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Fekete-Szegö Problem and Upper Bound of Second Hankel Determinant for a New Class of Analytic Functions

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Abstract. In the present investigation we consider Fekete-Szegö problem with complex parameter μ and also find upper bound of the second Hankel determinant $|a_2a_4 - a_3^2|$ for functions belonging to a new class $S_\gamma^\tau(A,B)$ using Toeplitz determinants.

1. Introduction and Preliminaries

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and $\mathcal{S}$ denote the subclass of $\mathcal{A}$ that are univalent in $\mathbb{U}$. Further, let $\mathcal{P}$ be the family of functions $p(z) \in \mathcal{H}$ (class of analytic function in $\mathbb{U}$) satisfying $p(0) = 1$ and $\Re(p(z)) > 0$.

If $f, g \in \mathcal{H}$, then the function f is said to be subordinate to g, written as $f(z) \prec g(z)$ $(z \in \mathbb{U})$, if there exists a Schwarz function $w \in \mathcal{H}$ with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathbb{U})$ such that $f(z) = g(w(z))$.
In particular, if g is univalent in $\mathbb{U}$, then we have the following equivalence:

$$f(z) \prec g(z) \iff f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

We now introduce the following class of functions.
Definition 1.1. Let $0 \leqq \gamma \leqq 1$, $\tau \in \mathbb{C} \setminus \{0\}$. A function $f \in \mathcal{A}$ is said to be in the class $S_\gamma^\tau(A,B)$, if

$$1 + \frac{1}{\tau}\left[(1-\gamma)\frac{f(z)}{z} + \gamma f'(z) - 1\right] \prec \frac{1+Az}{1+Bz} \; (-1 \leqq B < A \leqq 1; z \in \mathbb{U}), \tag{1.2}$$

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which is equivalent to saying that

$$\left|\frac{(1-\gamma)\frac{f(z)}{z}+\gamma f'(z)-1}{\tau(A-B)-B[(1-\gamma)\frac{f(z)}{z}+\gamma f'(z)-1]}\right|<1. \tag{1.3}$$

The class $S_{\gamma}^{\tau}(A,B)$ is essentially motivated by Swaminathan [25]. We list few particular cases of this class discussed in the literature
[1] $S_{\gamma}^{\tau}(1-2\beta,-1)=P_{\gamma}^{\tau}(\beta)$ for $0\leqq\beta<1,\ \tau=\mathbb{C}\setminus\{0\}$ is discussed recently by Swaminathan [25].
[2] The class $S_{\gamma}^{\tau}(1-2\beta,-1)$ for $\tau=e^{i\eta}cos\eta$ where $-\pi/2<\eta<\pi/2$ is considered in [11].
[3] The class $S_{1}^{\tau}(1-2\beta,-1)$ is considered in [3].
[4] The class $S_{1}^{\tau}(A,B)=\mathcal{R}^{\tau}(A,B)$ is considered in [6].
[5] The class $S_{1}^{\tau}(A,B)=\mathcal{R}_{0}^{\tau}(A,B)$ is considered in [1, 2].
For more details about these classes see the corresponding references.

Fekete and Szegö proved a noticeable result that the estimate

$$\left|a_3-\lambda a_2^2\right|\leqq 1+2\,exp\left(\frac{-2\lambda}{1-\lambda}\right) \tag{1.4}$$

holds for any normalized univalent function $f(z)$ of the form (1.1) in the open unit disk $\mathbb{U}$ and for $0\leqq\lambda\leqq 1$. This inequality is sharp for each λ (see [5]). The coefficient functional

$$\phi_{\lambda}(f)=a_3-\lambda{a_2}^2=\frac{1}{6}\left(f'''(0)-\frac{3\lambda}{2}[f''(0)]^2\right), \tag{1.5}$$

