

HYPERBOLIC EQUATIONS RELATED TO p -LAPLACIAN OPERATORS VIA TIME DISCRETIZATION METHODS

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ABSTRACT. In this paper we consider hyperbolic equations related to the p -Laplacian with a nonsmooth kernel. By exploiting a suitable implicit time-discretization technique we obtain the existence of global strong solution.

1. Introduction

A variety of phenomena arising in elasticity theory, molecular dynamics, mechanics and quantum mechanics leads to nonlinear partial differential equations of second order in time. Thus, in this paper, we study a hyperbolic differential equation related to the p -Laplacian operator. More precisely, we are interested in the existence and uniqueness of the solution of problem

$$\begin{aligned} u_{tt} + \beta(u_t) - \Delta_p u &\ni f(t) && \text{in } \Omega \times [0, T], \\ u &= 0 && \text{on } \partial\Omega \times [0, T], \\ u(\cdot, 0) = u^0, \quad u'(\cdot, 0) &= v^0 && \text{in } \Omega, \end{aligned} \quad (1)$$

where $-\Delta_p u = -\operatorname{div}(|\nabla u|^{p-2} \nabla u)$, β is linear growth condition and $f \in L^\infty(\Omega)$. $T > 0$ denotes some reference final time and Ω is a regular open bounded set of \mathbb{R}^d ($d \geq 1$) with its boundary $\partial\Omega$.

The functional analytic formulation of (1) leads to the initial value problem $u'' + Au' + Bu = f$, $u(0) = u^0$, $u'(0) = v^0$. For the case that A is maximal monotone and B is linear, the existence and uniqueness of the solution has been established by Lions and Strauss [6] by using a Faedo-Galerkin method. Another proof has been given by Brézis [4] in the theory of nonlinear semigroups. In this paper, we prove the the existence and uniqueness of the solution in case that A is linear and B is nonlinear by time-discretization method. It should be remarked that such a result applies to various significant cases of hyperbolic partial differential equations in refer to [6], [7].

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The object of our work is not only to establish existence and uniqueness properties for the solution of (1), but also to introduce a discrete approximation of the problem, then prove convergence of the discrete solution to the continuous one. Thus we recall our assumptions and state main results in Section 2. In Section 3, we show the existence of discrete scheme. After showing some estimates on the approximations, the passage to the limit and the existence results are given in Section 4.

2. Assumptions and Main results

We let $\|\cdot\|_p, \|\cdot\|_{1,p}$ and $\|\cdot\|_{-1,p}$ denote the norm in $L^p(\Omega)$, $W_0^{1,p}(\Omega)$ and $W^{-1,p}(\Omega)$ for $1 < p < \infty$, respectively ([1]). And $\langle \cdot, \cdot \rangle$ denotes the duality between $W_0^{1,p}(\Omega)$ and $W^{-1,p}(\Omega)$ or denotes inner product of $L^2(\Omega)$. For $1 < p < \infty$, we define the conjugate p' of p by $1/p + 1/p' = 1$. In this paper, C_i and C will denote positive constants and λ_i the imbedding constants of [1].

Now, we present our assumptions which are used throughout this paper. We suppose $d^* \leq p < \infty$ where $d^* = (2d)/(d+2)$, $u^0 \in L^\infty(\Omega)$ with $u^0 = 0$ on $\partial\Omega$, $v^0 \in L^\infty(\Omega)$ with $v^0 = 0$ on $\partial\Omega$ and the following:

- (H) The function $\beta : \mathbb{R} \rightarrow \mathbb{R}$ is linear with $(\beta(u)u) \geq 0$ and $|\beta(u)| \leq Cu$ where C is a positive constant.

Definition 1. ([2]) Let X be a reflexive Banach space and $A : X \rightarrow X'$. We say that A is *monotone* if $\langle Ay - Az, y - z \rangle \geq 0$ for all $y, z \in X$, and A is *hemicontinuous* if for each $y, z, w \in X$ the real-valued function $t \rightarrow \langle A(y + tz), w \rangle$ is continuous.

Lemma 2.1. (Minty's Theorem [9]) *Let X be a reflexive Banach space. If $A : X \rightarrow X'$ is monotone and hemicontinuous, then*

$$Ay = f \quad \text{if and only if} \quad \langle f - Az, y - z \rangle \geq 0$$

for all $z \in X$.

