Bull. Korean Math. Soc. **51** (2014), No. 5, pp. 1527–1538
http://dx.doi.org/10.4134/BKMS.2014.51.5.1527

A CYCLIC AND SIMULTANEOUS ITERATIVE ALGORITHM FOR THE MULTIPLE SPLIT COMMON FIXED POINT PROBLEM OF DEMICONTRACTIVE MAPPINGS

Yu-Chao Tang, Ji-Gen Peng, and Li-Wei Liu

Abstract. The purpose of this paper is to address the multiple split common fixed point problem. We present two different methods to approximate a solution of the problem. One is cyclic iteration method; the other is simultaneous iteration method. Under appropriate assumptions on the operators and iterative parameters, we prove both the proposed algorithms converge to the solution of the multiple split common fixed point problem. Our results generalize and improve some known results in the literatures.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. The multiple split common fixed point problem (MSCFPP) was first introduced in [8], which requires to find a common fixed point of a family of operators in one space whose image under a linear transformation is a common fixed point of another family of operators in the image space. The (MSCFPP) includes the well-known of the multiple-sets split feasibility problem (MSSFP) (see for example [6, 16]), the split feasibility problem (SFP) (see for example [4, 18, 19, 21] etc.) and the convex feasibility problem (CFP)([1]). The (MSCFPP) can be stated as follows:

$$\text{Find a point } x^* \in \bigcap_{i=1}^{p} Fix(U_i) \text{ such that } Ax^* \in \bigcap_{j=1}^{r} Fix(T_j), \tag{1.1}$$

where $p, r \geq 1$ are integers, $Fix(T)$ denotes the fixed point set of T, $A : H_1 \to H_2$ is a bounded linear operator, $\{U_i\}_{i=1}^{p} : H_1 \to H_1$, $\{T_j\}_{j=1}^{r} : H_2 \to H_2$ are nonlinear operators. In particular, if $p = r = 1$, then (1.1) reduces to the following

$$\text{Find a point } x^* \in Fix(U) \text{ such that } Ax^* \in Fix(T), \tag{1.2}$$

Received August 27, 2013.
2010 *Mathematics Subject Classification.* 49J53, 65K10.
Key words and phrases. demicontractive mappings, cyclic, simultaneous, split common fixed point.

which is usually called the two-sets of (SCFPP).

Under what conditions on the operators $\{U_i\}_{i=1}^p$, $\{T_j\}_{j=1}^r$ and the matrix A to guarantee the convergence of the designed algorithm to a solution of (MSCFPP) (1.1), Censor and Segal [8] first constructed an iterative algorithm to solve the two sets of (SCFPP) for directed operators (the definition can be found in Definition 2.4) in finite- dimensional spaces.

Algorithm 1. Let $x_0 \in H_1$ be arbitrary, the sequence $\{x_n\}$ defined by:

$$x_{n+1} = U(x_n - \gamma A^t(I - T)Ax_n),\ n \geq 0, \tag{1.3}$$

where $\gamma \in (0, \frac{2}{\lambda})$ with λ being the largest eigenvalue of the matrix A^tA (t stands for matrix transposition). By using the product space technique, they introduced a parallel algorithm to solve the (MSCFPP) as follows:

$$x_{n+1} = x_n + \gamma \left[\sum_{i=1}^{p} \alpha_i (U_i(x_n) - x_n) + \sum_{j=1}^{r} \beta_j A^t (T_j - I) A x_n \right],\ n \geq 0, \tag{1.4}$$

where $0 < \gamma < \frac{2}{L}$ with $L = \sum_{i=1}^p \alpha_i + \lambda \sum_{j=1}^r \beta_j$.

In 2011, Wang and Xu [14] converted the (MSCFPP) (1.1) to a common fixed point problem, and introduced a cyclic iterative algorithm to solve the (MSCFPP) under the assumption that T and U are directed operators. The advantage of this method is that one can apply the some exists method for solving the common fixed point problem to (MSCFPP). They proposed the cyclic iterative algorithm as follows.

