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Hyers–Ulam Stability of Jensen Functional Equation in p-Banach Spaces

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Abstract. In this paper, we prove the generalized Hyers–Ulam stability of the following Jensen type functional equation

$$f(\frac{x-y}{n}+z)+f(\frac{y-z}{n}+x)+f(\frac{z-x}{n}+y)=f(x)+f(y)+f(z)$$

in p-Banach spaces for any fixed nonzero integer n.

1. Introduction

In 1940, S. M. Ulam [21] suggested the stability problem of functional equations concerning the stability of group homomorphisms as follows:

We are given a group G_1 and a metric group G_2 with metric $\rho(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if $f : G_1 \to G_2$ satisfies $\rho(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G_1$, then a homomorphism $h : G_1 \to G_2$ exists with $\rho(f(x), h(x)) < \epsilon$ for all $x \in G_1$?

In 1941, D.H. Hyers [7] considered the case of approximately additive mappings between Banach spaces and proved the following result.

Suppose that E_1 and E_2 are Banach spaces and $f : E_1 \to E_2$ satisfies the following condition: if there is an $\epsilon \geq 0$ such that

$$\|f(x+y)-f(x)-f(y)\| \leq \epsilon$$

for all $x, y \in E_1$, then the limit $h(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ exists for all $x \in E_1$ and there exists a unique additive mapping $h : E_1 \to E_2$ such that

$$\|f(x)-h(x)\| \leq \epsilon.$$

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Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each $x \in E_1$, then the mapping h is $\mathbb{R}$-linear.

The method which was provided by Hyers, and which produces the additive mapping h, is called a direct method. This method is the most important and most powerful tool for studying the stability of various functional equations. Hyers' theorem was generalized by T. Aoki [1] and D.G. Bourgin [3] for additive mappings by considering an unbounded Cauchy difference. In 1978, Th.M. Rassias [15] also provided a generalization of Hyers' theorem for linear mappings which allows the Cauchy difference to be unbounded.

Let E_1 and E_2 be two Banach spaces and $f : E_1 \to E_2$ be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed x. Assume that there exists $\epsilon > 0$ and $0 \leq p < 1$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p), \quad \forall x, y \in E_1.$$

Then, there exists a unique $\mathbb{R}$-linear mapping $T : E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \leq \frac{2\epsilon}{2 - 2^p}\|x\|^p$$

for all $x \in E_1$. A generalized result of Rassias' theorem was obtained by P. Găvruta in [6] and S. Jung in [8]. In 1990, Th.M. Rassias [16] during the 27th International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geq 1$. In 1991, Z. Gajda [4], following the same approach as in [15], gave an affirmative solution to this question for $p > 1$. It was shown by Z. Gajda [4], as well as by Th.M. Rassias and P. Šemrl [17], that one cannot prove a Rassias' type theorem when $p = 1$. The counterexamples of Z. Gajda [4], as well as of Th.M. Rassias and P. Šemrl [17], have stimulated several mathematicians to invent new approximately additive or approximately linear mappings. In particular, J.M. Rassias [18, 19] proved a similar stability theorem in which he replaced the unbounded Cauchy difference by the factor $\|x\|^p\|y\|^q$ for $p, q \in \mathbb{R}$ with $p + q \neq 1$.

We recall some basic facts concerning quasi-normed spaces and some preliminary results. Let X be a real linear space. A quasi-norm is a real-valued function on X satisfying the following:

(1) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.

(2) $\|\lambda x\| = |\lambda|\|x\|$ for all $\lambda \in \mathbb{R}$ and all $x \in X$.

(3) There is a constant $M \geq 1$ such that $\|x + y\| \leq M(\|x\| + \|y\|)$ for all $x, y \in X$.

The pair $(X, \|\cdot\|)$ is called a quasi-normed space if $\|\cdot\|$ is a quasi-norm on X [2, 20]. The smallest possible M is called the modulus of concavity of $\|\cdot\|$. A quasi-Banach space is a complete quasi-normed space. In addition, a quasi-norm $\|\cdot\|$ is called a $p-$norm $(0 < p \leq 1)$ if

$$\|x + y\|^p \leq \|x\|^p + \|y\|^p$$

for all $x, y \in X$. In this case, a quasi-Banach space is called a p-Banach space.