Lemma 2.2. ([3]) *Let Ω be a bounded set in \mathbb{R}^d . Let $1 < p < \infty$ be fixed and $A : W_0^{1,p}(\Omega) \rightarrow W^{1,p'}(\Omega)$ a nonlinear operator defined by*

$$A(u) = -\operatorname{div} a(x, u, Du)$$

where $a(x, s, \xi)$ is a Carathéodory function $a : \Omega \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ such that

$$\begin{aligned} |a(x, s, \xi)| &\leq \beta[|s|^{p-1} + |\xi|^{p-1} + k(x)], \\ [a(x, s, \xi) - a(x, s, \eta)](\xi - \eta) &> 0, \quad \xi \neq \eta, \\ a(x, s, \xi)\xi &\geq \alpha|\xi|^p, \end{aligned}$$

where $k(x) \in L^{p'}(\Omega)$, $k \geq 0$, $\beta > 0$ and $\alpha > 0$.

Let $g(x, s, \xi)$ be a Carathéodory function such that

$$\begin{aligned} g(x, s, \xi)s &\geq 0, \\ |g(x, s, \xi)| &\leq b(|s|)(|\xi|^p + c(x)), \end{aligned}$$

where b is a continuous and increasing function with (finite) values on \mathbb{R}^+ and $c \in L^1(\Omega)$, $c \geq 0$. Then, for $h \in W^{-1,p'}(\Omega)$, the problem

$$Au + g(x, u, \nabla u) = h,$$

has at least one solution $u \in W_0^{1,p}(\Omega)$.

Lemma 2.3. ([8]) *If $u \in W_0^{1,p}(\Omega)$ is a solution to the equation*

$$-\Delta_p u + F(x, u) = h,$$

where $h \in W^{-1,r}$, $r > \frac{d}{p-1}$ and F satisfies $\xi F(x, \xi) \geq 0$ in $\Omega \times \mathbb{R}$, then $u \in L^\infty(\Omega)$.

Now, we state our main results as follows :

Theorem 2.4. *Under the assumptions (H), there exists a unique solution u of (1) such that $u \in L^p(0, T; W_0^{1,p}(\Omega)) \cap L^\infty(0, T; L^\infty(\Omega))$.*

3. Existence of scheme

For the problem (1), we consider the discrete scheme (DS) for $i = 1, 2, \dots, N$,

$$(DS) \quad \begin{cases} \frac{v_i - v_{i-1}}{\tau} + \beta\left(\frac{u_i - u_{i-1}}{\tau}\right) - \Delta_p u_i = f_i & \text{in } \Omega, \\ \frac{u_i - u_{i-1}}{\tau} = v_i & \\ u_i = 0, v_i = 0 & \text{in } \partial\Omega, \\ u_0 = u^0, v_0 = v^0 & \text{in } \Omega. \end{cases}$$

where $N\tau = T$ and T is a fixed positive real. We shall show that (DS) has a solution u_i for $i = 1, 2, \dots, N$.

Theorem 3.1. *Let (H) hold. Then for $i = 1, 2, \dots, N$, there exists a unique solution $u_i \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ of (DS) for sufficiently small τ .*

Proof. First of all, we rewrite (DS) as

$$-\tau^2 \Delta_p u_i + F(x, u_i) = \varphi_{i-1},$$

where $F(x, u_i) = (1 + \tau\beta)(u_i)$ and $\varphi_{i-1} = u_{i-1} - \tau v_{i-1} + \tau\beta(u_{i-1}) + \tau^2 f_i$.

Now, we consider the equation

$$-\tau^2 \Delta_p u + F(x, u) = \varphi_1 = u_0 - \tau v_0 + \tau\beta(u_0) + \tau^2 f_1, \tag{2}$$

where $F(x, u) = (1 + \tau\beta)(u)$ for fixed $\tau = T/N$. It is obvious that $a(x, u, Du) := |\nabla u|^{p-2} \nabla u$ satisfies all the three conditions of a in Lemma 2.3 (cf [3]). And, by assumption, Lemma 2.3 and Lemma 2.4, there exists a solution $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. We put $u_1 := u$ and we obtain v_1 by scheme. Then we consider the equation $-\tau \Delta_p u + F(x, u) = \varphi_2 = u_1 - \tau v_1 + \tau\beta(u_1) + \tau^2 f_2$ where $F(x, u) = (1 + \tau\beta)(u)$. Continuing this process, we have a solution u_i of (DS) for $i = 1, 2, \dots, N$ such that $u_i \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ ($i = 1, 2, \dots, N$).

