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ON THE STABILITY OF RADICAL FUNCTIONAL EQUATIONS IN QUASI-β-NORMED SPACES

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Abstract. In this paper, we prove the generalized Hyers-Ulam stability results controlled by considering approximately mappings satisfying conditions much weaker than Hyers and Rassias conditions for radical quadratic and radical quartic functional equations in quasi-β-normed spaces.

1. Introduction

In 1960, the stability problem of functional equations originated from the question of Ulam [44] concerning the stability of group homomorphisms. The famous Ulam stability problem was partially solved by Hyers [22] in Banach spaces. Hyers's theorem was generalized by Aoki [2] for additive mapping and by Rassias [31] for linear mapping by considering unbounded Cauchy differences. Rassias [32], [35] provided a generalization of Hyers' theorem by proving the existence of unique linear mappings near approximate additive mappings. On the other hand, Rassias [36], [37] considered the Cauchy difference controlled by a product of different powers of norm. The above results has been generalized by Forti [13] and Găvruta [15] who permitted the Cauchy difference to become arbitrary unbounded. Gajda and Ger [14] showed that one can get analogous stability results for subadditive multifunctions. Gruber [21] remarked that Ulam's problem is of particular interest in probability theory and in the case of functional equations of different types. During the last two decades, a number of papers and research monographs have been published on various generalizations and applications of the generalized Hyers-Ulam stability to a number of functional equations and mappings in various spaces ([1], [3]-[10], [16], [17], [24], [26], [34], [40], [41]).

The quadratic function $f(x) = cx^2$ satisfies the functional equation

$$(\mathcal{E}) \qquad f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

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and therefore the equation $(\mathcal{E})$ is called the *quadratic functional equation*. The Hyers-Ulam stability theorem for the quadratic functional equation was proved by Skof [42] and Czerwik [12]. Since then, the stability problem of various quadratic functional equations have been extensively investigated by a number of authors ([11], [18], [20], [23], [27], [29], [30], [33], [39]).

Before we present our results, we introduce some basic facts concerning quasi-β-normed space and some preliminary results. We fix a real number β with $0 < \beta \leq 1$ and let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$. Let X be a linear space over $\mathbb{K}$. A *quasi-β-norm* $\|\cdot\|$ is a real-valued function on X satisfying the following:

(1) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$;

(2) $\|\lambda x\| = |\lambda|^\beta \cdot \|x\|$ for all $\lambda \in \mathbb{K}$ and $x \in X$;

(3) there exists a constant $K \geq 1$ such that $\|x + y\| \leq K(\|x\| + \|y\|)$ for all $x, y \in X$.

The pair $(X, \|\cdot\|)$ is called a *quasi-β-normed space* if $\|\cdot\|$ is a quasi-β-norm on X. The smallest possible K is called the *module of concavity* of $\|\cdot\|$. A *quasi-β-Banach space* is a complete quasi-β-normed space.

A quasi-β-norm $\|\cdot\|$ is called a *(β, p)-norm* $(0 < p \leq 1)$ if $\|x + y\|^p \leq \|x\|^p + \|y\|^p$ for all $x, y \in X$. In this case, a quasi-β-Banach space is called a *(β, p)-Banach space*. For further details on quasi-β-normed spaces and (β, p)-Banach spaces, refer to the papers [19], [25], [28], [38] and [43].

Recall that a function $\varphi : A \to B$ with a domain A and a codomain $(B, \leq)$ which is closed under the addition is a *subadditive* (*superadditive*) *function* if $\varphi(x + y) \leq (\geq)\ \varphi(x) + \varphi(y)$ and a *subquadratic* (*superquadratic*) *function* with $\varphi(0) = 0$ if $\varphi(x+y)+\varphi(x-y) \leq (\geq)\ 2\varphi(x)+2\varphi(y)$ for all $x, y \in A$, respectively.

Let $\ell \in \{-1, 1\}$ be fixed. If there exists a constant L with $0 < L < 1$ such that a function $\varphi : A \to B$ satisfies

$$\ell\varphi(x + y) \leq \ell L^\ell(\varphi(x) + \varphi(y))$$

for all $x, y \in A$, then we say that φ is *contractively subadditive* if $\ell = 1$ and φ is *expansively superadditive* if $\ell = -1$. Similarly, if there exists a constant L with $0 < L < 1$ such that a function $\varphi : A \to B$ with $\varphi(0) = 0$ satisfies

$$\ell\varphi(x + y) + \ell\varphi(x - y) \leq 2\ell L^\ell(\varphi(x) + \varphi(y))$$

for all $x, y \in A$, then we say that φ is *contractively subquadratic* if $\ell = 1$ and φ is *expansively superquadratic* if $\ell = -1$.

In this paper, we point out the generalized Hyers-Ulam stability results controlled by approximately mappings for the radical quadratic and radical quartic functional equations which is introduced in [27],

$$f(\sqrt{ax^2 + by^2}) = af(x) + bf(y) \tag{1.1}$$

and

$$f(\sqrt{ax^2 + by^2}) + f(\sqrt{|ax^2 - ay^2|}) = 2a^2 f(x) + 2b^2 f(y) \tag{1.2}$$

in quasi-β-Banach spaces, and new theorems about the generalized Hyers-Ulam stability by using subadditive and subquadratic functions for those functional equations in (β, p)-Banach spaces.

2. Stability of the radical quadratic functional equation (1.1)

In this section, we are modified the generalized Hyers-Ulam stability of radical functional equations (1.1) in quasi-β-normed spaces and (β, p)-Banach spaces, respectively.