Algorithm 2. For any $x_0 \in H_1$, define a sequence $\{x_n\}$ by the following iterative procedure:

$$x_{n+1} = U_{[n]}\left(x_n + \lambda(V_{[n]}(x_n) - x_n)\right),\ n \geq 0, \tag{1.5}$$

where $[n] := n(\text{mod } p)$ with the mod function taking values in $\{1, \dots, p\}$, and $V := I + \sigma A^*(T - I)A$ with $\sigma \in (0, 1/\rho(A^*A))$ and $\lambda \in (0, 2)$.

To generalize the (MSCFPP) to a general type of operators, Moudafi [10] proposed an algorithm for solving the two-sets of (SCFPP) (1.2) for the quasi-nonexpansive operators in Hilbert spaces. The algorithm is summarized as follows:

Algorithm 3. Let $x_0 \in H_1$, and

$$x_{n+1} = (1 - \alpha_n)u_n + \alpha_n U(u_n),\ n \geq 0, \tag{1.6}$$

where $u_n = x_n + \gamma\beta A^*(T - I)(Ax_n)$, $\beta \in (0, 1)$, $\alpha_n \in (0, 1)$ and $\gamma \in (0, \frac{1}{\lambda\beta})$ with λ being the spectral radius of the operator A^*A, i.e., $\lambda = \rho(A^*A)$.

Further, Moudafi [9] generalized the Algorithm 3 to solve the solution set of the two-sets of (SCFPP) when the operators U and T are demicontractive.

Algorithm 4. Let $x_0 \in H_1$ be arbitrary, the sequence $\{x_n\}$ is defined by:

$$x_{n+1} = (1 - \alpha_n)u_n + \alpha_n U(u_n),\ n \geq 0, \tag{1.7}$$

where $u_n = x_n + \gamma A^*(T - I)Ax_n$, $\gamma \in (0, \frac{1-\mu}{\lambda})$ with λ being the spectral radius of the operator A^*A and $\{\alpha_n\} \subset (0, 1)$.

In [12], we extended the work of Moudafi [9] to the (MSCFPP) (1.1) and introduced a cyclic iterative algorithm to solve it.

Algorithm 5. Let $x_0 \in H_1$ be arbitrary, for $n \geq 0$, calculate

$$x_{n+1} = (1 - \alpha_n)u_n + \alpha_n U_{i(n)}(u_n),\ n \geq 0, \tag{1.8}$$

where $u_n = x_n + \gamma A^*(T_{j(n)} - I)Ax_n$, $i(n) = n(\text{mod}\, p) + 1$ and $j(n) = n(\text{mod}\, r) + 1$. $\gamma \in (0, \frac{1-\mu}{\lambda})$ with λ being the spectral radius of the operator A^*A and $\{\alpha_n\} \subset (0, 1)$.

Although the two-sets of (SCFPP) (1.2) is a special case of (MSCFPP) (1.1), the algorithm (1.4) can not reduce to the algorithm (1.3). The advantage of the cyclic iterative algorithm (1.5) and (1.8) is not only can solve the (MSCFPP), but also can reduce to the corresponding algorithms which are used to solve the two-sets of (SCFPP).

Inspired and motivated by the above works, we propose two iteration schemes which can be applied directly to (MSCFPP) (1.1).

1. Simultaneous iteration schemes

For any $x_0 \in H_1$, define the following iterative sequences

$$\begin{aligned} y_{j,n} &= x_n + \gamma A^*(T_j - I)Ax_n,\ j = 1, 2, \ldots, r. \\ u_n &= \sum_{j=1}^{r} \eta_j y_{j,n}. \\ z_{i,n} &= (1 - \alpha_n)u_n + \alpha_n U_i(u_n),\ i = 1, 2, \ldots, p. \end{aligned} \tag{1.9}$$

The update sequence $\{x_n\}$ is defined by

$$x_{n+1} = \sum_{i=1}^{p} \omega_i z_{i,n},\ n \geq 0, \tag{1.10}$$

where the constant $\gamma > 0$, $\{\alpha_n\} \subset (0, 1)$, and $\{\eta_j\}_{j=1}^r \subset (0, 1)$ and $\{\omega_i\}_{i=1}^p \subset (0, 1)$ with $\sum_{j=1}^r \eta_j = 1$ and $\sum_{i=1}^p \omega_i = 1$. The equivalent form of the parallel iterative sequence can be represented by

$$x_{n+1} = (1 - \alpha_n)u_n + \alpha_n \sum_{i=1}^{p} \omega_i U_i \left(x_n + \gamma A^* \sum_{j=1}^{r} \eta_j (T_j - I)(Ax_n) \right). \tag{1.11}$$

The simultaneous algorithmic structures favor parallel computing platforms. It also called the parallel iteration method.