on normalized analytic functions f in the unit disk represents various geometric quantities, for example, when $\lambda=1$, $\phi_{\lambda}(f)=a_3-\lambda{a_2}^2$, becomes $S_f(0)/6$, where S_f denote the Schwarzian derivative $(f''/f')'-(f''/f')^2/2$ of locally univalent functions f in $\mathbb{U}$. In literature, there exists a large number of results about inequalities for $\phi_{\lambda}(f)$ corresponding to various subclasses of $\mathcal{S}$. The problem of maximizing the absolute value of the functional $\phi_{\lambda}(f)$ is called the Fekete-Szegö problem; see [5]. In [12], Koepf solved the Fekete-Szegö problem for close-to-convex functions and the largest real number λ for which $\phi_{\lambda}(f)$ is maximized by the Koebe function $z/(1-z)^2$ is $\lambda=1/3$, and later in [13] (see also [15]), this result was generalized for functions that are close-to-convex of order β. One can see [1], [16] and [24] for result concerning to Fekete-Szegö problem for other classes.

In 1976, Noonan and Thomas [19] discussed the q^{th} Hankel determinant of a locally univalent analytic function $f(z)$ for $q\geqq 1$ and $n\geqq 1$ which is defined by

$$H_q(n)=\begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1}\\ a_{n+1} & a_{n+2} & \cdots & a_{n+q}\\ \cdot & \cdot & & \cdot\\ \cdot & \cdot & & \cdot\\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2}\end{vmatrix}.$$

For our present discussion, we consider the Hankel determinant in the case $q = 2$ and $n = 2$ i.e. $H_2(2) = a_2a_4 - a_3^2$. This is popularly known as the second Hankel determinant of f.

In the present paper we obtain an upper bound to the functional $H_2(2)$ for $f(z) \in S_\gamma^\tau(A, B)$. Earlier Janteng et al. [8, 9], Mishra and Gochhayat [17], Mishra and Kund [18], Bansal [2] and many other author have obtained sharp upper bounds of $H_2(2)$ for different classes of analytic functions. To prove our main results we need the following Lemma:

Lemma 1.1. ([23]) *Let*

$$h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \prec 1 + \sum_{n=1}^{\infty} C_n z^n = H(z) \ (z \in \mathbb{U}). \tag{1.6}$$

If the function H is univalent in $\mathbb{U}$ and $H(\mathbb{U})$ is a convex set, then

$$|c_n| \leqq |C_1|. \tag{1.7}$$

Lemma 1.2. ([4]) *Let a function $p \in \mathcal{P}$ be given by the series*

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + ... \ (z \in \mathbb{U}), \tag{1.8}$$

then, the sharp estimate

$$|c_n| \leqq 2 \ (n \in \mathbb{N}), \tag{1.9}$$

holds.

Lemma 1.3. ([10, 14]) *Let $p \in \mathcal{P}$ be given by the power series* (1.8)*, then for any complex number μ,*

$$\left|c_2 - \mu {c_1}^2\right| \leqq 2 \max\{1, |2\mu - 1|\} \tag{1.10}$$

and the result is sharp for the functions given by

$$p(z) = \frac{1 + z^2}{1 - z^2}; \ p(z) = \frac{1 + z}{1 - z} \ (z \in \mathbb{U}). \tag{1.11}$$

Lemma 1.4. ([7]) *Let a function $p \in \mathcal{P}$ be given by the power series* (1.8)*, then*

$$2c_2 = c_1^2 + x(4 - c_1^2) \tag{1.12}$$

for some x, $|x| \leqq 1$, and

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1 x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z, \tag{1.13}$$

for some z, $|z| \leqq 1$.

2. Main Results

We first give the following result related to the coefficient of $f(z) \in S_\gamma^\tau(A,B)$.

Theorem 2.1. *Let $f(z) \in \mathcal{A}$ is of the form (1.1). If $f(z)$ belongs to the class $S_\gamma^\tau(A,B)$, then*

$$|a_n| \leqq \frac{|\tau|(A-B)}{[1+\gamma(n-1)]} \ (n \in N\backslash\{1\}). \tag{2.1}$$