Let X be a normed space and $\phi : \mathbb{R}^2 \to \mathbb{R}^+ \cup \{0\}$ be a function. A function $f : \mathbb{R} \to X$ is called a *ϕ-approximatively radical quadratic function* if

$$\left\| f(\sqrt{ax^2 + by^2}) - af(x) - bf(y) \right\|_X \le \phi(x, y) \tag{2.1}$$

for all $x, y \in \mathbb{R}$, where $a, b \in \mathbb{R}^+$ are such that $a + b \ne 1$.

First, using the idea of Găvruta, we prove the generalized Hyers-Ulam stability of radical functional equations (1.1) in the spirit of Ulam, Hyers and Rassias.

Theorem 2.1. *Let X be a quasi-β-Banach space and $f : \mathbb{R} \to X$ be a ϕ-approximatively radical quadratic function with $f(0) = 0$. If a function $\phi : \mathbb{R}^2 \to \mathbb{R}^+ \cup \{0\}$ satisfies the following:*

$$\sum_{j=0}^{\infty} \left(\frac{K}{2^\beta}\right)^j \left(\phi\Big(0, \sqrt{\frac{a}{b}} 2^{\frac{j}{2}} x\Big) + \phi\Big(2^{\frac{j}{2}} x, \sqrt{\frac{a}{b}} 2^{\frac{j}{2}} x\Big) + \phi\Big(2^{\frac{j}{2}} x, 0\Big) + \phi\Big(2^{\frac{(j+1)}{2}} x, 0\Big) \right) < \infty, \tag{2.2}$$

and

$$\lim_{n \to \infty} \frac{1}{2^{\beta n}} \phi\Big(2^{\frac{n}{2}} x, 2^{\frac{n}{2}} y\Big) = 0 \tag{2.3}$$

for all $x, y \in \mathbb{R}$, then there exists a unique quadratic mapping $\mathcal{F} : \mathbb{R} \to X$ satisfying the functional equation (1.1) and the following inequality:

$$\begin{aligned} &\left\| f(x) - \mathcal{F}(x) \right\|_X \\ &\le \frac{K^3}{(2a)^\beta} \sum_{j=0}^{\infty} \left(\frac{K}{2^\beta}\right)^j \left(\phi\Big(0, \sqrt{\frac{a}{b}} 2^{\frac{j}{2}} x\Big) + \phi\Big(2^{\frac{j}{2}} x, \sqrt{\frac{a}{b}} 2^{\frac{j}{2}} x\Big) + \phi\Big(2^{\frac{j}{2}} x, 0\Big) + \phi\Big(2^{\frac{(j+1)}{2}} x, 0\Big) \right) \end{aligned} \tag{2.4}$$

for all $x \in \mathbb{R}$.

Proof. Replacing x and y with $\frac{x}{\sqrt{a}}$ and $\frac{y}{\sqrt{b}}$ in (2.1), respectively, we get

$$\left\| f(\sqrt{x^2 + y^2}) - af\Big(\frac{x}{\sqrt{a}}\Big) - bf\Big(\frac{y}{\sqrt{b}}\Big) \right\|_X \le \phi\Big(\frac{x}{\sqrt{a}}, \frac{y}{\sqrt{b}}\Big) \tag{2.5}$$

for all $x, y \in \mathbb{R}$. Setting $x = 0$ and $y = 0$ in (2.5), respectively, we get

$$\left\| f(\sqrt{y^2}) - bf\Big(\frac{y}{\sqrt{b}}\Big) \right\|_X \le \phi\Big(0, \frac{y}{\sqrt{b}}\Big), \quad \left\| f(\sqrt{x^2}) - af\Big(\frac{x}{\sqrt{a}}\Big) \right\|_X \le \phi\Big(\frac{x}{\sqrt{a}}, 0\Big)$$

for all $x, y \in \mathbb{R}$. Then we obtain

$$\left\| f(x) - \frac{b}{a} f\Big(\sqrt{\frac{a}{b}}x\Big)\right\|_X \le \frac{K}{a^\beta}\left(\phi\Big(x,0\Big) + \phi\Big(0, \sqrt{\frac{a}{b}}x\Big)\right) \tag{2.6}$$

for all $x \in \mathbb{R}$. Also, substituting x and y for $\frac{x+y}{\sqrt{2a}}$ and $\frac{x-y}{\sqrt{2b}}$ in (2.1), respectively, we get

$$\left\| f(\sqrt{x^2+y^2}) - af\Big(\frac{x+y}{\sqrt{2a}}\Big) - bf\Big(\frac{x-y}{\sqrt{2b}}\Big)\right\|_X \le \phi\Big(\frac{x+y}{\sqrt{2a}}, \frac{x-y}{\sqrt{2b}}\Big) \tag{2.7}$$

for all $x, y \in \mathbb{R}$. It follows from (2.5) and (2.7) that

$$\begin{aligned} &\left\| f\Big(\frac{x+y}{\sqrt{2a}}\Big) + \frac{b}{a} f\Big(\frac{x-y}{\sqrt{2b}}\Big) - f\Big(\frac{x}{\sqrt{a}}\Big) - \frac{b}{a} f\Big(\frac{y}{\sqrt{b}}\Big)\right\|_X \\ &\le \frac{K}{a^\beta}\left(\phi\Big(\frac{x}{\sqrt{a}}, \frac{y}{\sqrt{b}}\Big) + \phi\Big(\frac{x+y}{\sqrt{2a}}, \frac{x-y}{\sqrt{2b}}\Big)\right) \end{aligned} \tag{2.8}$$

