RANDOM ATTRACTOR FOR STOCHASTIC PARTIAL FUNCTIONAL DIFFERENTIAL EQUATIONS WITH INFINITE DELAY

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ABSTRACT. In this paper we are concerned with a class of stochastic partial functional differential equations with infinite delay. Supposing that the linear part is a Hille-Yosida operator but not necessarily densely defined and employing the integrated semigroup and random dynamics theory, we present some appropriate conditions to guarantee the existence of a random attractor.

1. Introduction

The purpose of this paper is to investigate the asymptotic behavior of solutions to the following stochastic partial functional differential equation

$$(1.1) \qquad dx(t) = Ax(t)dt + f(x_t)dt + \sigma dW(t),$$

where A is a linear operator defined on E (a separable Banach space with the norm $\|\cdot\|$), f is a nonlinear operator satisfying the global Lipschitz condition, $\sigma \in D(A)$ and $W(t)$ is a real-valued two-sided Winer process.

In order to capture the essential dynamics of such stochastic systems, the concept of random attractor was introduced in [7], then generalized in [6], as an extension to stochastic systems of the theory of attractors for deterministic system [13]. Many works are devoted to the existence of random attractors of, for example, stochastic PDEs on unbounded domain [3, 21, 22, 23] with applications to such as stochastic reaction-diffusion equation and Navier-Stocks equation, quasilinear stochastic PDEs [11, 12] with applications to stochastic porous media equation, quasi-continuous random dynamical system [15] and stochastic damped sine-Gordon equation [19]. We also mention some studies on the determined case, see for example [5, 16].

Notice that, in the works mentioned above, the linear part is a densely defined operator. Little is known for the non-densely defined case. In fact, operators with non-dense domain occur in many situations due to restrictions on the space where the equations are considered. For example, periodic continuous functions and Hölder continuous functions are not dense in the space of continuous functions, see more examples in [18]. Besides, the boundary conditions also give rise to operators with non-dense domains, e.g., the age-structured problem we give in the last section of the present paper.

Motivated by the previous works on the random attractor of the explicit partial differential equations, in this paper, we consider the existence of random attractors for a more general form as that in Eq. (1.1), where the linear operator $A : D(A) \subset E \to E$ is not necessarily densely defined but satisfies the following Hille-Yosida condition:

(H1) there exist two constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that $(\omega, +\infty) \subset \rho(A)$ and
$$\|(\lambda I - A)^{-n}\|_{\mathcal{L}} \leq \frac{M}{(\lambda - \omega)^n}, \quad \lambda > \omega,$$

where $\rho(A)$ is the resolvent set of A and $\| \cdot \|_{\mathcal{L}}$ denotes the operator norm.

Here we should mention paper [9], in which the authors studied a stochastic retarded reaction-diffusion equation on all d-dimensional space with additive white noise. With the help of strongly continuous semigroup theory, they considered the mild solution of the equation. Then utilizing a cut-off method, they obtained a uniform estimation on solutions. Thus the pullback asymptotic compactness was proved, and consequently the existence of a unique random attractor was got. Similar method was also used in [8] to the lattice dynamical system.

Return to Eq. (1.1), since the linear part A is not densely defined, we could not use the theory of strongly continuous semigroup directly. Fortunately, it is known that the non-densely defined Hille-Yosida operator generates an integrated semigroup, which was introduced in [1] and more properties about which will be enumerated later.

Now we consider the equation in the fading memory space C_γ, which is a separable Banach space defined by

$$(1.2) \quad C_\gamma = \left\{ \phi \,|\, \phi : (-\infty, 0] \to E \text{ is continuous and } \lim_{s \to -\infty} e^{\gamma s} \phi(s) \text{ exists} \right\},$$

with norm $\|\phi\|_\gamma = \sup_{-\infty < s \leq 0} e^{\gamma s} \|\phi(s)\|$, $\gamma > 0$. The aim of this paper is to provide some sufficient conditions for the existence of a random attractor of Eq. (1.1). For convenience, we suppose that the nonlinear function $f : C_\gamma \to E$ satisfies the global Lipschitz condition:

(H2) there exists a constant $L > 0$ such that
$$\|f(\phi_1) - f(\phi_2)\| \leq L\|\phi_1 - \phi_2\|_\gamma \quad \text{for any} \ \ \phi_1, \phi_2 \in C_\gamma.$$

The rest of the paper is organized as follows. In Section 2, we present some basic concepts and properties of the integrated semigroup theory and random dynamical systems. In Section 3, we convert Eq. (1.1) to a deterministic equation with a random parameter. In Section 4, we prove the existence of a random attractor. In the last section, as an application of our theory, we use the age-structured problem with white noise to illustrate our result.

