying Arguments

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Abstract. In this paper we derive some results for certain new class of analytic functions defined by using Ruscheweyh derivative with varying arguments.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic and univalent in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. Given

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two functions f, $g \in \mathcal{A}$, where $f(z)$ is given by (1.1) and $g(z)$ is given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k, \tag{1.2}$$

the Hadamard product $(f * g)(z)$ is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z). \tag{1.3}$$

By using the Hadamard product, Ruscheweyh [7] defined

$$D^{\gamma} f(z) = \frac{z}{(1-z)^{\gamma+1}} * f(z) \quad (\gamma \geq -1). \tag{1.4}$$

Also Ruscheweyh [7] observed that

$$D^n f(z) = \frac{z(z^{n-1} f(z))^{(n)}}{n!} \quad (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; \mathbb{N} = \{1, 2, ...\}). \tag{1.5}$$

The symbol $D^n f(z)$ $(n \in \mathbb{N}_0)$ was called the $n-th$ order Ruscheweyh derivative of $f(z)$ by Al-Amiri [1]. We note that

$$D^0 f(z) = f(z) \text{ and } D^1 f(z) = z f^{'}(z).$$

It is easy to see that

$$D^n f(z) = z + \sum_{k=2}^{\infty} \delta(n,k) a_k z^k, \tag{1.6}$$

where

$$\delta(n,k) = \binom{n+k-1}{n}. \tag{1.7}$$

In [4] Attiya and Aouf defined the class $Q(n, \lambda, A, B)$ as follows:

Definition 1.([4]) Let $Q(n, \lambda, A, B)$ denote the subclass of $\mathcal{A}$ consisting of functions $f(z)$ of the form (1.1) such that

$$(1-\lambda)\left(D^n f(z)\right)^{'} + \lambda \left(D^{n+1} f(z)\right)^{'} \prec \frac{1+Az}{1+Bz} \tag{1.8}$$

$$\left(\lambda \geq 0; -1 \leq A < B \leq 1; 0 < B \leq 1; n \in \mathbb{N}_0; z \in U\right).$$

Specializing the parameters λ, A, B and n, we can obtained different classes studied by various authors:

(i) $Q(0, \lambda, 2\alpha - 1, 1) = R(\lambda, \alpha)$ $(0 \leq \alpha < 1, \lambda \geq 0)$ (see Altintas [3])

(ii) $Q(0, 0, 2\alpha - 1, 1) = T^{**}(\alpha)$ $(0 \leq \alpha < 1)$ (see Sarangi and Uralegaddi [8] and Al-Amiri [2])

(iii) $Q(n, 0, 2\alpha - 1, 1) = Q_n(\alpha)$ $(0 \leq \alpha < 1, n \in \mathbb{N}_0)$ (see Uralegaddi and Sarangi [11])

(iv) $Q(0, 0, (2\alpha - 1)\beta, \beta) = P^*(\alpha, \beta)$ $(0 \leq \alpha < 1, 0 < \beta \leq 1)$ (see Gupta and Jain[5])

(v) $Q(0, 0, ((1 + \mu)\alpha - 1)\beta, \mu\beta) = P^*(\alpha, \beta, \mu)$ $(0 \leq \alpha < 1, 0 < \beta \leq 1, 0 < \mu \leq 1)$ (see Owa and Aouf [6]).

Also we note that:

(i) $Q(0, \lambda, A, B) = R(\lambda, A, B)$ (1.9)

$$= \left\{ f(z) \in \mathcal{A} : f'(z) + \lambda z f''(z) \prec \frac{1 + Az}{1 + Bz} \quad (\lambda \geq 0; -1 \leq A < B \leq 1; 0 < B \leq 1; z \in U) \right\}$$

(ii) $Q(n, 0, A, B) = Q_n(A, B) =$ (1.10)

$$\left\{ f(z) \in \mathcal{A} : (D^n f(z))' \prec \frac{1 + Az}{1 + Bz} \quad (\lambda \geq 0; -1 \leq A < B \leq 1; 0 < B \leq 1; n \in \mathbb{N}_0; z \in U) \right\}$$

Silverman [9] defined the class of univalent functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ for which $arg(a_k)$ prescribed in such a way that $f(z)$ is univalent if and only if $f(z)$ is starlike as follows:

Definition 2.([9]) A function $f(z)$ of the form (1.1) is said to be in the class $V(\theta_k)$ if $f \in \mathcal{A}$ and $arg(a_k) = \theta_k$ for all $k \geq 2$. If further more there exist a real number δ such that $\theta_k + (k - 1)\delta \equiv \pi \ (mod\ 2\pi)$ $(k \geq 2)$, then $f(z)$ is said to be in the class $V(\theta_k, \delta)$. The union of $V(\theta_k, \delta)$ taken over all possible sequences $\{\theta_k\}$ and all possible real numbers δ is denoted by V.