for all $x, y \in \mathbb{R}$. Letting $x = y = \sqrt{a}x$ in (2.8), we get

$$\left\| f(\sqrt{2}x) - f(x) - \frac{b}{a} f\Big(\sqrt{\frac{a}{b}}x\Big)\right\|_X \le \frac{K}{a^\beta}\left(\phi\Big(x, \sqrt{\frac{a}{b}}x\Big) + \phi\Big(\sqrt{2}x, 0\Big)\right) \tag{2.9}$$

for all $x \in \mathbb{R}$. It follows from (2.6) and (2.9) that

$$\begin{aligned} &\left\| f(x) - \frac{1}{2} f(\sqrt{2}x)\right\|_X \\ &\le \frac{K^2}{(2a)^\beta}\left(\phi\Big(x,0\Big) + \phi\Big(\sqrt{2}x, 0\Big) + \phi\Big(0, \sqrt{\frac{a}{b}}x\Big) + \phi\Big(x, \sqrt{\frac{a}{b}}x\Big)\right). \end{aligned} \tag{2.10}$$

Let $\Phi(x) = \frac{K^2}{(2a)^\beta}\big(\phi(x,0) + \phi(\sqrt{2}x,0) + \phi(0,\sqrt{\frac{a}{b}}x) + \phi(x, \sqrt{\frac{a}{b}}x)\big)$. Then, by the iterative method, we get

$$\left\| f(x) - \frac{1}{2^m} f(2^{\frac{m}{2}}x)\right\|_X \le K \sum_{j=0}^{m-1}\left(\frac{K}{2^\beta}\right)^j \Phi(2^{\frac{j}{2}}x) \tag{2.11}$$

for all $x \in \mathbb{R}$ and $n \in \mathbb{Z}^+$. For all $k, m \in \mathbb{Z}^+$ with $m > k \ge 0$, we have

$$\left\| \frac{1}{2^k} f(2^{\frac{k}{2}}x) - \frac{1}{2^m} f(2^{\frac{m}{2}}x)\right\|_X \le K \sum_{j=k}^{m-1}\left(\frac{K}{2^\beta}\right)^j \Phi(2^{\frac{j}{2}}x) \tag{2.12}$$

for all $x \in \mathbb{R}$. By (2.2) and (2.12), the sequence $\{\frac{1}{2^n} f(2^{\frac{n}{2}}x)\}$ is a Cauchy sequence for all $x \in \mathbb{R}$. Since X is the quasi-β-Banach space, it converges for all $x \in \mathbb{R}$. We can define a mapping $\mathcal{F} : \mathbb{R} \to X$ by $\mathcal{F}(x) := \lim_{n\to\infty} \frac{1}{2^n} f(2^{\frac{n}{2}}x)$ for all $x \in \mathbb{R}$. Then, by (2.2)

$$\left\| \mathcal{F}(\sqrt{ax+by}) - a\mathcal{F}(x) - b\mathcal{F}(y)\right\|_X \le \lim_{n\to\infty} \frac{1}{2^{\beta n}} \phi(2^{\frac{n}{2}}x, f(2^{\frac{n}{2}}y) = 0$$

and $\mathcal{F}(\sqrt{ax+by}) - a\mathcal{F}(x) - b\mathcal{F}(y) = 0$, that is, $\mathcal{F}$ is a quadratic mapping [27]. Taking $m \to \infty$ in (2.12) with $k = 0$, it follows that $\mathcal{F}$ satisfies (2.4) near the approximate function f of (1.1).

Next, we assume that there exists another quadratic mapping $\mathcal{G} : \mathbb{R} \to X$ which satisfies the functional equations (1.1) and (2.4). Since $\mathcal{G}$ satisfies (1.1), we have $\mathcal{G}(2^{\frac{n}{2}}x) = 2^n\mathcal{G}(x)$ for all $x \in X$ and $n \in \mathbb{Z}^+$. Thus we get

$$\left\|\frac{1}{2^n}f(2^{\frac{n}{2}}x) - \mathcal{G}(x)\right\|_X \le \frac{1}{2^{\beta n}}\Phi\left(2^{\frac{n}{2}}x\right)$$

for all $x \in \mathbb{R}$. Letting $n \to \infty$, we establishes $\mathcal{F}(x) = \mathcal{G}(x)$ for all $x \in \mathbb{R}$. This completes the proof. □

From Theorem 2.1, we obtain the following corollary concerning the stability for approximate mappings controlled by a sum of powers of norms and a product of powers of norms.

Corollary 2.2. *Let X be a quasi-β-Banach space, let $p, q \in \mathbb{R}^+ \cup \{0\}$, $\varepsilon \ge 0$ and $f : \mathbb{R} \to X$ be a function satisfying the following*:

$$\left\|f(\sqrt{ax^2+by^2})+af(x)-bf(y)\right\|_X \le \begin{cases}\varepsilon|x|^p|y|^q, & p+q < 2(\beta - \log_2 K);\\ \varepsilon(|x|^p+|y|^q), & p,q < 2(\beta-\log_2 K)\end{cases}$$

for all $x, y \in \mathbb{R}$. If a function $\phi : \mathbb{R}^2 \to \mathbb{R}^+ \cup \{0\}$ satisfies (2.2) *and* (2.3)*, then there exists a unique quadratic mapping $\mathcal{F} : \mathbb{R} \to X$ satisfying the functional equation* (1.1) *and the following inequality*:

$$\left\|f(x)-\mathcal{F}(x)\right\|_{\mathcal{X}} \le \begin{cases}\frac{\varepsilon K^3}{2^\beta a^\beta}\cdot\frac{(\frac{a}{b})^{\frac{q}{2}}|x|^{p+q}}{1-K2^{\frac{p+q}{2}-\beta}}, & p+q < 2(\beta-\log_2 K);\\ \\ \frac{\varepsilon K^3}{2^\beta a^\beta}\cdot\left(\frac{(2+2^{\frac{p}{2}})|x|^p}{1-K2^{\frac{p}{2}-\beta}} + \frac{2(\frac{a}{b})^{\frac{q}{2}}|x|^q}{1-K2^{\frac{q}{2}-\beta}}\right), & p,q < 2(\beta-\log_2 K)\end{cases}$$

for all $x \in \mathbb{R}$.