2. Preliminary results

In this section, we recall some basic concept related to the integrated semigroup and random dynamical systems, see [1, 2, 14, 20] for more details.

Consider an abstract evolution equation on a general Banach space E

$$(2.1) \qquad \frac{dx(t)}{dt} = Ax(t) + f(x_t), \quad t > 0$$

with initial function $x_0 = \xi$, where A is a Hille-Yosida operator, that is, A satisfies (H1).

Definition 2.1 ([1])**.** Let $T > 0$. A continuous function $x : (-\infty, T] \to E$ is called an integral solution of equation (2.1) if

(i) $\int_0^t x(s)ds \in D(A)$ for $t \in [0, T]$;
(ii) $x(t) = \xi(0) + A(\int_0^t x(s)ds) + \int_0^t f(x_s)ds$;
(iii) $x_0 = \xi$.

Remark 2.2. From (i) we know that if x is an integral solution of (2.1), then $x_t(0) = x(t) \in \overline{D(A)}$ for $t \in [0, T]$. In particular, $\xi(0) \in \overline{D(A)}$, which is a necessary condition for the existence of an integral solution.

Definition 2.3 ([1])**.** An integrated semigroup is a family $S(t)$, $t \geq 0$, of bounded linear operators on E with the following properties:

(i) $S(0) = 0$;
(ii) $t \mapsto S(t)$ is strongly continuous;
(iii) $S(s)S(t) = \int_0^s (S(t + r) - S(r)) \, dr$ for all $t, s \geq 0$.

Lemma 2.4 ([14])**.** *The following assertions are equivalent*:
(i) *A is the generator of a locally Lipschitz continuous integrated semigroup*;
(ii) *A is a Hille-Yosida operator.*

Now we introduce the part A_0 of A in $\overline{D(A)}$ as follows:

$$A_0 = A \quad \text{on} \quad D(A_0) = \{x \in D(A); \; Ax \in \overline{D(A)} \}.$$

Proposition 2.5 ([20])**.** *The part A_0 of A in $\overline{D(A)}$ generates a strongly continuous semigroup on $\overline{D(A)}$.*

Now we turn to random dynamical systems. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, where $\mathcal{F}$ is the Borel σ-algebra on Ω and $\mathbb{P}$ is the corresponding Wiener measure on $\mathcal{F}$. $(X, \| \cdot \|_X)$ is a separable Banach space with Borel σ-algebra $\mathcal{B}(X)$.

Definition 2.6 ([2]). $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$ is called a metric dynamical system, if $\theta : \mathbb{R} \times \Omega \to \Omega$ is $(\mathcal{B}(\mathbb{R}) \times \mathcal{F}, \mathcal{F})$ measurable, $\theta_0 = id$, $\theta_{t+s} = \theta_t \circ \theta_s$ for all t, $s \in \mathbb{R}$, and $\theta_t \mathbb{P} = \mathbb{P}$ for all $t \in \mathbb{R}$.

Definition 2.7 ([2]). A continuous random dynamical system over $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$ is a $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))$-measurable mapping

$$\phi : \mathbb{R}^+ \times \Omega \times X \to X, \quad (t, \omega, x) \mapsto \phi(t, \omega, x),$$

such that the following properties hold:

(1) $\phi(0, \omega, x) = x$ for all $\omega \in \Omega$ and $x \in X$;

(2) $\phi(t + s, \omega, \cdot) = \phi(t, \theta_s \omega, \phi(s, \omega, \cdot))$ for all $t, s \geq 0$ and $\omega \in \Omega$;

(3) ϕ is continuous in t and x.

Definition 2.8 ([19]). (1) A set-valued mapping $\omega \mapsto D(\omega) : \Omega \to 2^X$ is said to be a random set if the mapping $\omega \mapsto d(x, D(\omega))$ is measurable for any $x \in X$. If $D(\omega)$ is also closed (compact) for each $\omega \in \Omega$, the mapping $\omega \mapsto D(\omega)$ is called a random closed (compact) set. A random set $\omega \mapsto D(\omega)$ is said to be bounded if there exist $x_0 \in X$ and a random variable $R(\omega) > 0$ such that

$$D(\omega) \subset \{x \in X : \|x - x_0\|_X \leq R(\omega)\} \quad \text{for all } \omega \in \Omega.$$

(2) A random set $\omega \mapsto D(\omega)$ is called tempered provided for $\mathbb{P}$-a.s. $\omega \in \Omega$,

$$\lim_{t \to +\infty} e^{-\beta t} \sup\{\|b\|_X : b \in D(\theta_{-t}\omega)\} = 0 \quad \text{for all } \beta > 0.$$