Let $VQ(n, \lambda, A, B)$ denote the subclass of V consisting of functions $f(z) \in Q(n, \lambda, A, B)$.
We note that $VQ(0, 0, 2\alpha - 1, 1) = C_\alpha$ $(0 \leq \alpha < 1) = \left\{ f \in V : \mathbf{Re}\{f'(z)\} > \alpha \right\}$, studied by Srivastava and Owa [10].
Also we note that by specializing the parameters λ, A, B and n we can obtain different classes with varying arguments:

(i) $VQ(0, \lambda, 2\alpha - 1, 1) = VR(\lambda, \alpha)$ $(0 \leq \alpha < 1, \lambda \geq 0)$

(ii) $VQ(0, 0, 2\alpha - 1, 1) = VT^{**}(\alpha)$ $(0 \leq \alpha < 1)$

(iii) $VQ(n, 0, 2\alpha - 1, 1) = VQ_n(\alpha)$ $(0 \leq \alpha < 1, n \in \mathbb{N}_0)$

(iv) $VQ(0,0,(2\alpha-1)\beta,\beta) = VP^*(\alpha,\beta)$ $(0 \le \alpha < 1, 0 < \beta \le 1)$

(v) $VQ(0,0,((1+\mu)\alpha-1)\beta,\mu\beta) = VP^*(\alpha,\beta,\mu)$ $(0 \le \alpha < 1, 0 < \beta \le 1, 0 < \mu \le 1)$

(vi) $VQ(0,\lambda,A,B)$=$VR(\lambda,A,B)$ $(\lambda \ge 0, -1 \le A < B \le 1, 0 < B \le 1)$

(vii) $VQ(n,0,A,B)$=$VQ_n(A,B)$ $(-1 \le A < B \le 1, 0 < B \le 1, n \in \mathbb{N}_0)$

In this paper we obtain coefficient bounds for functions in the class $VQ(n,\lambda,A,B)$, further we obtain distortion bounds and the extreme points for functions in this class.

2. Coefficient Estimates

Unless otherwise mentioned, we assume in the reminder of this paper that, $\lambda \ge 0$, $-1 \le A < B \le 1$, $0 < B \le 1$, $n \in \mathbb{N}_0$, $\delta(n,k)$ and C_k are given by (1.7) and (2.2) respectively and $z \in U$.

Theorem 1. *Let the function $f(z)$ defined by (1.1) be in V. Then $f(z) \in VQ(n,\lambda,A,B)$, if and only if*

$$\sum_{k=2}^{\infty} k\delta(n,k)C_k |a_k| \le (B-A)(n+1), \tag{2.1}$$

where

$$C_k = (1+B)[n+1+\lambda(k-1)]. \tag{2.2}$$

Proof. Suppose that $f(z) \in VQ(n,\lambda,A,B)$. Then

$$h(z) = (1-\lambda)(D^n f(z))' + \lambda\left(D^{n+1}f(z)\right)' = \frac{1+Aw(z)}{1+Bw(z)}, \tag{2.3}$$

where

$$w \in H = \{w \text{ } analytic, \text{ } w(0)=0 \text{ } and \text{ } |w(z)|<1, \text{ } z \in U\}.$$

Thus we get

$$w(z) = \frac{1-h(z)}{Bh(z)-A}.$$

Therefore

$$h(z) = 1 + \sum_{k=2}^{\infty} \frac{k[n+1+\lambda(k-1)]\delta(n,k)}{n+1} a_k z^{k-1},$$

and $|w(z)| < 1$ implies

$$\left| \frac{\sum\limits_{k=2}^{\infty} \frac{k[n+1+\lambda(k-1)]\delta(n,k)}{n+1} a_k z^{k-1}}{(B-A) + B\sum\limits_{k=2}^{\infty} \frac{k[n+1+\lambda(k-1)]\delta(n,k)}{n+1} a_k z^{k-1}} \right| < 1. \tag{2.4}$$

Since $f(z) \in V$, $f(z)$ lies in the class $V(\theta_k, \delta)$ for some sequence $\{\theta_k\}$ and a real number δ such that

$$\theta_k + (k-1)\delta \equiv \pi (mod\ 2\pi) \quad (k \geq 2)\,.$$

Set $z = re^{i\delta}$ in (2.4), we get

$$\left| \frac{\sum\limits_{k=2}^{\infty} \frac{k[n+1+\lambda(k-1)]\delta(n,k)}{n+1} |a_k| \, r^{k-1}}{(B-A) - B\sum\limits_{k=2}^{\infty} \frac{k[n+1+\lambda(k-1)]\delta(n,k)}{n+1} |a_k| \, r^{k-1}} \right| < 1. \tag{2.5}$$