Next, we show the uniqueness of u_i ($i = 1, 2, \dots, N$). Let u_i and u_i^* be two solutions of (DS) for $i = 1, 2, \dots, N$. Using the result which we will establish below (see Theorem 3.2), we have

$$\|u_i\|_{1,p} + \|u_i^*\|_{1,p} \leq M$$

where M is a suitable positive constant independent of τ . And, from [5] we have

$$\langle -\Delta_p u + \Delta_p v, u - v \rangle \geq C_p \|u - v\|_{1,p}^p \tag{3}$$

for all $u, v \in W_0^{1,p}(\Omega)$.

Since u_i and u_i^* are solutions of (DS), we have

$$-\tau^2 \Delta_p u_i + \tau^2 \Delta_p u_i^* + (I + \tau\beta)(u_i - u_i^*) = 0. \tag{4}$$

Multiplying (4) by $u_i - u_i^*$ and integrating over Ω , we get

$$\langle -\tau^2 \Delta_p u_i + \tau^2 \Delta_p u_i^*, u_i - u_i^* \rangle + \int_{\Omega} ((I + \tau\beta)(u_i - u_i^*))(u_i - u_i^*) dx = 0. \tag{5}$$

By (3) and (H), the equation (5) becomes

$$(\tau^2 C_p - \tau C_p) \|u_i - u_i^*\|_{1,p}^p \leq -\frac{1}{2} \int_{\Omega} |u_i - u_i^*|^2 dx.$$

Thus, it implies that for sufficiently small τ , i.e., $\tau < \frac{1}{C_p}$, $u_i = u_i^*$ holds. \square

Now, we consider the bounds of $\{u_i\}$ ($i = 1, 2, \dots, N$), which is constructed in Theorem 3.1 as solutions of (DS).

Theorem 3.2. *We assume (H). Then there exist C_1 – C_3 , which are positive constants and independent of τ , such that for all $i = 1, 2, \dots, m$,*

- (a) $\|v_i\|_{\infty} \leq C_1$,
- (b) $\tau \sum_{i=1}^m \|u_i\|_{1,p}^p \leq C_2$,
- (c) $\|v_m\|_2^2 + \sum_{i=1}^m \|v_i - v_{i-1}\|_2^2 \leq C_3$,

where $m = 1, 2, \dots, N$.

Proof. (a) Multiplying (DS) by $|v_i|^k v_i$ and integrating over Ω , we have by (H), $f \in L^{\infty}(\Omega)$ and Hölder's inequality,

$$\int_{\Omega} |v_i|^k v_i v_i dx \leq \|v_i\|_{k+2}^{k+1} \|v_{i-1}\|_{k+2} + m(\Omega)^{1/(k+2)} \tau C_4 \|v_i\|_{k+2}^{k+1}.$$

Then $\|v_i\|_{k+2} \leq m(\Omega)^{1/k+2} \tau C_4 + \|v_{i-1}\|_{k+2}$. By induction, we have $\|v_i\|_{k+2} \leq m(\Omega)^{1/(k+2)} C_4 T + \|v_0\|_{k+2}$. Letting $k \rightarrow \infty$, $\|v_i\|_{\infty} \leq C(C_4, T, u_0) =: C_1$.

(b) Let $z \in W_0^{1,p}(\Omega)$ be fixed. Multiplying the equation (DS) by $u_i - z$ and integrating over Ω , we have

$$\begin{aligned} & \langle v_i - v_{i-1}, u_i \rangle + \tau \|u_i\|_{1,p}^p \\ &= \langle v_i - v_{i-1}, z \rangle - \tau \langle \Delta_p u_i, z \rangle - \tau \langle \beta(v_i), u_i - z \rangle + \langle f_i, u_i - z \rangle. \end{aligned} \tag{6}$$

Now, we apply Young's inequality to (6) to get

$$\begin{aligned} & \langle v_i - v_{i-1}, u_i \rangle + \frac{\tau}{4} \|u_i\|_{1,p}^p \\ & \leq \langle v_i - v_{i-1}, z \rangle + \tau \left\{ \left(\frac{1}{4} \frac{p}{p-1} \right)^{-(p-1)} p^{-1} + CC_1 \lambda + C_f \lambda \right\} \|z\|_{1,p}^p \\ & \quad + \tau \left(\frac{1}{4} \frac{p}{\lambda_1} \right)^{-\frac{p}{p'}} (p')^{-1} C_f^{p'} m(\Omega) + \tau (C_1 C)^{p'} m(\Omega) + \left(\frac{p}{4\lambda_1} \right)^{-\frac{p}{p'}} (p') \end{aligned}$$

