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INSERTION-OF-IDEAL-FACTORS-PROPERTY

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Abstract. Due to Bell, a ring R is usually said to be *IFP* if $ab = 0$ implies $aRb = 0$ for $a, b \in R$. It is shown that if $f(x)g(x) = 0$ for $f(x) = a_0 + a_1x$ and $g(x) = b_0 + \cdots + b_nx^n$ in $R[x]$, then $(f(x)R[x])^{2n+2}g(x) = 0$. Motivated by this results, we study the structure of the IFP when proper ideals are taken in place of R, introducing the concept of *insertion-of-ideal-factors-property* (simply, *IIFP*) as a generalization of the IFP. A ring R will be called an *IIFP ring* if $ab = 0$ (for $a, b \in R$) implies $aIb = 0$ for some proper nonzero ideal I of R, where R is assumed to be non-simple. We in this note study the basic structure of IIFP rings.

1. Introduction

Insertion-of-Factors-Property has done important roles in noncommutative ring theory and module theory. Throughout this note every ring is an associative ring with identity unless otherwise stated. Given a ring R, let $N(R)$ and $N_*(R)$ denote the set of all nilpotent elements and the prime radical in R, respectively. The polynomial ring with an indeterminate x over R is denoted by $R[x]$. The n by n full (resp. upper triangular) matrix ring over R is denoted by $Mat_n(R)$ (resp. $U_n(R)$), and denote by e_{ij} the matrix with (i, j)-entry 1 and elsewhere zero. $\mathbb{Z}$ denotes the ring of integers, and $\mathbb{Z}_n$ denotes the ring of integers modulo n.

Due to Bell [4], a ring R (possibly without identity) is called to satisfy the *insertion-of-factors-property* (simply, an *IFP* ring) if $ab = 0$ implies $aRb = 0$ for $a, b \in R$. Narbonne [8] and Shin [10] used the terms *semicommutative* and *SI* for the IFP, respectively. A ring R (possibly without identity) is called *reduced* if $N(R) = 0$. This insertion-of-factors-property unifies the commutativity and the reduced condition. But there exist many non-reduced commutative rings

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(e.g., $\mathbb{Z}_{n^l}$ for $n, l \geq 2$), and many noncommutative reduced rings (e.g., direct products of noncommutative domains). A ring is usually called *Abelian* if each idempotent is central. A simple computation yields that IFP rings are Abelian. It is also easily checked that $N(R) = N_*(R)$ for an IFP ring R.

Proposition 1.1. *Let R be an IFP ring.*

(1) *If $f(x)g(x) = 0$ for $f(x) = a_0 + a_1x$ and $g(x) = b_0 + \cdots + b_nx^n$ in $R[x]$, then*

$$(f(x)R[x])^{2n+2}g(x) = 0.$$

(2) *If $f(x)g(x) = 0$ for $f(x) = a_0 + \cdots + a_mx^m$ and $g(x) = b_0 + b_1x$ in $R[x]$, then*

$$f(x)(R[x]g(x))^{2m+2} = 0.$$

Proof. (1) Let $f(x) = a_0 + a_1x, g(x) = b_0 + \cdots + b_nx^n \in R[x]$ such that $f(x)g(x) = 0$. Then

$$\begin{aligned}&a_0b_0 = 0,\\ &a_0b_i + a_1b_{i-1} = 0 \text{ for } i = 1, \ldots, n,\\ &a_1b_n = 0.\end{aligned}$$

We will use the IFP of R freely. Note $a_0Rb_0 = 0$ and $a_1Rb_n = 0$. Then we also obtain

$$\begin{aligned}a_0^2b_1 &= a_0(a_0b_1 + a_1b_0) = 0,\\ a_0^3b_2 &= a_0^2(a_0b_2 + a_1b_1) = 0,\\ &\cdots\\ a_0^{i+1}b_i &= a_0^i(a_0b_i + a_1b_{i-1}) = 0 \text{ for } i = 3, 4, \ldots, n,\\ &\cdots\\ a_0^{n+1}b_n &= a_0^n(a_0b_n + a_1b_{n-1}) = 0.\end{aligned}$$

Similarly we can obtain

$$a_1^ib_{n-(i-1)} = 0 \text{ for } i = 2, \ldots, n+1.$$

Next consider the case of $n = 1$, i.e., $g(x) = b_0 + b_1x$. Then we obtain

$$a_0Rb_0 = a_0Ra_0Rb_1 = 0 \text{ and } a_1Rb_1 = a_1Ra_1Rb_0 = 0$$

from $a_0b_0 = 0$, $a_0^2b_1 = 0$, $a_1b_1 = 0$, and $a_1^2b_0 = 0$. These yield

$$a_0r_1a_0r_2b_1 = a_1r_3a_1r_4b_0 = 0$$

for all r_i's in R; hence we moreover obtain

$$\begin{aligned}&f(x)s_1f(x)s_2f(x)s_3f(x)s_4g(x)\\ =&(a_0 + a_1x)s_1(a_0 + a_1x)s_2(a_0 + a_1x)s_3(a_0 + a_1x)s_4(b_0 + b_1x) = 0\end{aligned}$$

for all s_i's in R since every coefficient of the expansion of $f(x)s_1f(x)s_2f(x)s_3f(x)s_4g(x)$ contains at least two a_0's or two a_1's. Thus we now have

$$f(x)R[x]f(x)R[x]f(x)R[x]f(x)R[x]g(x) = 0.$$

Proceeding by induction on n, we can finally obtain

$$(f(x)R[x])^{2n+2}g(x) = 0.$$

The proof of (2) is a symmetry one of (1). □

In Proposition 1.1, consider the case of $m = n = 1$. Then we have $f(x)R[x]g(x) = 0$ when $f(x)g(x) = 0$ by [7, Proposition 1.3]. So $f(x)Ig(x) = 0$ for all ideals I of R.