Theorem 2.3. *Let X and f be same as Theorem 2.1. If a function $\phi : \mathbb{R}^2 \to \mathbb{R}^+ \cup \{0\}$ satisfies the following*:

$$\sum_{j=1}^{\infty}\left(2^\beta K\right)^j\left(\phi\Big(0,\sqrt{\frac{a}{b}}2^{-\frac{j}{2}}x\Big)+\phi\Big(2^{-\frac{j}{2}}x,\sqrt{\frac{a}{b}}2^{-\frac{j}{2}}x\Big)+\phi(2^{-\frac{j}{2}}x,0)+\phi\Big(2^{-\frac{j+1}{2}}x,0\Big)\right) < \infty$$

and

$$\lim_{n\to\infty}2^{\beta n}\phi\Big(2^{-\frac{n}{2}}x,2^{-\frac{n}{2}}y\Big) = 0$$

for all $x, y \in \mathbb{R}$, then there exists a unique quadratic mapping $\mathcal{F} : \mathbb{R} \to X$ satisfying the functional equation (1.1) *and the following inequality*:

(2.13)

$$\begin{aligned}&\|f(x)-\mathcal{F}(x)\|_X\\ &\le \frac{K^2}{2^\beta a^\beta}\sum_{j=1}^{\infty}\left(2^\beta K\right)^j\left(\phi\Big(0,\sqrt{\frac{a}{b}}2^{-\frac{j}{2}}x\Big)+\phi\Big(2^{-\frac{j}{2}}x,\sqrt{\frac{a}{b}}2^{-\frac{j}{2}}x\Big)+\phi(2^{-\frac{j}{2}}x,0)+\phi\Big(2^{-\frac{j+1}{2}}x,0\Big)\right)\end{aligned}$$

for all $x \in \mathbb{R}$.

Proof. If x is replaced with $\frac{x}{\sqrt{2}}$ in (2.10), then the proof follows from the proof of Theorem 2.1. $\square$

Corollary 2.4. *Let X, p, q and $\varepsilon \geq 0$ be as Corollary 2.2. If a function $f : \mathbb{R} \to X$ satisfies the following inequality:*

$$\left\| f(\sqrt{ax^2+by^2})+af(x)-bf(y)\right\|_X \leq \begin{cases} \varepsilon|x|^p|y|^q, & 2(\beta+\log_2 K) < p+q; \\ \varepsilon(|x|^p+|y|^q), & 2(\beta+\log_2 K) < p, q \end{cases}$$

for all $x, y \in \mathbb{R}$, then there exists a unique quadratic mapping $\mathcal{F} : \mathbb{R} \to X$ satisfying the functional equation (1.1) *and the following inequality:*

$$\left\| f(x)-\mathcal{F}(x)\right\|_X \leq \begin{cases} \frac{\varepsilon K^3}{2^\beta a^\beta} \cdot \frac{(\frac{a}{b})^{\frac{q}{2}}|x|^{p+q}}{2^{\frac{p+q}{2}-\beta}-K}, & 2(\beta+\log_2 K)<p+q; \\ \\ \frac{\varepsilon K^3}{2^\beta a^\beta} \cdot \left(\frac{(2+2^{\frac{-p}{2}})|x|^p}{2^{\frac{p}{2}-\beta}-K} + \frac{2(\frac{a}{b})^{\frac{q}{2}}|x|^q}{2^{\frac{q}{2}-\beta}-K}\right), & 2(\beta+\log_2 K)<p, q \end{cases}$$

for all $x \in \mathbb{R}$.

Now, we investigate the generalized Hyers-Ulam stability of radical functional equations (1.1) in (β, p)-Banach spaces using contractively subadditive and expansively superadditive.

Theorem 2.5. *Let X be a (β, p)-Banach space and $f : \mathbb{R} \to X$ be a ϕ-approximatively radical quadratic function with $f(0) = 0$. Assume that the function ϕ is contractively subadditive with a constant L satisfying $2^{1-2\beta}L < 1$. Then there exists a unique quadratic mapping $\mathcal{F} : \mathbb{R} \to X$ satisfying the functional equation* (1.1) *and the following inequality:*

$$\left\| f(x) - \mathcal{F}(x)\right\|_X \leq \frac{\widehat{\widehat{\Phi}}(x)}{\sqrt[p]{(4a)^{\beta p} - (2a^\beta L)^p}} \tag{2.14}$$

for all $x \in \mathbb{R}$, where

$$\widehat{\Phi}(x) = \phi(x, 0) + \phi(\sqrt{2}x, 0) + \phi\Big(0, \sqrt{\frac{a}{b}}x\Big) + \phi\Big(x, \sqrt{\frac{a}{b}}x\Big)$$

and

$$\widehat{\widehat{\Phi}}(x) = K^3(2^\beta \widehat{\Phi}(x) + \widehat{\Phi}(\sqrt{2}x)).$$