Since $\mathbf{Re}\,\{w(z)\} < |w(z)| < 1$, we have

$$\mathbf{Re} \left\{ \frac{\sum\limits_{k=2}^{\infty} \frac{k[n+1+\lambda(k-1)]\delta(n,k)}{n+1} |a_k| \, r^{k-1}}{(B-A) - B\sum\limits_{k=2}^{\infty} \frac{k[n+1+\lambda(k-1)]\delta(n,k)}{n+1} |a_k| \, r^{k-1}} \right\} < 1. \tag{2.6}$$

Hence

$$\sum_{k=2}^{\infty} kC_k\delta(n,k)\, |a_k| \, r^{k-1} \leq (B-A)\,(n+1). \tag{2.7}$$

Letting $r \longrightarrow 1$ in (2.7), we get (2.1).
Conversely, $f(z) \in V$ and satisfies (2.1). Since $r^{k-1} < 1$. So we have

$$\begin{aligned}
\left| \sum_{k=2}^{\infty} \tfrac{k[n+1+\lambda(k-1)]\delta(n,k)}{n+1} |a_k| \, z^{k-1} \right| \quad &\leq \quad \sum_{k=2}^{\infty} \frac{k[n+1+\lambda(k-1)]\delta(n,k)}{n+1} |a_k| \, r^{k-1} \\
&\leq \quad (B-A) - B\sum_{k=2}^{\infty} \frac{k[n+1+\lambda(k-1)]\delta(n,k)}{n+1} |a_k| \, r^{k-1} \\
&\leq \quad \left| (B-A) - B\sum_{k=2}^{\infty} \frac{k[n+1+\lambda(k-1)]\delta(n,k)}{n+1} a_k z^{k-1} \right| \\
&\leq \quad \left| (B-A) + B\sum_{k=2}^{\infty} \frac{k[n+1+\lambda(k-1)]\delta(n,k)}{n+1} a_k z^{k-1} \right|
\end{aligned}$$

which gives (2.4) and hence follows that

$$(1-\lambda)\left(D^n f(z)\right)' + \lambda \left(D^{n+1} f(z)\right)' = \frac{1 + Aw(z)}{1 + Bw(z)}$$

that is $f(z) \in VQ(n, \lambda, A, B)$. This completes the proof of Theorem 1.

Corollary 1. Let the function $f(z)$ defined by (1.1) be in the class $VQ(n,\lambda,A,B)$. Then

$$|a_k| \leq \frac{(B-A)(n+1)}{kC_k\delta(n,k)} \ (k \geq 2).$$

The result (2.1) is sharp for the function $f(z)$ defined by

$$f(z) = z + \frac{(B-A)(n+1)}{kC_k\delta(n,k)} e^{i\theta_k} z^k \ (k \geq 2). \tag{2.8}$$

3. Distortion Theorems

Theorem 2. *Let the function $f(z)$ defined by (1.1) be in the class $VQ(n,\lambda,A,B)$. Then*

$$|z| - \frac{B-A}{C_2}|z|^2 \leq |f(z)| \leq |z| + \frac{B-A}{C_2}|z|^2 . \tag{3.1}$$

The result is sharp.

Proof. We employ the same technique as used by Silverman [9]. In view of Theorem 1, since

$$\Phi(k) = C_k\delta(n,k), \tag{3.2}$$

is an increasing function of $k (k \geq 2)$, we have

$$\Phi(2)\sum_{k=2}^{\infty}|a_k| \leq \sum_{k=2}^{\infty}\Phi(k)|a_k| \leq (B-A)(n+1),$$

that is

$$\sum_{k=2}^{\infty}|a_k| \leq \frac{(B-A)(n+1)}{\Phi(2)} = \frac{(B-A)}{2C_2}. \tag{3.3}$$

Thus we have

$$|f(z)| \leq |z| + \sum_{k=2}^{\infty}|a_k||z|^k \leq |z| + |z|^2\sum_{k=2}^{\infty}|a_k|,$$

Thus

$$|f(z)| \leq |z| + \frac{(B-A)}{2C_2}|z|^2 .$$

Similarly, we get

$$|f(z)| \geq |z| - \sum_{k=2}^{\infty}|a_k||z|^k \geq |z| - |z|^2\sum_{k=2}^{\infty}|a_k| .$$

Thus

$$|f(z)| \geq |z| - \frac{(B-A)}{2C_2}|z|^2 .$$

This completes the proof of Theorem 2. Finally the result is sharp for the function

$$f(z) = z + \frac{(B-A)}{2C_2} e^{i\theta_2} z^2, \tag{3.4}$$

at $z = \pm |z| e^{-i\theta_2}$.