Proof. If $f(z)$ of the form (1.1) belongs to in $S_\gamma^\tau(A,B)$, then by definition

$$1+\frac{1}{\tau}\left[(1-\gamma)\frac{f(z)}{z}+\gamma f'(z)-1\right] \prec \frac{1+Az}{1+Bz} = h(z) \ (-1 \leqq B < A \leqq 1; z \in \mathbb{U}), \tag{2.2}$$

where $h(z)$ is obviously convex univalent in $\mathbb{U}$ under the stated conditions on A and B. Using (1.1) and doing Binomial expansion of $(1+Bz)^{-1}$ in (2.2), we have

$$1+\frac{1}{\tau}\left[(1-\gamma)\frac{f(z)}{z}+\gamma f'(z)-1\right]$$

$$=1+\sum_{n=1}^{\infty}\frac{(1+n\gamma)}{\tau}\ a_{n+1}z^n \prec 1+(A-B)z-B(A-B)z^2+...(z \in \mathbb{U}).$$

Now, by applying Lemma 1.1, we get the desired result. □

It is easy to derive a sufficient condition for $f(z)$ to be in $S_\gamma^\tau(A,B)$ using standard techniques (see [22]). Hence we state the following result without proof.

Theorem 2.2. *Let $f(z) \in \mathcal{A}$. Then a sufficient condition for $f(z)$ to be in $S_\gamma^\tau(A,B)$ is*

$$\sum_{n=2}^{\infty}[1+\gamma(n-1)]\,|a_n| \leqq \frac{|\tau|(A-B)}{|B|+1}. \tag{2.3}$$

In the next two theorems we give the result concerning Fekete-Szegö problem and upper bound of Hankel determinant for the class $S_\gamma^\tau(A,B)$.

Theorem 2.3. *Let a function $f(z)$ be given by (1.1) belongs to the class $S_\gamma^\tau(A,B)$, where*

$$0 \leqq \gamma \leqq 1,\ \tau \in \mathbb{C}\setminus\{0\},\ -1 \leqq B < A \leqq 1; z \in \mathbb{U}, \tag{2.4}$$

then for any complex number μ

$$|a_3-\mu a_2^2| \leqq \frac{(A-B)\,|\tau|}{(1+2\gamma)} \max\left\{1, \left|B+\frac{\mu\tau(A-B)(1+2\gamma)}{(1+\gamma)^2}\right|\right\}. \tag{2.5}$$

The result is sharp.

Proof. If $f(z) \in S^{\tau}_{\gamma}(A,B)$, then there exists a Schwarz function $w(z)$ analytic in $\mathbb{U}$ with $w(0)=0$ and $|w(z)|<1$ in $\mathbb{U}$, such that

$$1+\frac{1}{\tau}\left[(1-\gamma)\frac{f(z)}{z}+\gamma f'(z)-1\right]=\phi(w(z))\ (z\in\mathbb{U}). \tag{2.6}$$

where

$$\phi(z)=\frac{1+Az}{1+Bz}=1+(A-B)z-B(A-B)z^2+B^2(A-B)z^3+.... \tag{2.7}$$

$$=1+B_1z+B_2z^2+B_3z^3+.....\ (z\in\mathbb{U}).$$

Define the function $p_1(z)$ by

$$p_1(z)=\frac{1+w(z)}{1-w(z)}=1+c_1z+c_2z^2+...(z\in\mathbb{U}). \tag{2.8}$$

Since $w(z)$ is a Schwarz function, we see that $\Re p_1(z)>0$ and $p_1(0)=1$. Define the function $h(z)$ by

$$h(z)=1+\frac{1}{\tau}\left[(1-\gamma)\frac{f(z)}{z}+\gamma f'(z)-1\right]=1+b_1z+b_2z^2+....(z\in\mathbb{U}). \tag{2.9}$$

In view of the equations (2.6), (2.8) and (2.9), we have

$$h(z)=\phi\left(\frac{p_1(z)-1}{p_1(z)+1}\right)=\phi\left(\frac{c_1z+c_2z^2+c_3z^3+...}{2+c_1z+c_2z^2+c_3z^3+...}\right) \tag{2.10}$$

$$=\phi\left(\frac{1}{2}c_1z+\frac{1}{2}(c_2-c_1^2/2)z^2+\frac{1}{2}(c_3-c_1c_2+c_1^3/4)z^3+...\right) \tag{2.11}$$

$$=1+\frac{B_1c_1}{2}z+\left[\frac{B_1}{2}(c_2-c_1^2/2)+\frac{B_2c_1^2}{4}\right]z^2 \tag{2.12}$$

$$+\left[\frac{B_1}{2}\left(c_3-c_1c_2+c_1^3/4\right)+\frac{B_2c_1}{2}\left(c_2-c_1^2/2\right)+\frac{B_3c_1^3}{8}\right]z^3+....$$