Proof. It follows from (2.10) in the proof of Theorem 2.1 that

$$\begin{aligned} &\left\| 2f(x) - f(\sqrt{2}x)\right\|_X \\ &\leq \frac{K^2}{a^\beta}\left(\phi\Big(x, 0\Big) + \phi\Big(\sqrt{2}x, 0\Big) + \phi\Big(0, \sqrt{\frac{a}{b}}x\Big) + \phi\Big(x, \sqrt{\frac{a}{b}}x\Big)\right). \end{aligned} \tag{2.15}$$

Let $\widehat{\Phi}(x) = \phi(x, 0) + \phi(\sqrt{2}x, 0) + \phi(0, \sqrt{\frac{a}{b}}x) + \phi(x, \sqrt{\frac{a}{b}}x)$. Then we obtain

$$\left\| f(x) - \frac{1}{4}f(2x)\right\|_X \leq \frac{1}{(4a)^\beta}\widehat{\widehat{\Phi}}(x), \tag{2.16}$$

where $\widehat{\widehat{\Phi}}(x) = K^3(2^\beta\widehat{\Phi}(x) + \widehat{\Phi}(\sqrt{2}x))$. It follows from (2.16) with 2^jx in the place of x and the iterative method that

$$\begin{aligned}
\left\|\frac{1}{4^k}f(2^kx) - \frac{1}{4^m}f(2^mx)\right\|_X^p &\le \sum_{j=k}^{m-1}\frac{1}{4^{\beta pj}}\left\|f(2^jx) - \frac{1}{4}f(2^{j+1}x)\right\|_X^p \\
&\le \frac{1}{(4a)^{\beta p}}\sum_{j=k}^{m-1}\frac{1}{4^{j\beta p}}\widehat{\widehat{\Phi}}(2^jx)^p \qquad (2.17)\\
&\le \left(\frac{\widehat{\widehat{\Phi}}(x)}{4^\beta a^\beta}\right)^p\sum_{j=k}^{m-1}\left(2^{1-2\beta}L\right)^{jp}
\end{aligned}$$

for all $x \in \mathbb{R}$ and $m, k \in \mathbb{Z}^+$ with $m > k \ge 0$. Then the sequence $\left\{\frac{1}{4^n}f(2^nx)\right\}$ is a Cauchy sequence in a (β, p)-Banach space X and so we can define a mapping $\mathcal{F} : \mathbb{R} \to X$ by

$$\mathcal{F}(x) := \lim_{n\to\infty}\frac{1}{4^n}f(2^nx)$$

for all $x \in \mathbb{R}$. Then we get

$$\left\|\mathcal{F}(\sqrt{ax^2+by^2}) - a\mathcal{F}(x) - b\mathcal{F}(y)\right\|_X^p \le \phi(x,y)^p\lim_{n\to\infty}(2^{1-2\beta}L)^{np} = 0$$

for all $x, y \in \mathbb{R}$. Then $\mathcal{F}(\sqrt{ax^2+by^2}) - a\mathcal{F}(x) - b\mathcal{F}(y) = 0$, that is, $\mathcal{F}$ is a quadratic mapping. Taking $m \to \infty$ in (2.17) with $k = 0$, we can show that $\mathcal{F}$ satisfies (2.14) near the approximate function f of the functional equation (1.1).

Next, we assume that there exists anther quadratic mapping $\mathcal{G} : \mathbb{R} \to X$ which satisfies the functional equation (1.1) and (2.14). Then we have

$$\left\|\mathcal{G}(x) - \frac{1}{4^n}f(2^nx)\right\|_X^p \le \frac{\widehat{\widehat{\Phi}}(x)^p}{(4a)^{\beta p} - (2a^\beta L)^p}\left(2^{1-2\beta}L\right)^{np}$$

for all $x \in \mathbb{R}$ and $n \in \mathbb{Z}^+$. Letting $n \to \infty$, the uniqueness of $\mathcal{F}$ follows. This completes the proof. □

Theorem 2.6. *Let $X, f, \widehat{\Phi}(x)$ be same as in Theorem 2.5. Assume that the function ϕ is expansively superadditive with a constant L satisfying $2^{2\beta-1}L < 1$. Then there exists a unique quadratic mapping $\mathcal{F} : \mathbb{R} \to X$ satisfying the functional equation* (1.1) *and the following inequality*:

$$\left\|f(x) - \mathcal{F}(x)\right\|_X \le \frac{\widehat{\widehat{\Phi}}_2(x)}{\sqrt[p]{(2a^\beta L^{-1})^p - (4a)^{\beta p}}} \qquad (2.18)$$

for all $x \in \mathbb{R}$, where $\widehat{\widehat{\Phi}}_2(x) = K^3\left(\widehat{\Phi}(2^{-\frac{1}{2}}x) + 2^\beta\widehat{\Phi}(2^{-1}x)\right)$.

Proof. It follows from (2.15) of the proof of Theorem 2.5 that

$$\|f(x)-4f(2^{-1}x)\|_X \le \frac{K^3}{a^\beta}\Big(\widehat{\Phi}(2^{-\frac{1}{2}}x)+2^\beta\widehat{\Phi}(2^{-1}x)\Big)=\frac{\widehat{\widehat{\Phi}}_2(x)}{a^3} \tag{2.19}$$

for all $x\in\mathbb{R}$. Then, in (2.19), replacing x by $2^{-j}x$ and using the iterative method, we have

$$\|4^kf(2^{-k}x)-4^mf(2^{-m}x)\|_X^p \le \Big(\frac{\widehat{\widehat{\Phi}}_2(x)}{a^\beta}\Big)^p\sum_{j=k}^{m-1}\big(2^{2\beta-1}L\big)^{jp} \tag{2.20}$$

for all $x\in\mathbb{R}$ and $k,m\in\mathbb{Z}^+$ with $m>k\ge 0$. The remains follow the proof of Theorem 2.5. This completes the proof. □

3. Stability of the radical quartic functional equation (1.2)

In this section, we are modified the generalized Hyers-Ulam stability of radical functional equations (1.2) in quasi-β-normed spaces and (β,p)-Banach spaces, respectively.