Now we define $\varphi(v) = \langle v, v \rangle / 2$ and $\varphi^*(v) = \sup_z \{ \langle v, z \rangle - \varphi(z) \}$, then we have

$$\begin{aligned} & \varphi^*(v_i) - \varphi^*(v_{i-1}) + \frac{\tau}{4} \|u_i\|_{1,p}^p \\ & \leq \langle v_i - v_{i-1}, z \rangle + \tau C (\|z\|_{1,p}, p, \lambda, \lambda_1, m(\Omega), C, C_f, C_1) + \tau^2 \sum_{j=1}^i \tau \|u_j\|_{1,p}^p \end{aligned} \tag{7}$$

for $i = 1, 2, \dots, m$ and for arbitrary $m = 1, 2, \dots, N$. By summing (7) for $i = 1, 2, \dots, m$,

$$\begin{aligned} & \varphi^*(v_m) - \langle v_m, z \rangle + \frac{\tau}{4} \sum_{i=1}^m \|u_i\|_{1,p}^p \\ & \leq \varphi^*(v_0) - \langle v_0, z \rangle + C_5 + \tau \sum_{i=1}^m \sum_{j=1}^i \tau \|u_j\|_{1,p}^p, \end{aligned} \tag{8}$$

where $C_5 := C(\|z\|_{1,p}, p, \lambda, \lambda_1, m(\Omega), C, C_f, C_1)$ for $m = 1, 2, \dots, N$. Applying the discrete Gronwall's lemma to (8),

$$\varphi^*(v_m) - \langle v_m, z \rangle + \frac{\tau}{4} \sum_{i=1}^m \|u_i\|_{1,p}^p \leq C(v_0, \|z\|_{1,p}, p, \lambda, \lambda_1, m(\Omega), C, C_f, C_1).$$

Hence $\tau \sum_{i=1}^m \|u_i\|_{1,p}^p \leq C_3$.

(c) Multiplying (DS) by v_i and integrating over Ω , we get by (H) and using $a(a-b) = \frac{1}{2}a^2 - \frac{1}{2}b^2 + \frac{1}{2}(a-b)^2$,

$$\|v_i\|_2^2 - \|v_{i-1}\|_2^2 + \|v_i - v_{i-1}\|_2^2 \leq \tau(1 + C_f)C_1.$$

Summing the above inequality for $i = 1, 2, \dots, m$, we have

$$\|v_m\|_2^2 + \sum_{i=1}^m \|v_i - v_{i-1}\|_2^2 \leq T(1 + C_f)C_1 + \|v_0\|_2^2,$$

for $m = 1, 2, \dots, N$. Thus, $\|v_m\|_2^2 + \sum_{i=1}^m \|v_i - v_{i-1}\|_2^2 \leq C_3$. □

In the forthcoming discussion, the following notations will be used extensively. For vectors u_i ($i = 0, 1, \dots, N$) in Theorem 3.1, we define two functions u_τ and \bar{u}_τ on $[0, T]$ by

$$\begin{aligned} u_\tau(0) &:= u_0, & u_\tau(t) &:= u_i + \frac{u_i - u_{i-1}}{\tau}(t - i\tau), \\ \bar{u}_\tau(0) &:= u_0, & \bar{u}_\tau(t) &:= u_i, \end{aligned}$$

for $t \in ((i - 1)\tau, i\tau]$ ($i = 1, 2, \dots, N$) and $\tau = T/N$. Similarly, we define

$$\begin{aligned} v_\tau(0) &:= v_0, & v_\tau(t) &:= v_i + \frac{v_i - v_{i-1}}{\tau}(t - i\tau), \\ \bar{v}_\tau(0) &:= v_0, & \bar{v}_\tau(t) &:= v_i, \end{aligned}$$

for $i = 0, 1, \dots, N$. Also, we let $\bar{f}_\tau(t) := f_i$ for $t \in ((i - 1)\tau, i\tau]$ ($i = 1, 2, \dots, N$).