Thus,

$$b_1=\frac{1}{2}B_1c_1;\ \ b_2=\frac{1}{2}B_1(c_2-c_1^2/2)+\frac{1}{4}B_2c_1^2 \tag{2.13}$$

and

$$b_3 = \frac{B_1}{2}\left(c_3 - c_1c_2 + c_1^3/4\right) + \frac{B_2c_1}{2}\left(c_2 - c_1^2/2\right) + \frac{B_3c_1^3}{8}. \tag{2.14}$$

Using (2.7) and (2.9) in (2.13) and (2.14), we obtain

$$a_2 = \frac{(A-B)c_1\tau}{2(1+\gamma)};\ \ a_3 = \frac{\tau(A-B)}{4(1+2\gamma)}\left[2c_2 - c_1^2(1+B)\right] \tag{2.15}$$

and

$$a_4 = \frac{\tau(A-B)}{8(1+3\gamma)}\left[4c_3 - 4c_1c_2(1+B) + c_1^3(1+B)^2\right]. \tag{2.16}$$

Therefore we have

$$a_3 - \mu {a_2}^2 = \frac{(A-B)\tau}{2(1+2\gamma)}\left[c_2 - \nu {c_1}^2\right], \tag{2.17}$$

where

$$\nu = \left(1 + B + \frac{\mu\tau(A-B)(1+2\gamma)}{(1+\gamma)^2}\right). \tag{2.18}$$

Our result now follows by an application of Lemma 1.2. Also by the application of Lemma 1.2 equality in (2.5) is obtained when

$$p_1(z) = \frac{1+z^2}{1-z^2};\ p_1(z) = \frac{1+z}{1-z} \tag{2.19}$$

but

$$h(z) = 1 + \frac{1}{\tau}\left[(1-\gamma)\frac{f(z)}{z} + \gamma f'(z) - 1\right] = \phi\left(\frac{p_1(z)-1}{p_1(z)+1}\right). \tag{2.20}$$

Putting value of $p_1(z)$ we get the desired results. □

Theorem 2.4. *Let a function $f(z)$ given by (1.1) be in the class $S_\gamma^\tau(A,B)$, where*

$$0 \leqq \gamma \leqq 1,\ \tau \in \mathbb{C}\setminus\{0\},\ -1 \leqq B < A \leqq 1; z \in \mathbb{U}, \tag{2.21}$$

then

$$|a_2a_4 - a_3^2| \leqq \frac{|\tau|^2(A-B)^2}{(1+2\gamma)^2}. \tag{2.22}$$

Proof. Using (2.15) and (2.16), we have

$$|a_2a_4-a_3^2| = \frac{|\tau|^2(A-B)^2}{16(1+\gamma)(1+3\gamma)}|4c_1c_3-4c_1^2c_2(1+B)+c_1^4(1+B)^2$$

$$-\frac{(1+\gamma)(1+3\gamma)}{(1+2\gamma)^2}[4c_2^2-4c_1^2c_2(1+B)+c_1^4(1+B)^2]|$$

$$= T\left|4c_1c_3-4c_1^2c_2(1+B)+c_1^4(1+B)^2-p[4c_2^2-4c_1^2c_2(1+B)+c_1^4(1+B)^2]\right|$$

$$(2.23) \qquad = T\left|4c_1c_3-4c_1^2c_2(1+B)(1-p)-4pc_2^2+c_1^4(1+B)^2(1-p)\right|.$$

where

$$(2.24) \qquad T = \frac{|\tau|^2(A-B)^2}{16(1+\gamma)(1+3\gamma)} \text{ and } p = \frac{(1+\gamma)(1+3\gamma)}{(1+2\gamma)^2}.$$