2. Cyclic iteration schemes

For any $x_0 \in H_1$, the iterative sequence $\{x_n\}$ is defined by

$$x_{n+1} = (1 - \alpha_n)u_n + \alpha_n U_{[n]}(u_n),\ n \geq 0, \tag{1.12}$$

where u_n is given as (1.9), and $[n] = n(\text{mod } p) + 1$, the mod function takes value in $\{1, 2, \ldots, p\}$, the constant $\gamma > 0$, $\{\alpha_n\} \subset (0, 1)$, and $\{\eta_j\}_{j=1}^r \subset (0, 1)$ with $\sum_{j=1}^r \eta_j = 1$.

Under mild assumptions on the iterative parameters, we prove both the algorithms converge weakly to a solution of the (MSCFPP) (1.1).

2. Preliminaries

In this section, we collect some important definitions and prove some useful lemmas which will be used in the following section. We introduce the following notations. Ω denotes the solution set of (MSCFPP) (1.1). $\omega_w(x_n) = \{x : \exists x_{n_j} \rightharpoonup x\}$ denotes the weak ω-limit set of $\{x_n\}$. The symbol $\rightharpoonup$ for weak convergence and $\to$ for strong convergence, respectively.

Definition 2.1. Assume that $T : H \to H$ is an operator with $Fix(T) \neq \emptyset$,

(i) T is said to be nonexpansive, if

$$\|Tx - Ty\| \leq \|x - y\| \quad \text{for all } x, y \in H.$$

(ii) T is said to be quasi-nonexpansive, if

$$\|Tx - q\| \leq \|x - q\| \quad \text{for all } x \in H,\ q \in Fix(T).$$

(iii) T is said to be strictly pseudocontractive, if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2$$

for all $x, y \in H$, and some $k \in (0, 1)$.

It is easily observed that if T is nonexpansive with nonempty $Fix(T)$, then T is quasi-nonexpansive.

Definition 2.2 ([9])**.** An operator $T : H \to H$ is called k-demicontractive, if there exists a constant $k \in (0, 1)$ such that

$$\|Tx - q\|^2 \leq \|x - q\|^2 + k\|x - Tx\|^2$$

for all $x \in H$ and $q \in Fix(T)$.

If T is strictly pseudocontractive with $Fix(T) \neq \emptyset$, then T is k-demicontractive. The next lemma shows two equivalent definition of demicontractive operator.

Lemma 2.1 ([12])**.** *Let $T : H \to H$ be k-demicontractive operator such that $Fix(T) \neq \emptyset$. Then it is equivalent to the following inequalities*:

(i) $\langle x - Tx, x - q\rangle \geq \frac{1-k}{2}\|x - Tx\|^2,\ q \in Fix(T),\ x \in H;$

(ii) $\langle x - Tx, q - Tx\rangle \leq \frac{1+k}{2}\|x - Tx\|^2,\ q \in Fix(T),\ x \in H.$

The demiclosedness of the mapping T is important to deal with the convergence of fixed point algorithm.

Definition 2.3. $I - T$ is called demiclosed at zero, if for any sequence $\{x_n\} \subset H$ and $x \in H$, we have $x_n \rightharpoonup x$ and $x_n - Tx_n \to 0$, then $x \in Fix(T)$.

We recall the definition of directed operator which properties can be found in [8] and [2].

Definition 2.4 ([8])**.** T is a directed operator, if

$$\langle q - Tx, x - Tx \rangle \leq 0$$

for all $x \in H$ and $q \in Fix(T)$.

The directed operator is also included by the demicontractive operator. We shall use the notion of Fejér-monotone sequences in the following.