Given a p-norm, the formula $d(x, y) := \|x - y\|^p$ gives us a translation invariant metric on X. By the Aoki–Rolewicz theorem [20], each quasi-norm is equivalent to some p-norm (see also [2]). Since it is much easier to work with p-norms, henceforth, we restrict our attention mainly to p-norms. We observe that if $x_1, x_2, \ldots, x_n$ are non-negative real numbers, then

$$\left(\sum_{i=1}^{n} x_i\right)^p \leq \sum_{i=1}^{n} {x_i}^p,$$

where $0 < p \leq 1$ [14].

In 2009, M.S Moslehian and A. Najati [10] introduced the Hyers–Ulam stability of an additive functional inequality

$$\|f(\frac{x-y}{2} + z) + f(\frac{y-z}{2} + x) + f(\frac{z-x}{2} + y)\| \leq \|f(x + y + z)\| \tag{1.1}$$

and then have investigated the general solution and the Hyers-Ulam stability problem for the functional inequality. The stability problems of several functional equations in quasi-normed spaces and several functional inequalities have been investigated by a number of authors and there are many interesting result concerning the stability of various functional inequalities [5, 10, 11, 12, 13, 14].

Recently, H. Kim and E. Son [9], proved the generalized Hyers–Ulam stability for the following Jensen type functional inequality

$$\|f(\frac{x-y}{n} + z) + f(\frac{y-z}{n} + x) + f(\frac{z-x}{n} + y)\| \leq \|f(x + y + z)\|$$

for any fixed nonzero integer n. Now, let's consider a modified and general Jensen type functional equation

$$f(\frac{x-y}{n} + z) + f(\frac{y-z}{n} + x) + f(\frac{z-x}{n} + y) = f(x) + f(y) + f(z) \tag{1.2}$$

for any fixed nonzero integer n. First of all, it is easy to see that a function f satisfies the equation (1.2) if and only if $f(x) - f(0)$ is additive. Thus the equation (1.2) may be called the Jensen type functional equation and the general solution of equation (1.2) may be called the Jensen type function. In this paper, we investigate the generalized Hyers–Ulam stability of (1.2) in p-Banach spaces for any fixed nonzero integer n.

2. Generalized Hyers-Ulam Stability of (1.2) by Direct Method

From now on, assume that X is a quasi-normed space with quasi-norm $\|.\|_X$ and that Y is a p-Banach space with p-norm $\|.\|$.

Before taking up the main subject, given a mapping $f : X \to Y$, we define the difference operator $Df : X^3 \to Y$ by

$$Df(x, y, z) := f(\frac{x-y}{n} + z) + f(\frac{y-z}{n} + x) + f(\frac{z-x}{n} + y) - f(x) - f(y) - f(z),$$

for all $x, y, z \in X$ and for any fixed nonzero integer n.

Theorem 2.1. Suppose that a mapping $f : X \to Y$ satisfies the functional inequality

$$\|Df(x,y,z)\|_Y \leq \varphi(x,y,z) \tag{2.1}$$

for all $x, y, z \in X$, and that the perturbing function $\varphi : X^3 \to \mathbb{R}^+$ satisfies

$$\sum_{i=0}^{\infty} \frac{1}{2^{(i+1)p}} \varphi(2^i x, 2^i y, 2^i z)^p < \infty \tag{2.2}$$

for all $x, y, z \in X$. Then, there exists a unique additive mapping $h : X \to Y$ defined by $h(x) = \lim_{k\to\infty} \frac{1}{2^k} f(2^k x)$ such that

$$\|f(x) - f(0) - h(x)\|_Y \leq \Big[\sum_{i=0}^{\infty} \frac{1}{2^{(i+1)p}} \varphi(2^i x, 2^i x, (1-n)2^i x)^p\Big]^{\frac{1}{p}} \tag{2.3}$$

for all $x \in X$.