Now we consider the case of substituting proper ideals for the whole ring in the definition of IFP rings, extending Proposition 1.1 to general situations.

Definition 1. A ring R is said to satisfy the *insertion-of-ideal-factors-property* (simply, called *IIFP* ring) if there exists a nonzero proper ideal I (if any) of R such that $aIb = 0$ whenever $ab = 0$ for $a, b \in R$. Simple rings are assumed to be IIFP.

IFP rings are clearly IIFP. But there exists IIFP rings but not IFP. For example, $Mat_n(D)$ is non-Abelian and so this ring is not IFP where D is a simple ring and $n \geq 2$; but $Mat_n(D)$ is IIFP by definition.

Lemma 1.2. *Let R be a simple ring. Then R is IFP if and only if R is a domain.*

Proof. Let R be IFP and assume $ab = 0$ for $a, b \in R$. Then $aRb = 0$ and so $(RaR)(RbR) = 0$. Thus we get $a = 0$ or $b = 0$ since R is simple. The converse is obvious. □

For a ring R and $n \geq 2$, consider the subring

$$D_n(R) = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \in U_n(R) \;\middle|\; a, a_{ij} \in R \right\}$$

of $U_n(R)$. $U_n(R)$ for $n \geq 2$ need not be IIFP over an IIFP ring R by help of Example 1.4 to follow. But we can argue about the IIFP for $D_n(R)$ affirmatively.

Proposition 1.3. *If a non-simple ring R is IIFP then $D_n(R)$ is IIFP for $n \geq 2$.*

Proof. Let R be a non-simple IIFP ring and suppose that $AB = 0$ for $A = (a_{ij}), B = (b_{ij}) \in D_n(R)$. Then $a_{11}b_{11} = 0$. Since R is IIFP, there exists a nonzero proper ideal I of R such that $a_{11}Ib_{11} = 0$. Set

$$J = \begin{pmatrix} 0 & 0 & 0 & \cdots & I \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Then J is a nonzero proper ideal of $D_n(R)$ which satisfies

$$AJB = \begin{pmatrix} 0 & 0 & 0 & \cdots & a_{11}Ib_{11} \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} = 0.$$

This implies that $D_n(R)$ is also IIFP. □

Note that $D_n(R)$ cannot be IFP for $n \geq 4$ over any ring R by [6, Example 1.3], comparing with Proposition 1.3.

It is natural to ask whether R is an IIFP ring if for any nonzero proper ideal I of R, R/I and I are IIFP, where I is considered as an IIFP ring without identity. However the following example provides a negative answer.

Example 1.4. Let D be a division ring and $R = U_2(D)$. Then R is clearly not IFP and all ideals of R are

$$I_1 = \begin{pmatrix} D & D \\ 0 & 0 \end{pmatrix}, I_2 = \begin{pmatrix} 0 & D \\ 0 & D \end{pmatrix}, \text{ and } I_3 = \begin{pmatrix} 0 & D \\ 0 & 0 \end{pmatrix}.$$

Note that each I_k is IFP as a subring of R without identity, and that each R/I_k is IFP, by [3, Example 5]. However R is not IIFP. For, $e_{11}e_{22} = 0$ but $e_{11}I_ke_{22} \neq 0$ for all $k = 1, 2, 3$.

Moreover Example 1.4 illuminates that the ring R is not IIFP too when we take the stronger condition "I is IFP" instead of "I is IIFP". However if we take the condition "I is reduced" then we may have an affirmative answer as in the following.

Proposition 1.5. *If a ring R has a proper ideal which is reduced as a subring of R without identity, then R is IIFP.*

Proof. Assume that I is a proper ideal of R which is reduced as a subring of R without identity. Let $ab = 0$ for $a, b \in R$. Then $(bIa)^2 = 0$ and $bIa \subseteq I$. This yields $bIa = 0$ since I is reduced. Accordingly, $((aIb)I)^2 = aI(bIa)IbI = 0$ and so $aIbI = 0$ since I is reduced. This yields $(aIb)^2 \subseteq aIbI = 0$. But $aIb \subseteq I$, so we get $aIb = 0$ since I is reduced. Thus R is IIFP. □

IFP rings are both Abelian and IIFP. But the concepts of Abelian and IIFP are independent of each other by the following.

Example 1.6. (1) $Mat_2(D)$ is non-Abelian for any simple ring D; but $Mat_2(D)$ is IIFP by definition.