Definition 2.5. Let C be a nonempty closed convex subset of H and $\{x_n\}$ is a sequence in H. The sequence $\{x_n\}$ is called Fejér-monotone with respect to C, if

$$\|x_{n+1} - z\| \leq \|x_n - z\|, \quad n \geq 0,\ z \in C.$$

The next lemma can be found in the Chapter 2 of [3].

Lemma 2.2. *Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. Then*

(i) $\|x + y\|^2 = \|x\|^2 + 2\langle x, y \rangle + \|y\|^2.$

(ii) $\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2,\ \forall x, y \in H$ *and* $\forall \alpha \in [0, 1]$.

(iii) $\left\|\sum_{i=1}^{n} \lambda_i x_i\right\|^2 = \sum_{i=1}^{n} \lambda_i \|x_i\|^2 - \frac{1}{2}\sum_{i,j=1}^{n} \lambda_i\lambda_j\|x_i - x_j\|^2,\ n \geq 2,$

where $\lambda_i \in [0, 1]$*, for all* $i = 1, 2, \ldots, n$ *with* $\sum_{i=1}^{n} \lambda_i = 1$.

To facilitate our proof, we will make use of the following lemmas.

Lemma 2.3 ([9])**.** *Let T be a k-demicontractive self mapping on H with $Fix(T) \neq \emptyset$ and set $T_\alpha := (1-\alpha)I + \alpha T$ for $\alpha \in (0, 1]$. Then, T_α is quasi-nonexpansive provided that $\alpha \in [0, 1-k]$ and $\|T_\alpha x - q\|^2 \leq \|x - q\|^2 - \alpha(1 - k - \alpha)\|Tx - x\|^2,\ x \in H,\ q \in Fix(T)$.*

Lemma 2.4 ([1])**.** *If a sequence $\{x_n\}$ is Féjer-monotone respect to a closed subset of C, then $x_n \rightharpoonup x^* \in C$ if and only if $\omega_w(x_n) \subset C$.*

3. Main results

Let $\{U_i\}_{i=1}^{p}$ and $\{T_j\}_{j=1}^{r}$ be a finite family of demicontractive mappings. Then there exists $\{\beta_i\}_{i=1}^{p} \subset (0, 1)$ and $\{\mu_j\}_{j=1}^{r} \subset (0, 1)$, such that

$$\|U_i x - q\|^2 \leq \|x - q\|^2 + \beta_i\|x - U_i x\|^2,\ x \in H,\ q \in Fix(U_i),\ i = 1, 2, \ldots, p,$$

and

$$\|T_j x - p\|^2 \leq \|x - p\|^2 + \mu_j\|x - T_j x\|^2,\ x \in H,\ p \in Fix(T_j),\ j = 1, 2, \ldots, r.$$

Let $\beta = \max_{1 \leq i \leq p}\{\beta_i\},\ \mu = \max_{1 \leq j \leq r}\{\mu_j\}$. Then we have

$$\|U_i x - q\|^2 \leq \|x - q\|^2 + \beta\|x - U_i x\|^2 \text{ for all } x \in H,\ q \in Fix(U_i),\ i = 1, 2, \ldots, p,$$

and

$\|T_jx-p\|^2 \leq \|x-p\|^2+\mu\|x-T_jx\|^2$ for all $x \in H,\ p \in Fix(T_j),\ j = 1, 2, \ldots, r.$

First, we prove the following lemma.

Lemma 3.1. *Let $A : H_1 \to H_2$ be a bounded linear operator. Assume that $\{U_i\}_{i=1}^p : H_1 \to H_1$ be β_i-demicontractive and $\{T_j\}_{j=1}^r : H_2 \to H_2$ be μ_j-demicontractive mappings. If the solution set Ω of (1.1) is nonempty, then the iterative sequence $\{x_n\}$ generated by (1.10) is the Fejér-monotone, i.e., for any $x \in \Omega$,*

$$\|x_{n+1} - x\| \leq \|x_n - x\|, \quad \forall n \geq 0,$$

provided that $\gamma \in (0, \frac{1-\mu}{\lambda}]$ and $\alpha_n \in (0, 1-\beta]$, where λ is the spectral radius of the operator A^*A.