Let X be a normed space and $\psi:\mathbb{R}^2\to\mathbb{R}^+\cup\{0\}$ be a function. A function $f:\mathbb{R}\to X$ is called a *ψ-approximatively radical quartic function* if

$$\Big\|f(\sqrt{ax^2+by^2})+f(\sqrt{|ax^2-by^2|})-2a^2f(x)-2b^2f(y)\Big\|_X\le\psi(x,y) \tag{3.1}$$

for all $x,y\in\mathbb{R}$, where $a,b\in\mathbb{R}^+$ are fixed with $a^2+b^2\neq 1$.

First, we prove the generalized Hyers-Ulam stability of the radical functional equations (1.2) in quasi-β-normed spaces using the idea of Găvruta.

Theorem 3.1. *Let X be a quasi-β-Banach space and $f:\mathbb{R}\to X$ be a ψ-approximatively radical quartic function with $f(0)=0$. If a mapping $\psi:\mathbb{R}^2\to\mathbb{R}^+\cup\{0\}$ satisfy the following:*

(3.2)
$$\sum_{j=0}^{\infty}\Big(\frac{K}{4^\beta}\Big)^j\Big(\psi\Big(0,\sqrt{\frac{a}{b}}2^{\frac{j}{2}}x\Big)+\psi\Big(2^{\frac{j}{2}}x,\sqrt{\frac{a}{b}}2^{\frac{j}{2}}x\Big)+\psi\Big(2^{\frac{j}{2}}x,0\Big)+\frac{1}{2^\beta}\psi\Big(2^{\frac{j+1}{2}}x,0\Big)\Big)<\infty,$$

and

$$\lim_{n\to\infty}\frac{1}{4^{\beta n}}\psi\Big(2^{\frac{n}{2}}x,2^{\frac{n}{2}}y\Big)=0 \tag{3.3}$$

for all $x,y\in\mathbb{R}$, then there exists a unique quartic mapping $\mathcal{H}:\mathbb{R}\to X$ satisfying the functional equation (1.2) and the following inequality:

(3.4)
$$\begin{aligned}&\Big\|f(x)-\mathcal{H}(x)\Big\|_X\\ \le&\frac{K^3}{(4a^2)^\beta}\sum_{j=0}^{\infty}\Big(\frac{K}{4^\beta}\Big)^j\Big(\psi\Big(0,\sqrt{\frac{a}{b}}2^{\frac{j}{2}}x\Big)+\psi\Big(2^{\frac{j}{2}}x,\sqrt{\frac{a}{b}}2^{\frac{j}{2}}x\Big)+\psi\Big(2^{\frac{j}{2}}x,0\Big)+\frac{1}{2^\beta}\psi\Big(2^{\frac{j+1}{2}}x,0\Big)\Big)\end{aligned}$$

for all $x\in\mathbb{R}$.

Proof. Replacing x and y with $\frac{x}{\sqrt{a}}$ and $\frac{y}{\sqrt{b}}$ in (3.1), respectively, we get

$$\left\|f(\sqrt{x^2+y^2})+f(\sqrt{|x^2-y^2|})-2a^2f\Big(\frac{x}{\sqrt{a}}\Big)-2b^2f\Big(\frac{y}{\sqrt{b}}\Big)\right\|_X\le\psi\Big(\frac{x}{\sqrt{a}},\frac{y}{\sqrt{b}}\Big)\tag{3.5}$$

for all $x,y\in\mathbb{R}$. Setting $x=y=\sqrt{a}x$ in (3.5), we get

$$\left\|f(\sqrt{2ax^2})-2a^2f(x)-2b^2f\Big(\sqrt{\frac{a}{b}}x\Big)\right\|_X\le\psi\Big(x,\sqrt{\frac{a}{b}}x\Big)\tag{3.6}$$

for all $x\in\mathbb{R}$. Replacing x and y with $\sqrt{2a}x$ and 0 in (3.5), respectively, we obtain

$$\left\|f(\sqrt{2ax^2})-a^2f(\sqrt{2}x)\right\|_X\le\frac{1}{2^\beta}\psi\Big(\sqrt{2}x,0\Big)\tag{3.7}$$

for all $x\in\mathbb{R}$. It follows from (3.6) and (3.7) that

$$\left\|a^2f(\sqrt{2}x)-2a^2f(x)-2b^2f\Big(\sqrt{\frac{a}{b}}x\Big)\right\|_X\le K\left(\psi\Big(x,\sqrt{\frac{a}{b}}x\Big)+\frac{1}{2^\beta}\psi\Big(\sqrt{2}x,0\Big)\right)\tag{3.8}$$

for all $x\in\mathbb{R}$. Substituting $x=\sqrt{a}x$ and $y=0$ in (3.5), we get

$$\left\|2f(\sqrt{ax^2})-2a^2f(x)\right\|_X\le\psi\,(x,0)\tag{3.9}$$

for all $x\in\mathbb{R}$. Also, substituting $x=0$ and $y=\sqrt{a}x$ in (3.5), we get

$$\left\|2f(\sqrt{ax^2})-2b^2f(\sqrt{\frac{a}{b}}x)\right\|_X\le\psi\Big(0,\sqrt{\frac{a}{b}}x\Big)\tag{3.10}$$