Proof. Letting $y = x$ in (2.1), we have

$$\|f(\frac{x-z}{n} + x) + f(\frac{z-x}{n} + x) - 2f(x)\|_Y \leq \varphi(x,x,z) \tag{2.4}$$

for all $x, z \in X$. Replacing z by $x - nx$ in (2.4), we get

$$\|f(2x) + f(0) - 2f(x)\|_Y \leq \varphi(x,x,(1-n)x), \tag{2.5}$$

for all $x \in X$. Setting $g(x) = f(x) - f(0)$, we have

$$\|g(2x) - 2g(x)\|_Y \leq \varphi(x,x,(1-n)x), \tag{2.6}$$

that is,

$$\|g(x) - \frac{1}{2}g(2x)\|_Y \leq \frac{1}{2}\varphi(x,x,(1-n)x) \tag{2.7}$$

for all $x \in X$. It follows from (2.7) that

$$\begin{aligned}
\left\|\frac{g(2^l x)}{2^l} - \frac{g(2^m x)}{2^m}\right\|_Y^p &\leq \sum_{i=l}^{m-1} \left\|\frac{1}{2^i} g(2^i x) - \frac{1}{2^{i+1}} g(2^{i+1} x)\right\|_Y^p \\
&= \sum_{i=l}^{m-1} \frac{1}{2^{ip}} \left\|g(2^i x) - \frac{1}{2} g(2^{i+1} x)\right\|_Y^p \\
&\leq \sum_{i=l}^{m-1} \frac{1}{2^{(i+1)p}} \varphi(2^i x, 2^i x, (1-n)2^i x)^p \qquad (2.8)
\end{aligned}$$

for all non-negative integers m and l with $m > l \geq 0$ and $x \in X$. Since the right-hand side of (2.8) tends to zero as $l \to \infty$, by the convergence of the series (2.2), we obtain that the sequence $\{\frac{g(2^m x)}{2^m}\}$ is Cauchy for all $x \in X$. Because of the fact that Y is complete, it follows that the sequence $\{\frac{g(2^m x)}{2^m}\}$ converges in Y. Therefore, one can define a mapping $h : X \to Y$ as

$$h(x) = \lim_{m\to\infty} \frac{g(2^m x)}{2^m} = \lim_{m\to\infty} \frac{f(2^m x)}{2^m}, \quad x \in X.$$

Moreover, letting $l = 0$ and taking $m \to \infty$ in (2.8), we get

$$\|f(x) - f(0) - h(x)\|_Y \leq \Big[\sum_{i=0}^{\infty} \frac{1}{2^{(i+1)p}} \varphi(2^i x, 2^i x, (1-n)2^i x)^p\Big]^{\frac{1}{p}}$$

for all $x \in X$.

It follows from (2.1) and (2.2) that

$$\begin{aligned} \| \quad h \quad & (\frac{x-y}{n} + z) + h(\frac{y-z}{n} + x) + h(\frac{z-x}{n} + y) - h(x) - h(y) - h(z) \quad \|_Y^p \\ = & \lim_{k\to\infty} \frac{1}{2^{kp}} \|Df(2^k x, 2^k y, 2^k z)\|_Y^p \\ \leq & \lim_{k\to\infty} \frac{1}{2^{kp}} \varphi(2^k x, 2^k y, 2^k z)^p = 0 \end{aligned}$$

for all $x, y, z \in X$. So the mapping h is additive.

Next, let $h' : X \to Y$ be another additive mapping satisfying

$$\|f(x) - f(0) - h'(x)\|_Y \leq \big[\sum_{i=0}^{\infty} \frac{1}{2^{(i+1)p}} \varphi(2^i x, 2^i x, (1-n)2^i x)^p\big]^{\frac{1}{p}}$$

for all $x \in X$. Then, we have

$$\begin{aligned} \|h(x) - h'(x)\|_Y^p &= \left\| \frac{1}{2^k} h(2^k x) - \frac{1}{2^k} h'(2^k x) \right\|_Y^p \\ &\leq \frac{1}{2^{kp}} (\|h(2^k x) - f(2^k x) + f(0)\|_Y^p + \|f(2^k x) - f(0) - h'(2^k x)\|_Y^p) \\ &\leq \sum_{i=0}^{\infty} \frac{2}{2^{(i+k+1)p}} \varphi(2^{i+k} x, 2^{i+k} x, (1-n)2^{i+k} x)^p \\ &= \sum_{i=k}^{\infty} \frac{2}{2^{(i+1)p}} \varphi(2^i x, 2^i x, (1-n)2^i x)^p \end{aligned}$$

for all $k \in \mathbf{N}$ and all $x \in X$. Taking the limit as $k \to \infty$, we conclude that

$$h(x) = h'(x)$$

for all $x \in X$. This proves the uniqueness of the additive function h satisfying (2.3). □

If we put $\varphi(x,y,z) := \theta(\|x\|_X^{r_1}\|y\|_X^{r_2}\|z\|_X^{r_3})$ and $\varphi(x,y,z) := \theta_1\|x\|_X^{r_1} + \theta_2\|y\|_X^{r_2} + \theta_3\|z\|_X^{r_3}$ in the following corollaries, respectively, then we lead to the desired results.