It can be easily verified that for $0 \leqq \gamma \leqq 1$, $p \in \left[\frac{8}{9}, 1\right]$. The above equation (2.23) is equivalent to

$$(2.25) \qquad |a_2a_4-a_3^2| = T|d_1c_1c_3+d_2c_1^2c_2+d_3c_2^2+d_4c_1^4|,$$

where

$$(2.26) \qquad d_1 = 4;\ d_2 = -4(1+B)(1-p);\ d_3 = -4p\ ;\ d_4 = (1-p)(1+B)^2.$$

Since the functions $p(z)$ and $p(e^{i\theta}z)$ $(\theta \in \mathbb{R})$ are members of the class $\mathcal{P}$ simultaneously, we assume without loss of generality that $c_1 > 0$. For convenience of notation, we take $c_1 = c$ $(c \in [0,2]$, see(1.7)). Also, substituting the values of c_2 and c_3 respectively, from (1.12) and (1.13) in (2.25), we have

$$|a_2a_4-a_3^2| =$$

$$\frac{T}{4}|c^4(d_1+2d_2+d_3+4d_4)+2xc^2(4-c^2)(d_1+d_2+d_3)+(4-c^2)x^2(-d_1c^2+d_3(4-c^2))$$

$$+2d_1c(4-c^2)(1-|x|^2)z|$$

An application of triangle inequality, replacement of $|x|$ by μ and substituting the values of d_1, d_2, d_3 and d_4 from (2.26), we have

$$|a_2a_4-a_3^2| \leqq \frac{T}{4}[4c^4(1-p)B^2$$

$$+8|B|(1-p)\mu c^2(4-c^2)+(4-c^2)\mu^2(4c^2+4p(4-c^2))+8c(4-c^2)(1-\mu^2)]$$

$$= T[c^4(1-p)B^2+2c(4-c^2)+2\mu|B|(1-p)c^2(4-c^2)+\mu^2(4-c^2)(c^2(1-p)-2c+4p)], \tag{2.27}$$

$$= F(c,\mu)(\text{say}), \tag{2.28}$$

Next, we assume that the upper bound for (2.28) occurs at an interior point of the rectangle $[0,2]\times[0,1]$. Differentiating $F(c,\mu)$ in (2.28) partially with respect to μ, we have

$$\frac{\partial F}{\partial \mu} = T\left[2|B|(1-p)c^2(4-c^2)+2\mu(4-c^2)(c^2(1-p)-2c+4p)\right]. \tag{2.29}$$

For $0<\mu<1$ and for any fixed c with $0<c<2$, from (2.29), we observe that $\frac{\partial F}{\partial \mu}>0$. Therefore $F(c,\mu)$ is an increasing function of μ, which contradicts our assumption that the maximum value of $F(c,\mu)$ occurs at an interior point of the rectangle $[0,2]\times[0,1]$. Moreover, for fixed $c\in[0,2]$,

$$Max\ F(c,\mu)=F(c,1)=G(c)(say). \tag{2.30}$$

Thus

$$G(c)=T\left[c^4(1-p)(B^2-2|B|-1)+4c^2(2|B|(1-p)+1-2p)+16p\right]. \tag{2.31}$$

Next,

$$G'(c)=4cT\left[c^2(1-p)(B^2-2|B|-1)+2(2|B|(1-p)+1-2p)\right] \tag{2.32}$$

$$=4cT\left[c^2(1-p)(B^2-2|B|-1)+2\left\{(1-p)[2|B|+1]-p\right\}\right]. \tag{2.33}$$

So $G'(c)<0$ for $0<c<2$ and has real critical point at $c=0$. Also $G(c)>G(2)$. Therefore, maximum of $G(c)$ occurs at $c=0$. Therefore, the upper bound of $F(c,\mu)$ corresponds to $\mu=1$ and $c=0$. Hence,

$$|a_2a_4-a_3^2|\leqq 16pT=\frac{|\tau|^2(A-B)^2}{(1+2\gamma)^2}.$$

This completes the proof of the Theorem. □

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