Proof. Let x belongs to the solution set Ω. By Lemma 2.3, for any $i = 1, 2, \ldots, p$, we obtain

$$\|z_{i,n} - x\|^2 \leq \|u_n - x\|^2 - \alpha_n(1 - \beta - \alpha_n)\|U_i(u_n) - u_n\|^2. \tag{3.1}$$

On the other hand, for any $j = 1, 2, \ldots, r$, we have

$$\begin{aligned}
\|y_{j,n} - x\|^2 &= \|x_n + \gamma A^*(T_j - I)Ax_n - x\|^2 \\
&= \|x_n - x\|^2 + \gamma^2\|A^*(T_j - I)Ax_n\|^2 \\
&\quad + 2\gamma\langle x_n - x, A^*(T_j - I)Ax_n\rangle \\
&\leq \|x_n - x\|^2 + \lambda\gamma^2\|(T_j - I)(Ax_n)\|^2 + 2\gamma\langle x_n - x, A^*(T_j - I)Ax_n\rangle. \qquad (3.2)
\end{aligned}$$

For the last term of the above inequality, by Lemma 2.1(ii), we have

$$\begin{aligned}
&2\gamma\langle x_n - x, A^*(T_j - I)Ax_n\rangle \\
&= 2\gamma\langle A(x_n - x), (T_j - I)(Ax_n)\rangle \\
&= 2\gamma\langle A(x_n - x) + (T_j - I)(Ax_n) - (T_j - I)(Ax_n), (T_j - I)(Ax_n)\rangle \\
&= 2\gamma\left(\langle T_j(Ax_n) - Ax, (T_j - I)(Ax_n)\rangle - \|(T_j - I)(Ax_n)\|^2\right) \\
&\leq 2\gamma\left(\frac{1+\mu}{2}\|(T_j - I)(Ax_n)\|^2 - \|(T_j - I)(Ax_n)\|^2\right) \\
&= -\gamma(1 - \mu)\|(T_j - I)(Ax_n)\|^2. \qquad (3.3)
\end{aligned}$$

Substituting (3.3) into (3.2), we get

$$\|y_{j,n} - x\|^2 \leq \|x_n - x\|^2 - \gamma(1 - \mu - \lambda\gamma)\|(T_j - I)(Ax_n)\|^2. \tag{3.4}$$

It follows from Lemma 2.2, $u_n = \sum_{j=1}^r \eta_j y_{j,n}$ and (3.4), we obtain

$$\begin{aligned}
\|u_n - x\|^2 &= \left\|\sum_{j=1}^r \eta_j y_{j,n} - x\right\|^2 = \left\|\sum_{j=1}^r \eta_j (y_{j,n} - x)\right\|^2 \\
&\leq \sum_{j=1}^r \eta_j \|y_{j,n} - x\|^2
\end{aligned}$$

$$\leq \|x_n - x\|^2 - \gamma(1-\mu-\lambda\gamma)\sum_{j=1}^{r}\eta_j\|(T_j - I)(Ax_n)\|^2. \tag{3.5}$$

Finally, we prove the Fejér-monotone of the sequence $\{x_n\}$. In fact, by Lemma 2.2, (3.1) and (3.5), we obtain

$$\begin{aligned}
\|x_{n+1} - x\|^2 &= \left\|\sum_{i=1}^{p}\omega_i z_{i,n} - x\right\|^2 \\
&= \left\|\sum_{i=1}^{p}\omega_i(z_{i,n} - x)\right\|^2 \\
&\leq \sum_{i=1}^{p}\omega_i\|z_{i,n} - x\|^2 \\
&\leq \|u_n - x\|^2 - \alpha_n(1-\beta-\alpha_n)\sum_{i=1}^{p}\omega_i\|U_i(u_n) - u_n\|^2 \\
&\leq \|x_n - x\|^2 - \alpha_n(1-\beta-\alpha_n)\sum_{i=1}^{p}\omega_i\|U_i(u_n) - u_n\|^2 \\
&\quad - \gamma(1-\mu-\lambda\gamma)\sum_{j=1}^{r}\eta_j\|(T_j - I)(Ax_n)\|^2.
\end{aligned}$$

Since $\gamma \in (0, \frac{1-\mu}{\lambda}]$ and $\alpha_n \in (0, 1-\beta]$, so the sequence $\{x_n\}$ is Fejér-monotone. This completes the proof. □

Now, we prove the convergence of the simultaneous iteration scheme.