(2) We use the subring

$$R = \left\{ \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \mid a - b \equiv c \equiv 0 (\text{mod } 2) \right\}$$

of $Mat_2(\mathbb{Z})$ in [5, Example 13]. Then R is Abelian by the argument in [5, Example 13].

Let I be any nonzero proper ideal of R. Then I must contain a matrix $\begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix}$ with $\beta \neq 0$. So we have

$$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 4\beta \\ 0 & 0 \end{pmatrix} \neq 0.$$

This entails

$$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} I \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \neq 0.$$

But $\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} = 0$, so R is not IIFP.

2. Properties of IIFP rings

In this section we examine the IIFP of some ring extensions which have roles in ring theory. For a reduced ring R and $f(x), g(x) \in R[x]$, Armendariz [2, Lemma 1] proved that

$$ab = 0 \text{ for all } a \in C_{f(x)}, b \in C_{g(x)} \text{ whenever } f(x)g(x) = 0.$$

Chhawchharia and Rege [9] called a ring *Armendariz* if it satisfies this property. So reduced rings are clearly Armendariz. This fact will be used freely in this note. Armendariz rings are Abelian by the proof of [1, Theorem 6] (or [5, Lemma 7]). The concepts of Armendariz and IFP are independent of each other by [9, Example 3.2] and [3, Example 14]. Also there exists an Aramendariz ring which is not IIFP by help of [3, Example 14].

A ring R is said to have the *finite intersection property on ideals* provided that every intersection of finite number of nonzero ideals remains nonzero.

Proposition 2.1. *Let R be an Armendariz ring which has the finite intersection property on ideals. Then if R is IIFP then $R[x]$ is IIFP.*

Proof. Let R be an IIFP and assume $f(x)g(x) = 0$ for $f(x) = \sum_{i=0}^{m} a_i x^i, g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$. Then since R is Armendariz, $a_i b_j = 0$ for all i, j. But since R is IIFP, there exist nonzero proper ideal $I_{i,j}$ of R such that

$$a_i I_{i,j} b_j = 0$$

for any pair (i, j). Next set

$$I = \bigcap_{i,j} I_{i,j}.$$

Since R has the finite intersection property on ideals, I is a nonzero proper ideal of R which satisfies

$$a_i I b_j = 0 \text{ for all } i, j.$$

This yields $f(x)I[x]g(x) = 0$, noting that $I[x]$ is also a nonzero proper ideal of $R[x]$. □

An element u of a ring R is *right regular* if $ur = 0$ implies $r = 0$ for $r \in R$. Similarly, *left regular* is defined, and *regular* means if it is both left and right regular (i.e., not a zero-divisor).

Proposition 2.2. *Let R be a ring and M be an multiplicatively closed subset of central regular elements in R. Then R is IIFP if and only if RM^{-1} is IIFP, where R and RM^{-1} are both assumed to be non-simple.*

Proof. Let R be an IIFP ring and assume $am^{-1}bn^{-1} = 0$. Then clearly $ab = 0$. But since R is IIFP, there exists a nonzero proper ideal I of R such that $aIb = 0$. Set $J = IM^{-1}$. Note that every element of J is of the form st^{-1} with $s \in I$ and $t \in M$ since I is an ideal of R. Then clearly J is a nonzero ideal of RM^{-1} such that

$$am^{-1}st^{-1}bn^{-1} = asbm^{-1}t^{-1}n^{-1} = 0$$

for all $st^{-1} \in J$. Here if $J \subsetneq RM^{-1}$ then we are done. If $J = RM^{-1}$, then $am^{-1}Kbn^{-1} = 0$ for all nonzero proper ideals K of RM^{-1}.

Conversely, let RM^{-1} is IIFP and assume $ab = 0$ for $a, b \in R$. Then there exists a nonzero proper ideal J of RM^{-1} such that $aJb = 0$. Set

$$I = \{s \in R \mid st^{-1} \in J\}.$$

Then I is an ideal of R such that $J = IM^{-1}$ from the computation that

$$rst^{-1} = r(st^{-1}), srt^{-1} = (st^{-1})r, s = st^{-1}t \in J$$

for $r \in R$ and $st^{-1} \in J$. Since $I \subseteq J$, we have $aIb = 0$. Moreover from $J \subsetneq RM^{-1}$, we get $I \subsetneq R$. Thus R is IIFP. □

Recall the ring of *Laurent polynomials* in x, written by $R[x; x^{-1}]$. Let $M = \{1, x, x^2, \ldots\}$. Then M is clearly a multiplicatively closed subset of central regular elements in $R[x]$ such that $R[x; x^{-1}] = M^{-1}R[x]$. So Proposition 2.2 leads to the following.

Corollary 2.3. *Let R be a ring. Then $R[x]$ is IIFP if and only if $R[x; x^{-1}]$ is IIFP.*

The following is obtained from Proposition 2.1 and Corollary 2.3.

Corollary 2.4. *Let R be an Armendariz ring which has the finite intersection property on ideals. Then if R is IIFP then both $R[x]$ and $R[x; x^{-1}]$ are IIFP.*

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