Corollary 2.2. Let $r_i > 0$ for $i = 1,2,3$ with $\sum_{i=1}^3 r_i < 1$ and $\theta \geq 0$. If a mapping $f : X \to Y$ satisfies the following functional inequality

$$\|Df(x,y,z)\|_Y \leq \theta(\|x\|_X^{r_1}\|y\|_X^{r_2}\|z\|_X^{r_3})$$

for all $x,y,z \in X$, then there exists a unique additive mapping $h : X \to Y$ such that

$$\|f(x) - f(0) - h(x)\|_Y \leq \frac{\theta(1-n)^{r_3}\|x\|_X^r}{\sqrt[p]{2^p - 2^{rp}}}$$

for all $x \in X$, where $r = \sum_{i=1}^3 r_i$.

Corollary 2.3. Let $0 < r_i < 1$ and $\theta_i \geq 0$ for $i = 1,2,3$. If a mapping $f : X \to Y$ satisfies the following functional inequality

$$\|Df(x,y,z)\|_Y \leq \theta_1\|x\|_X^{r_1} + \theta_2\|y\|_X^{r_2} + \theta_3\|z\|_X^{r_3}$$

for all $x,y,z \in X$, then there exists a unique additive mapping $h : X \to Y$ such that

$$\|f(x) - f(0) - h(x)\|_Y \leq \Big(\frac{\theta_1^p\|x\|_X^{r_1p}}{2^p - 2^{r_1p}} + \frac{\theta_2^p\|x\|_X^{r_2p}}{2^p - 2^{r_2p}} + \frac{\theta_3^p(1-n)^{r_3p}\|x\|_X^{r_3p}}{2^p - 2^{r_3p}}\Big)^{\frac{1}{p}}$$

for all $x \in X$.

Theorem 2.4. Suppose that a mapping $f : X \to Y$ satisfies the functional inequality

$$\|Df(x,y,z)\|_Y \leq \varphi(x,y,z) \tag{2.9}$$

for all $x,y,z \in X$, and that the perturbing function $\varphi : X^3 \to \mathbb{R}^+$ satisfies

$$\sum_{i=0}^{\infty} 2^{ip}\varphi(\frac{x}{2^{i+1}}, \frac{y}{2^{i+1}}, \frac{z}{2^{i+1}})^p < \infty \tag{2.10}$$

for all $x,y,z \in X$. Then, there exists a unique additive mapping $h : X \to Y$ defined by $h(x) = \lim_{k\to\infty} 2^k\{f(\frac{x}{2^k}) - f(0)\}$ such that

$$\|f(x) - f(0) - h(x)\|_Y \leq \Big[\sum_{i=0}^{\infty} 2^{ip}\varphi(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}, \frac{(1-n)x}{2^{i+1}})^p\Big]^{\frac{1}{p}} \tag{2.11}$$

for all $x \in X$.

Proof. Now, if we replace x by $\frac{x}{2}$ in (2.7), then we have

$$\|g(x) - 2g(\frac{x}{2})\|_Y \le \varphi\Big(\frac{x}{2}, \frac{x}{2}, \frac{(1-n)x}{2}\Big)$$

for all $x \in X$. Thus it follows from the last inequality that

$$\|g(x) - 2^m g(\frac{x}{2^m})\|_Y^p \le \sum_{i=0}^{m-1} 2^{ip} \varphi\Big(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}, \frac{(1-n)x}{2^{i+1}}\Big)^p \tag{2.12}$$

for all nonnegative integer m and all $x \in X$.

The remaining proof is similar to the corresponding part of Theorem 2.1. □

If we put $\varphi(x, y, z) := \theta(\|x\|_X^{r_1}\|y\|_X^{r_2}\|z\|_X^{r_3})$ and $\varphi(x, y, z) := \theta_1\|x\|_X^{r_1} + \theta_2\|y\|_X^{r_2} + \theta_3\|z\|_X^{r_3}$ in the following corollaries, respectively, then we get lead to the desired results.