Theorem 3.1. *Let $A : H_1 \to H_2$ be a bounded linear operator. Let $\{U_i\}_{i=1}^{p} : H_1 \to H_1$ be β_i-demicontractive and $\{T_j\}_{j=1}^{r} : H_2 \to H_2$ be μ_j-demicontractive mapping. Assume that $\{I - U_i\}_{i=1}^{p}$ and $\{I - T_j\}_{j=1}^{r}$ are demiclosed at zero. If the solution set Ω of* (1.1) *is nonempty, then the sequence $\{x_n\}$ generated by* (1.10) *converges weakly to a solution of the Ω, provided that $\gamma \in (0, \frac{1-\mu}{\lambda})$ and $\alpha_n \in (\delta, 1-\beta-\delta)$ for a small $\delta > 0$.*

Proof. From the last inequality of Lemma 3.1, and the requirement of the parameters γ and α_n in Theorem 3.1, we conclude that

$$\sum_{n=0}^{\infty}\sum_{i=1}^{p}\omega_i\|U_i(u_n) - u_n\|^2 < +\infty \quad \text{for any } i = 1, 2, \ldots, p,$$

and

$$\sum_{n=0}^{\infty}\sum_{j=1}^{r}\eta_j\|(T_j - I)(Ax_n)\|^2 < +\infty \quad \text{for any } j = 1, 2, \ldots, r.$$

Therefore,

$$\lim_{n\to\infty} \|U_i(u_n) - u_n\| = 0 \tag{3.6}$$

for any $i = 1, 2, \ldots, p$ and

$$\lim_{n\to\infty} \|(T_j - I)(Ax_n)\| = 0 \tag{3.7}$$

for any $j = 1, 2, \ldots, r$.

It follows from the Fejér-monotonicity of the sequence $\{x_n\}$ that the sequence $\{x_n\}$ is bounded and $\omega_w(x_n)$ is nonempty. Let $x^* \in \omega_w(x_n)$. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x^*$. By the demiclosed of $\{I - T_j\}_{j=1}^r$ at 0 and (3.7), we obtain

$$(T_j - I)(Ax^*) = 0 \text{ for any } j = 1, 2 \ldots, r,$$

i.e.,

$$Ax^* \in \bigcap_{j=1}^r Fix(T_j).$$

Since $y_{j,n} = x_n + \gamma A^*(T_j - I)(Ax_n)$, it follows that $y_{j,n_k} \rightharpoonup x^*$ and $u_{n_k} = \sum_{j=1}^r \eta_j y_{j,n_k} \rightharpoonup x^*$. Notice that the (3.6), and $\{I - U_i\}_{i=1}^p$ are demiclosed at 0, therefore, $U_i(x^*) = x^*$ for all $i = 1, 2, \ldots, p$, i.e., $x^* \in \bigcap_{i=1}^p Fix(U_i)$. So $x^* \in \Omega$. Therefore, by the Fejér monotonicity of $\{x_n\}$ with respect to Ω, we can apply Lemma 2.4 to conclude that $\{x_n\}$ converges weakly to a solution of Ω. This completes the proof. □

We have proven the weak convergence of the parallel iterative method. Now, we are ready to prove the convergence of cyclic iterative sequence defined in (1.12) to the problem (1.1). Similarly, we need the following lemma to facilitate the main convergence theorem.

Lemma 3.2. *Let $A : H_1 \to H_2$ be a bounded linear operator. Assume that $\{U_i\}_{i=1}^p : H_1 \to H_1$ be β_i-demicontractive and $\{T_j\}_{j=1}^r : H_2 \to H_2$ be μ_j-demicontractive mappings. If the solution set Ω of* (1.1) *is nonempty, then the iterative sequence $\{x_n\}$ generated by* (1.12) *is the Fejér-monotone, i.e., for any $x \in \Omega$,*

$$\|x_{n+1} - x\| \le \|x_n - x\|, \quad \forall n \ge 0,$$

provided that $\gamma \in (0, \frac{1-\mu}{\lambda}]$ and $\alpha_n \in (0, 1 - \beta]$, where λ is the spectral radius of the operator A^*A.