for all $x\in\mathbb{R}$. It follows from (3.9) and (3.10) that

$$\left\|2b^2f(\sqrt{\frac{a}{b}}x)-2a^2f(x)\right\|_X\le K\left(\psi\Big(x,0\Big)+\psi\Big(0,\sqrt{\frac{a}{b}}x\Big)\right)\tag{3.11}$$

for all $x\in\mathbb{R}$. It follows from (3.8) and (3.11) that

$$\begin{aligned}&\left\|f(x)-\frac{1}{4}f(2^{\frac{1}{2}}x)\right\|_X\\&\le\frac{K^2}{(4a^2)^\beta}\left(\psi\Big(0,\sqrt{\frac{a}{b}}x\Big)+\psi\Big(x,\sqrt{\frac{a}{b}}x\Big)+\psi\Big(x,0\Big)+\frac{1}{2^\beta}\psi\Big(2^{\frac{1}{2}}x,0\Big)\right)\end{aligned}\tag{3.12}$$

for all $x\in\mathbb{R}$. Let $\Psi(x)=\psi(0,\sqrt{\frac{a}{b}}x)+\psi(x,\sqrt{\frac{a}{b}}x)+\psi(x,0)+\frac{1}{2^\beta}\psi(2^{\frac{1}{2}}x,0)$. Then, for all $m,k\in Z^+$ with $m>k\ge0$, we get

$$\left\|\frac{1}{4^k}f(2^{\frac{k}{2}}x)-\frac{1}{4^m}f(2^{\frac{m}{2}}x)\right\|_X\le\frac{K^3}{(4a^2)^\beta}\sum_{j=k}^{m-1}\Big(\frac{K}{2^\beta}\Big)^j\Psi(2^{\frac{j}{2}}x)\tag{3.13}$$

for all $x\in\mathbb{R}$. From (3.2) and (3.13), the sequence $\left\{\frac{1}{4^n}f(2^{\frac{n}{2}}x)\right\}$ is a Cauchy sequence for all $x\in\mathbb{R}$. Since X is the (β,p)-Banach space X, it converges and so we can define a mapping $\mathcal{H} : \mathbb{R} \to X$ by

$$\mathcal{H}(x) := \lim_{n\to\infty} \frac{1}{4^n} f(2^{\frac{n}{2}} x)$$

for all $x \in \mathbb{R}$. The remains are similar to that of Theorem 2.1. This completes the proof. □

Theorem 3.2. *Let $f : \mathbb{R} \to X$ be a ψ-approximatively radical quadratic function. If a mapping $\psi : \mathbb{R}^2 \to \mathbb{R}^+ \cup \{0\}$ satisfies the following*:

$$\sum_{j=1}^{\infty} (4^\beta K)^j \left(\psi\Big(0, \sqrt{\frac{a}{b}} 2^{-\frac{j}{2}} x\Big) + \psi\Big(2^{-\frac{j}{2}} x, \sqrt{\frac{a}{b}} 2^{-\frac{j}{2}} x\Big) + \psi\Big(2^{-\frac{j}{2}} x, 0\Big) + \frac{1}{2^\beta} \psi\Big(2^{-\frac{j+1}{2}} x, 0\Big)\right) < \infty$$

and

$$\lim_{n\to\infty} 2^n \psi\Big(2^{-\frac{n}{2}} x, 2^{-\frac{n}{2}} y\Big) = 0$$

for all $x, y \in \mathbb{R}$, then there exists a unique quadratic mapping $\mathcal{H} : \mathbb{R} \to X$ satisfying the functional equation (1.2) *and the following inequality*:

(3.14)

$$\begin{aligned} &\Big\| f(x) - \mathcal{H}(x) \Big\|_X \\ &\leq \frac{K^2}{(4a^2)^\beta} \sum_{j=1}^{\infty} (4^\beta K)^j \left(\psi\Big(0, \sqrt{\frac{a}{b}} 2^{-\frac{j}{2}} x\Big) + \psi\Big(2^{-\frac{j}{2}} x, \sqrt{\frac{a}{b}} 2^{-\frac{j}{2}} x\Big) + \psi\Big(2^{-\frac{j}{2}} x, 0\Big) + \frac{1}{2^\beta} \psi\Big(2^{-\frac{j+1}{2}} x, 0\Big)\right) \end{aligned}$$

for all $x \in \mathbb{R}$.

Proof. If x is replaced with $\frac{x}{\sqrt{2}}$ in the inequality (3.12), then the proof follows from that of Theorem 3.1. □

Corollary 3.3. *For any $p, q \in \mathbb{R}^+ \cup \{0\}$ and $\varepsilon \geq 0$, if a function $f : \mathbb{R} \to X$ satisfies the following inequality*:

$$\Big\| f(\sqrt{ax^2 + by^2}) + f(\sqrt{|ax^2 - by^2|}) - 2a^2 f(x) - 2b^2 f(y) \Big\|_X \leq \begin{cases} \varepsilon |x|^p |y|^q; \\ \varepsilon(|x|^p + |y|^q) \end{cases}$$

for all $x, y \in \mathbb{R}$, then there exists a unique quartic mapping $\mathcal{H} : \mathbb{R} \to X$ satisfying the functional equation (1.2) *and the following inequality*:

$$\|f(x) - \mathcal{H}(x)\|_X \leq \begin{cases} \frac{\varepsilon K^3 \sqrt{(\frac{a}{b})^q} |x|^{p+q}}{a^{2\beta}(4^\beta - K\sqrt{2^{p+q}})}, & p + q < 4^\beta - 2\log_2 K; \\ \frac{\varepsilon K^3}{a^{2\beta}} \left(\frac{(2+\sqrt{2^p})|x|^r}{4^\beta - K\sqrt{2^p}} + \frac{2\sqrt{(\frac{a}{b})^q}|x|^q}{4^\beta - K\sqrt{2^q}} \right), & p, q < 4^\beta - 2\log_2 K \end{cases}$$

for all $x \in \mathbb{R}$.