Corollary 2.5. Let $r_i > 0$ for $i = 1, 2, 3$ with $\sum_{i=1}^3 r_i > 1$ and $\theta \ge 0$. If a mapping $f : X \to Y$ satisfies the following functional inequality

$$\|Df(x, y, z)\|_Y \le \theta(\|x\|_X^{r_1}\|y\|_X^{r_2}\|z\|_X^{r_3})$$

for all $x, y, z \in X$, then there exists a unique additive mapping $h : X \to Y$ such that

$$\|f(x) - f(0) - h(x)\|_Y \le \frac{\theta(1-n)^{r_3}\|x\|_X^r}{\sqrt[p]{2^{rp} - 2^p}}$$

for all $x \in X$, where $r = \sum_{i=1}^3 r_i$.

Corollary 2.6. Let $r_i > 1$ and $\theta_i \ge 0$ for $i = 1, 2, 3$. If a mapping $f : X \to Y$ satisfies the following functional inequality

$$\|Df(x, y, z)\|_Y \le \theta_1\|x\|_X^{r_1} + \theta_2\|y\|_X^{r_2} + \theta_3\|z\|_X^{r_3}$$

for all $x, y, z \in X$, then there exists a unique additive mapping $h : X \to Y$ such that

$$\|f(x) - f(0) - h(x)\|_Y \le \Big(\frac{\theta_1^p\|x\|_X^{r_1 p}}{2^{r_1 p} - 2^p} + \frac{\theta_2^p\|x\|_X^{r_2 p}}{2^{r_2 p} - 2^p} + \frac{\theta_3^p(1-n)^{r_3 p}\|x\|_X^{r_3 p}}{2^{r_3 p} - 3^p}\Big)^{\frac{1}{p}}$$

for all $x \in X$.

3. Alternative Generalized Hyers–Ulam Stability of (1.2)

From now on, we investigate the generalized Hyers–Ulam stability of the functional equation (1.2) using the contractive property of perturbing term of the equation (1.2).

Theorem 3.1. Suppose that a mapping $f : X \to Y$ satisfies the functional inequality

$$\|Df(x,y,z)\|_Y \le \varphi(x,y,z)$$

for all $x,y,z \in X$ and that there exists a constant L with $0 < L < 1$ for which the perturbing function $\varphi : X^3 \to \mathbb{R}^+$ satisfies the property

$$\varphi(2x,2y,2z) \le 2L\varphi(x,y,z) \tag{3.1}$$

for all $x,y,z \in X$. Then, there exists a unique additive mapping $h : X \to Y$ given by $h(x) = \lim_{k\to\infty} \frac{1}{2^k} f(2^k x)$ such that

$$\|f(x) - f(0) - h(x)\|_Y \le \frac{1}{2\sqrt[p]{1-L^p}}\varphi(x,x,(1-n)x)$$

for all $x \in X$.

Proof. It follows from (2.8) and (3.1) that

$$\begin{aligned} \left\| \frac{g(2^l x)}{2^l} - \frac{g(2^m x)}{2^m} \right\|_Y^p &\le \sum_{i=l}^{m-1} \frac{1}{2^{(i+1)p}} \varphi(2^i x, 2^i x, (1-n)2^i x)^p \\ &\le \sum_{i=l}^{m-1} \frac{(2L)^{ip}}{2^{(i+1)p}} \varphi(x,x,(1-n)x)^p \\ &\le \frac{1}{1-L^p} \frac{\varphi(x,x,(1-n)x)^p}{2^p} \end{aligned}$$

for all nonnegative integers m and l with $m > l \ge 0$ and $x \in X$. Since the sequence $\{\frac{f(2^m x)}{2^m}\}$ is Cauchy for all $x \in X$, we can define a mapping $h : X \to Y$ by

$$h(x) = \lim_{m\to\infty} \frac{g(2^m x)}{2^m} = \lim_{m\to\infty} \frac{f(2^m x)}{2^m}, \quad x \in X.$$

Moreover, letting $l = 0$ and $m \to \infty$ in the last inequality yields

$$\|f(x) - f(0) - h(x)\|_Y \le \frac{1}{2\sqrt[p]{1-L^p}}\varphi(x,x,(1-n)x) \tag{3.2}$$

for all $x \in X$.