Proof. The proof is similar to the Lemma 3.1, we give the highlight for simple. Let $x \in \Omega$, since the definition of $\{u_n\}$ in the cyclic iterative method (1.12) is the same as in the parallel iterative method (1.10), by (3.5), we have

$$\|u_n - x\|^2 \le \|x_n - x\|^2 - \gamma(1 - \mu - \lambda\gamma) \sum_{j=1}^r \eta_j \|(T_j - I)(Ax_n)\|^2.$$

Therefore, we obtain

$$\begin{aligned}\|x_{n+1}-x\|^2 &\le \|u_n-x\|^2-\alpha_n(1-\beta-\alpha_n)\|U_{[n]}(u_n)-u_n\|^2\\ &\le \|x_n-x\|^2-\alpha_n(1-\beta-\alpha_n)\|U_{[n]}(u_n)-u_n\|^2\\ &\quad -\gamma(1-\mu-\lambda\gamma)\sum_{j=1}^{r}\eta_j\|(T_j-I)(Ax_n)\|^2. \end{aligned}\tag{3.8}$$

It follows from the restriction on the parameters of γ and α_n that $\{x_n\}$ is Fejér-monotone sequence. This completes the proof. □

Theorem 3.2. *Let $A : H_1 \to H_2$ be a bounded linear operator. Let $\{U_i\}_{i=1}^{p} : H_1 \to H_1$ be β_i-demicontractive and $\{T_j\}_{j=1}^{r} : H_2 \to H_2$ be μ_j-demicontractive mappings. Assume that $\{I-U_i\}_{i=1}^{p}$ and $\{I-T_j\}_{j=1}^{r}$ are demiclosed at zero, and $\{U_i\}_{i=1}^{p}$ are continuous. If the solution set Ω of* (1.1) *is nonempty, then the sequence $\{x_n\}$ generated by* (1.12) *converges weakly to a solution of the Ω, provided that $\gamma \in (0, \frac{1-\mu}{\lambda})$ and $\alpha_n \in (\delta, 1-\beta-\delta)$ for a small $\delta > 0$.*

Proof. From the inequality (3.8), and the fact that $\alpha_n \in (\delta, 1-\beta-\delta)$ and $\gamma \in (0, \frac{1-\mu}{\lambda})$, we have

$$\sum_{n=0}^{\infty}\|U_{[n]}(u_n)-u_n\|^2 < +\infty,$$

and

$$\sum_{n=0}^{\infty}\sum_{j=1}^{r}\eta_j\|(T_j-I)(Ax_n)\|^2 < +\infty.$$

Therefore,

$$\lim_{n\to\infty}\|U_{[n]}(u_n)-u_n\| = 0,\tag{3.9}$$

and

$$\lim_{n\to\infty}\|(T_j-I)(Ax_n)\| = 0 \;\text{ for any } j = 1,2,\ldots,r.\tag{3.10}$$

Since the sequence $\{x_n\}$ is Fejér-monotone, so it is bounded. Let $x^* \in w_\omega(x_n)$. Take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x^*$. By the demiclosedness of $\{I-T_j\}_{j=1}^{r}$ and the fact (3.10), we get

$$(T_j-I)(Ax^*) = 0 \;\text{ for any } j = 1,2,\ldots,r,$$

then

$$Ax^* \in \bigcap_{j=1}^{r} Fix(T_j).$$

Next, we show that $x^* \in \bigcap_{i=1}^{p} Fix(U_i)$. In fact, from the definition of $\{x_n\}$ and Lemma 2.2, we have

$$\begin{aligned} \|x_{n+1} - x_n\|^2 &= \left\| \alpha_n (U_{[n]}(u_n) - u_n) + \gamma \sum_{j=1}^{r} \eta_j A^*(T_j - I)(Ax_n) \right\|^2 \\ &\leq 2\alpha_n^2 \|U_{[n]}(u_n) - u_n\|^2 + 2\gamma^2 \left\| \sum_{j=1}^{r} \eta_j A^*(T_j - I)(Ax_n) \right\|^2 \\ &\leq 2\alpha_n^2 \|U_{[n]}(u_n) - u_n\|^2 + 2\gamma^2 \sum_{j=1}^{r} \eta_j \lambda \|(T_j - I)(Ax_n)\|^2. \end{aligned}$$