Now, we prove the generalized Hyers-Ulam stability of the radical functional equations (1.2) in (β, p)-Banach spaces using contractively subquadratic and expansively superquadratic.

Theorem 3.4. *Let X be a (β,p)-Banach space and $f:\mathbb{R}\to X$ be a ψ-approximatively radical quadratic function with $f(0)=0$. Assume that the function ψ is contractively subquadratic with a constant L satisfying $2^{2-4\beta}L<1$. Then there exists a unique quartic mapping $\mathcal{H}:\mathbb{R}\to X$ satisfying the functional equation* (1.2) *and the following inequality:*

$$\Big\|f(x)-\mathcal{H}(x)\Big\|_X \le \frac{\widehat{\widehat{\Psi}}(x)}{\sqrt[p]{(16a^2)^{\beta p}-(4a^{2\beta}L)^p}}, \tag{3.15}$$

where

$$\widehat{\Psi}(x)=\psi(x,0)+\psi\Big(0,\sqrt{\frac{a}{b}}x\Big)+\psi\Big(x,\sqrt{\frac{a}{b}}x\Big)+\frac{1}{2^\beta}\psi(\sqrt{2}x,0)$$

and

$$\widehat{\widehat{\Psi}}(x)=K^3\left(4^\beta\widehat{\Psi}(x)+\widehat{\Psi}(\sqrt{2}x)\right)$$

for all $x\in\mathbb{R}$.

Proof. Using (3.12) in the proof of Theorem 3.1, we have

$$\Big\|f(x)-\frac{1}{16}f(2x)\Big\|_X \le \frac{K^3(4^\beta\widehat{\Psi}(x)+\widehat{\Psi}(\sqrt{2}x))}{(4a)^{2\beta}}=\frac{\widehat{\widehat{\Psi}}(x)}{(4a)^{2\beta}} \tag{3.16}$$

for all $x\in\mathbb{R}$. Then, in (3.16), replacing x by $2^{-j}x$ and using the iterative method, we have

$$\begin{aligned}\Big\|\frac{1}{16^k}f(2^kx)-\frac{1}{16^m}f(2^mx)\Big\|_X^p &\le \sum_{j=k}^{m-1}\Big\|\frac{1}{16^j}f(2^jx)-\frac{1}{16^{j+1}}f(2^{j+1}x)\Big\|_X^p\\ &\le \Big(\frac{1}{4a}\Big)^{2\beta p}\sum_{j=k}^{m-1}\frac{1}{16^{\beta pj}}\,\widehat{\widehat{\Psi}}(2^jx)^p\\ &\le \Big(\frac{\widehat{\widehat{\Psi}}(x)}{4a}\Big)^{2\beta p}\sum_{j=k}^{m-1}\left(2^{2-4\beta}L\right)^{jp}\end{aligned} \tag{3.17}$$

for all $x\in\mathbb{R}$ and $m,k\in\mathbb{Z}^+$ with $m>k\ge 0$. The sequence $\left\{\frac{1}{16^n}f(2^nx)\right\}$ is a Cauchy sequence for all $x\in\mathbb{R}$. Since X is a (β,p)-Banach space, it converges for all $x\in\mathbb{R}$. Then we can define a mapping $\mathcal{H}:\mathbb{R}\to X$ by

$$\mathcal{H}(x):=\lim_{n\to\infty}\frac{1}{16^n}f(2^nx)$$

for all $x\in\mathbb{R}$. The remains are similar to the proof of Theorem 2.5. This completes the proof. ∎

Theorem 3.5. *Let $X,f,\widehat{\Psi}$ be same as in Theorem 3.4. Assume that the function ψ is expansively superquadratic with a constant L satisfying $2^{4\beta-2}L<1$.*

Then there exists a unique quartic mapping $\mathcal{H} : \mathbb{R} \to X$ satisfying the functional equation (1.2) *and the following inequality*:

$$\|f(x) - \mathcal{H}(x)\|_X \leq \frac{\widehat{\widehat{\Psi}}_2(x)}{\sqrt[p]{(4a^{2\beta}L^{-1})^p - (16a^2)^{\beta p}}} \tag{3.18}$$

for all $x \in \mathbb{R}$, where $\widehat{\widehat{\Psi}}_2(x) = K^3 \left(\widehat{\Phi}(2^{-\frac{1}{2}}x) + 4^\beta \widehat{\Phi}(2^{-1}x)\right)$.

Proof. It follows from (3.12) in the proof of Theorem 3.1 that

$$\left\|f(x) - 16f(2^{-1}x)\right\|_X \leq \frac{1}{a^{2\beta}} \widehat{\widehat{\Psi}}_2(2^{-1}x)$$

for all $x \in \mathbb{R}$ and so

$$\left\|16^k f(2^{-k}x) - 16^m f(2^{-m}x)\right\|_X^p \leq \Big(\frac{\widehat{\widehat{\Psi}}_2(x)}{a^{2\beta}}\Big)^p \sum_{j=k}^{m-1} \left(2^{4\beta-2}L\right)^{jp} \tag{3.19}$$

for all $x \in \mathbb{R}$ and $k, m \in \mathbb{Z}^+$ with $m > k \geq 0$. The remains follow the proof of Theorem 3.1. This completes the proof. □

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