The remaining proof is similar to the corresponding part of Theorem 2.1. □

Corollary 3.2. Let $\xi : [0,\infty) \to [0,\infty)$ be a nontrivial function satisfying

$$\xi(2t) \le \xi(2)\xi(t), \quad (t \ge 0), \qquad 0 < \xi(2) < 2.$$

If $f : X \to Y$ is a mapping satisfying the following functional inequality

$$\|Df(x,y,z)\|_Y \le \theta\{\xi(\|x\|_X) + \xi(\|y\|_X) + \xi(\|z\|_X)\}$$

for all $x, y, z \in X$ and for some $\theta \geq 0$, then there exists a unique additive mapping $h : X \to Y$ such that

$$\|f(x) - f(0) - h(x)\|_Y \leq \frac{1}{\sqrt[p]{2^p - \xi(2)^p}}\theta\{2\xi(\|x\|_X) + \xi(|1-n|\|x\|_X)\}$$

for all $x \in X$.

Proof. Letting $\varphi(x, y, z) = \theta\{\xi(\|x\|_X) + \xi(\|y\|_X) + \xi(\|z\|_X)\}$, and applying Theorem 3.1 with $L := \frac{\xi(2)}{2}$, we obtain the desired result. □

Theorem 3.3. Suppose that a mapping $f : X \to Y$ satisfies the functional inequality

$$\|Df(x, y, z)\|_Y \leq \varphi(x, y, z)$$

for all $x, y, z \in X$ and there exists a constant L with $0 < L < 1$ for which the perturbing function $\varphi : X^3 \to \mathbb{R}^+$ satisfies the property

$$\varphi(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}) \leq \frac{L}{2}\varphi(x, y, z) \tag{3.3}$$

for all $x, y, z \in X$. Then, there exists a unique additive mapping $h : X \to Y$ defined by $h(x) = \lim_{k\to\infty} 2^k\{f(\frac{x}{2^k}) - f(0)\}$ such that

$$\|f(x) - f(0) - h(x)\|_Y \leq \frac{L}{2\sqrt[p]{1 - L^p}}\varphi(x, x, (1-n)x)$$

for all $x \in X$.

Proof. It follows from (2.12) and (3.3) that

$$\begin{aligned}
\|g(x) - 2^m g(\frac{x}{2^m})\|_Y^p &\leq \sum_{i=0}^{m-1} 2^{ip}\varphi\Big(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}, \frac{(1-n)x}{2^{i+1}}\Big)^p \\
&\leq \sum_{i=0}^{m-1} \frac{L^{(i+1)p}}{2^p}\varphi(x, x, (1-n)x)^p \\
&\leq \frac{L^p}{1 - L^p}\frac{\varphi(x, x, (1-n)x)^p}{2^p}
\end{aligned}$$

for all non-negative integer m and all $x \in X$.

The remaining proof is similar to the corresponding part of Theorem 2.1. □

Corollary 3.4. Let $\xi : [0, \infty) \to [0, \infty)$ be a nontrivial function satisfying

$$\xi(\frac{t}{2}) \leq \xi(\frac{1}{2})\xi(t), \quad (t \geq 0), \qquad 0 < \xi(\frac{1}{2}) < \frac{1}{2}.$$

If $f : X \to Y$ is a mapping satisfying the following functional inequality

$$\|Df(x, y, z)\|_Y \leq \theta\{\xi(\|x\|_X) + \xi(\|y\|_X) + \xi(\|z\|_X)\}$$

for all $x, y, z \in X$ and for some $\theta \geq 0$, then there exists a unique additive mapping $h : X \to Y$ such that

$$\begin{aligned}\|f(x) - f(0) - h(x)\|_Y &\leq \frac{\xi(\frac{1}{2})}{\sqrt[p]{1 - 2^p \xi(\frac{1}{2})^p}} \theta\{2\xi(\|x\|_X) + \xi(|1-n|\|x\|_X)\} \\ &= \frac{1}{\sqrt[p]{\frac{1}{\xi(\frac{1}{2})^p} - 2^p}} \theta\{2\xi(\|x\|_X) + \xi(|1-n|\|x\|_X)\}\end{aligned}$$

for all $x \in X$.

Proof. Letting $\varphi(x, y, z) = \theta\{\xi(\|x\|_X) + \xi(\|y\|_X) + \xi(\|z\|_X)\}$ and applying Theorem 3.3 with $L := 2\xi(\frac{1}{2})$, we lead to the approximation. □

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