It follows from the (3.9) and (3.10) that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

Therefore,

$$\begin{aligned} &\|u_{n+1} - u_n\|^2 \\ &= \left\| \sum_{j=1}^{r} \eta_j (y_{i,n+1} - y_{j,n}) \right\|^2 \\ &\leq \sum_{j=1}^{r} \eta_j \|y_{j,n+1} - y_{j,n}\|^2 \\ &= \sum_{j=1}^{r} \eta_j \left\| x_{n+1} - x_n + \gamma A^*(T_j - I)(Ax_{n+1}) - \gamma A^*(T_j - I)(Ax_n) \right\|^2 \\ &\leq 2\|x_{n+1} - x_n\|^2 + 2\gamma^2 \sum_{j=1}^{r} \eta_j \|A^*(T_j - I)(Ax_{n+1} - Ax_n)\|^2 \\ &\leq 2\|x_{n+1} - x_n\|^2 + 2\gamma^2 \sum_{j=1}^{r} \eta_j \lambda \|(T_j - I)(Ax_{n+1} - Ax_n)\|^2 \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

Then, for any $i = 1, 2, \ldots, p$, $\|u_{n+i} - u_n\| \to 0$ as $n \to \infty$, and

$$\begin{aligned} \|u_n - U_{[n+i]}(u_n)\| \leq \|u_n - u_{n+i}\| &+ \|u_{n+i} - U_{[n+i]}(u_{n+i})\| \\ &+ \|U_{[n+i]}(u_{n+i}) - U_{[n+i]}(u_n)\|, \end{aligned}$$

From (3.9) and the continuity of $\{U_i\}_{i=1}^{p}$, we have

$$\lim_{n\to\infty} \|u_n - U_{[n+i]}(u_n)\| = 0.$$

It is now clear that for each $k \in \{1, 2, \ldots, p\}$, there exists $i \in \{1, 2, \ldots, p\}$ such that $k = (n + i)(\text{mod } p) + 1$, then

$$\lim_{n\to\infty} \|u_n - U_k u_n\| = \lim_{n\to\infty} \|u_n - U_{[n+i]}(u_n)\| = 0.$$

Since $I - U_k$ is demiclosedness at zero and $x^* \in \omega_w(u_n)$, so $x^* \in \bigcap_{i=1}^{p} Fix(U_i)$. Then the weak convergence of the iterative sequence $\{x_n\}$ can be obtained by Lemma 2.4. This completes the proof. □

Remark 3.1. The simultaneous (parallel) iteration method and cyclic iteration method are two common ways to solve the convex feasibility problem. Although the parallel iteration scheme (1.4) can not be reduced to (1.3), the simultaneous iteration scheme (1.10) for the multiple split common fixed point problem which can be reduced to the original iteration scheme (1.7) by letting $p = r = 1$. We also propose a new cyclic iteration scheme (1.12) which is different from (1.8).

Acknowledgements. This work was supported by the National Natural Science Foundations of China(11131006, 11201216, 11401293, 11461046), the Natural Science Foundations of Jiangxi Province(CA201107114, 20114BAB201004) and the Youth Science Funds of The Education Department of Jiangxi Province (GJJ14154).

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Yu-Chao Tang
Department of Mathematics
NanChang University
Nanchang 330031, P. R. China
and
School of Mathematics and Statistics
Xi'an Jiaotong University
Xi'an 710049, P. R. China
E-mail address: hhaaoo1331@gmail.com

Ji-Gen Peng
School of Mathematics and Statistics
Xi'an Jiaotong University
Xi'an 710049, P. R. China
E-mail address: pengjg@xjtu.edu.cn

Li-Wei Liu
Department of Mathematics
NanChang University
Nanchang 330031, P. R. China
E-mail address: liulws@yahoo.com.cn