Corollary 2. Under the hypotheses of Theorem 2, $f(z)$ is included in a disc with center at the origin and radius r_1 given by

$$r_1 = 1 + \frac{(B-A)}{2C_2}. \tag{3.5}$$

Theorem 3. *Let the function $f(z)$ defined by (1.1) be in the class $VQ(n, \lambda, A, B)$. Then*

$$1 - \frac{(B-A)}{C_2}|z| \leq \left|f^{'}(z)\right| \leq 1 + \frac{(B-A)}{C_2}|z| . \tag{3.6}$$

The result is sharp.

Proof. Similarly $\dfrac{\Phi(k)}{k}$ is an increasing function of $k(k \geq 2)$, where $\Phi(k)$ is defined by (3.2). In view of Theorem 1, we have

$$\frac{\Phi(2)}{2}\sum_{k=2}^{\infty} k|a_k| \leq \sum_{k=2}^{\infty}\Phi(k)|a_k| \leq (B-A)(n+1),$$

that is

$$\sum_{k=2}^{\infty} k|a_k| \leq \frac{(B-A)}{\Phi(2)} = \frac{(B-A)}{C_2}.$$

Thus we have

$$\left|f^{'}(z)\right| \leq 1 + |z|\sum_{k=2}^{\infty} k|a_k| \leq 1 + \frac{(B-A)}{C_2}|z| . \tag{3.7}$$

Similarly

$$\left|f^{'}(z)\right| \geq 1 - |z|\sum_{k=2}^{\infty} k|a_k| \geq 1 - \frac{(B-A)}{C_2}|z| . \tag{3.8}$$

Finally, we can see that the assertions of Theorem 3 are sharp for the function $f(z)$ defined by (3.4). This completes the proof of Theorem 3.

Corollary 3. Under the hypotheses of Theorem 3, $f^{'}(z)$ is included in a disc with center at the origin and radius r_2 given by

$$r_2 = 1 + \frac{(B-A)}{C_2}. \tag{3.9}$$

4. Extreme Points

Theorem 4. *Let the function $f(z)$ defined by (1.1) be in the class $VQ(n, \lambda, A, B)$, with* arg $a_k = \theta_k$, *where $\theta_k + (k-1)\delta \equiv \pi$ (mod 2π) $(k \geq 2)$. Define*

$$f_1(z) = z$$

and

$$f_k(z) = z + \frac{(B-A)(n+1)}{kC_k\delta(n,k)} e^{i\theta_k} z^k \quad (k \geq 2; z \in U).$$

Then $f(z) \in VQ(n, \lambda, A, B)$ if and only if $f(z)$ can expressed in the form $f(z) = \sum\limits_{k=1}^{\infty} \mu_k f_k(z)$, where $\mu_k \geq 0$ and $\sum\limits_{k=1}^{\infty} \mu_k = 1$.

Proof. If $f(z) = \sum\limits_{k=1}^{\infty} \mu_k f_k(z)$ with $\mu_k \geq 0$ and $\sum\limits_{k=1}^{\infty} \mu_k = 1$, then

$$\begin{aligned} & \sum_{k=2}^{\infty} [kC_k\delta(n,k)] \frac{(B-A)(n+1)}{kC_k\delta(n,k)} \mu_k = \sum_{k=2}^{\infty} (B-A)(n+1)\mu_k \\ = \; & (1-\mu_1)(B-A)(n+1) \leq (B-A)(n+1). \end{aligned}$$

Hence $f(z) \in VQ(n, \lambda, A, B)$.
Conversely, let the function $f(z)$ defined by (1.1) be in the class $VQ(n, \lambda, A, B)$, define

$$\mu_k = \frac{kC_k\delta(n,k)}{(B-A)(n+1)} |a_k| \qquad (k \geq 2)$$

and

$$\mu_1 = 1 - \sum_{k=2}^{\infty} \mu_k.$$

From Theorem 1, $\sum\limits_{k=2}^{\infty} \mu_k \leq 1$ and so $\mu_1 \geq 0$. Since $\mu_k f_k(z) = \mu_k z + a_k z^k$, then

$$\sum_{k=1}^{\infty} \mu_k f_k(z) = z + \sum_{k=2}^{\infty} a_k z^k = f(z).$$

This completes the proof of Theorem 4.

Remarks.

(i) Putting $\lambda = n = 0$, $A = 2\alpha - 1\,(0 \leq \alpha < 1)$ and $B = 1$ in all the above results, we obtain the corresponding results obtained by Srivastava and Owa [10];

(ii) Putting $n = 0$ in all the above results, we obtain the corresponding results for the class $VR(\lambda, A, B)$ of which $R(\lambda, A, B)$ is given by (1.9);

(iii) Putting $\lambda = 0$ in all the above results, we obtain the corresponding results for the class $VQ_n(A, B)$ of which $Q_n(A, B)$ is given by (1.10).

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