Hence we can rewrite (DS) in a more compact form as

$$v'_\tau + \beta(u'_\tau) - \Delta_p \bar{u}_\tau + \bar{f}_\tau = 0, \quad \text{a.e. in } [0, T]. \tag{9}$$

4. Estimates and Limits

4.1. Estimates

First of all, by the consequence of Theorem 3.2, the followings are very easily proven.

$$\bar{u}_\tau \text{ is bounded in } L^p(0, T; W_0^{1,p}(\Omega)) \cap L^\infty(0, T; L^\infty(\Omega)), \tag{10}$$

$$u'_\tau \text{ is bounded in } L^p(0, T; W_0^{1,p}(\Omega)), \tag{11}$$

$$\bar{v}_\tau \text{ is bounded in } L^\infty(0, T; L^\infty(\Omega)), \tag{12}$$

$$v'_\tau \text{ is bounded in } L^2(0, T; L^\infty(\Omega)). \tag{13}$$

Moreover, by (10) and boundedness of p -Laplacian operator,

$$-\Delta_p \bar{u}_\tau \text{ is bounded in } L^{p'}(0, T; W^{-1,p'}(\Omega)). \tag{14}$$

And by (9), (11), (13) and $f \in L^\infty(\Omega)$,

$$v'_\tau \text{ is bounded in } L^{p'}(0, T; W^{-1,p'}(\Omega)). \tag{15}$$

We emphasize that all of the above boundedness are independent of τ .

First of all, we have $u \in L^p(0, T; W_0^{1,p}(\Omega)) \cap L^\infty(0, T; L^\infty(\Omega))$ which is a weak limit of \bar{u}_τ as $\tau \rightarrow 0$ by (10). i.e., \bar{u}_τ converges weakly to u as $\tau \rightarrow 0$ in $L^p(0, T; W_0^{1,p}(\Omega)) \cap L^\infty(0, T; L^\infty(\Omega))$.

4.2. Limits and Proof of Theorem 2.4

As we mentioned in (10)-(15), we have u, v and f such that

$$\bar{u}_\tau \rightarrow u \text{ weakly in } L^p(0, T; W_0^{1,p}(\Omega)) \cap L^\infty(0, T; L^\infty(\Omega)), \tag{16}$$

$$v_\tau \rightarrow v \text{ weakly star in } W^{1,p'}(0, T; W^{-1,p'}(\Omega)), \tag{17}$$

$$v_\tau \rightarrow v \text{ weakly in } C(0, T; L^2(\Omega)),$$

$$\bar{v}_\tau \rightarrow v \text{ weakly star in } L^{p'}(0, T; W^{-1,p'}(\Omega)), \tag{18}$$

$$\bar{v}_\tau \rightarrow v \text{ strongly in } L^2(0, T; L^2(\Omega)),$$

$$\bar{f}_\tau \rightarrow f \text{ strongly in } L^{p'}(0, T; W_0^{-1,p'}(\Omega)). \tag{19}$$

We note that the above sequences with τ are for some not relabeled subsequence. Also we note that

$$\begin{aligned} \lim_{\tau \rightarrow 0} \int_0^T \langle \phi, \bar{v}_\tau - v_\tau \rangle dt &\leq \lim_{\tau \rightarrow 0} \tau \sum_{i=1}^N \int_{(i-1)\tau}^{i\tau} \int_\Omega \phi(x, t) \left(\frac{v_i - v_{i-1}}{\tau} \right) dx dt \\ &= \lim_{\tau \rightarrow 0} \tau \sum_{i=1}^N \int_{(i-1)\tau}^{i\tau} \int_\Omega \phi(x, t) v'_\tau(t) dx dt \\ &= \lim_{\tau \rightarrow 0} \tau \int_0^T \langle \phi, v'_\tau \rangle dt = 0, \end{aligned}$$

for all $\phi \in L^p(0, T; W_0^{1,p}(\Omega))$ by (15). Hence, v_τ and \bar{v}_τ have the same limit v in (17) and (18).

Therefore,

$$-\Delta_p \bar{u}_\tau \rightarrow -u_{tt} - \beta(u_t) - f \text{ weakly in } L^{p'}(0, T; W_0^{-1,p'}(\Omega)). \tag{20}$$

And

$$\begin{aligned} \limsup_{\tau \rightarrow 0} \int_0^T \langle -\Delta_p \bar{u}_\tau, \bar{u}_\tau \rangle dt &\leq \limsup_{\tau \rightarrow 0} \int_0^T \int_\Omega -v'_\tau \bar{u}_\tau dx dt \\ &+ \limsup_{\tau \rightarrow 0} \int_0^T \int_\Omega -\beta(u'_\tau) \bar{u}_\tau dx dt + \limsup_{\tau \rightarrow 0} \int_0^T \int_\Omega -\bar{f}_\tau \bar{u}_\tau dx dt. \end{aligned}$$

Therefore, by Lemma 2.1 (Minty's Theorem), (14) and (16)-(20), there exist a unique solution u of (1) such that $u \in L^p(0, T; W_0^{1,p}(\Omega)) \cap L^\infty(0, T; L^\infty(\Omega))$ if $p \geq 2$.

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