(3) A random set $\omega \mapsto B(\omega)$ is said to be a random absorbing set if for any tempered random set $\omega \mapsto D(\omega)$, there exists $t_0(\omega)$ such that

$$\phi(t, \theta_{-t}\omega, D(\theta_{-t}\omega)) \subset B(\omega) \quad \text{for all } t \geq t_0, \omega \in \Omega.$$

(4) A random set $\omega \mapsto B_1(\omega)$ is said to be a random attracting set if for any tempered random set $\omega \mapsto D(\omega)$, we have

$$\lim_{t \to \infty} d_H(\phi(t, \theta_{-t}\omega, D(\theta_{-t}\omega)), B_1(\omega)) = 0, \quad \text{for all } \omega \in \Omega.$$

(5) A random compact set $\omega \mapsto A(\omega)$ is said to be a random attractor if it is a random attracting set and $\phi(t, \omega, A(\omega)) = A(\theta_t \omega)$ for all $\omega \in \Omega$ and $t \geq 0$.

About the following definition, we remark that one can refer to [4] for its origin.

Definition 2.9 ([3]). ϕ is called pullback asymptotically compact on X if for $\mathbb{P}$-a.e. $\omega \in \Omega$, $\{\phi(t_n, \theta_{-t_n}\omega, x_n)\}_{n=1}^{\infty}$ has a convergent subsequence in X whenever $t_n \to \infty$, and $x_n \in B(\theta_{-t_n}\omega)$ with $\omega \mapsto B(\omega)$ is tempered.

In what follows, we recall the definition of the Kuratowski's measure of non-compactness for a bounded set B of a Banach space E, which is defined as

(2.2) $\kappa(B) = \inf\{d > 0 : B \text{ has a finite cover of diameter } < d\},$

see [13], and plays an important role in proving the pullback asymptotically compact in Section 4.

Theorem 2.10 ([3]). *Let ϕ be a continuous random dynamical system over $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$. Suppose that $\omega \mapsto K(\omega)$ is a closed random absorbing set, and ϕ is pullback asymptotically compact on X. Then ϕ has a unique random attractor $\omega \mapsto A(\omega)$, where*

$$A(\omega) = \bigcap_{\tau \geq 0} \overline{\bigcup_{t \geq \tau} \phi(t, \theta_{-t}\omega, K(\theta_{-t}\omega))}, \quad \omega \in \Omega.$$

3. Problem transformation

In this section, we focus our attention on associating a continuous random dynamical system with Eq. (1.1). To this end, we need to convert the stochastic equation into a deterministic equation with a random parameter. In the sequel, we take

$$\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\}$$

and identify $\omega(t) = W(t)$. Define the time shift by

$$\theta_t \omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad \omega \in \Omega, \ t \in \mathbb{R}.$$

Then $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$ is a metric dynamical system. Now, we consider the one-dimensional Ornstein-Uhlenbeck equation

$$d\tilde{z} + \tilde{z}dt = dW(t).$$

It is obvious that its unique stationary solution can be described by

$$(3.1) \qquad \tilde{z}(\theta_t \omega) = -\int_{-\infty}^{0} e^s \omega(t+s)ds + \omega(t), \quad t \in \mathbb{R}.$$

Note that the random variable $|\tilde{z}(\omega)|$ is tempered and $t \mapsto \log |\tilde{z}(\theta_t \omega)|$ is $\mathbb{P}$-a.e. continuous (see [3] and the generalization [10]). It follows from [2, Proposition 4.3.3] that for any $\epsilon > 0$, there is a tempered random variable $\tilde{r}(\omega) > 0$ such that

$$\frac{1}{\tilde{r}(\omega)} \leq |\tilde{z}(\omega)| \leq \tilde{r}(\omega),$$

where $\tilde{r}(\omega)$ satisfies for $\mathbb{P}$-a.s $\omega \in \Omega$,

$$(3.2) \qquad e^{-\epsilon|t|}\tilde{r}(\omega) \leq \tilde{r}(\theta_t \omega) \leq e^{\epsilon|t|}\tilde{r}(\omega).$$

Put $z(\theta_t \omega) = \sigma \tilde{z}(\theta_t \omega)$. Then it solves

$$dz + zdt = \sigma dW(t).$$

Moreover, the following lemma holds.

Lemma 3.1. *For any $\epsilon > 0$, there is a tempered random variable $r(\omega) > 0$ such that*

$$\|z(\theta_t \omega)\| \leq e^{\epsilon|t|}r(\omega),$$

where $r(\omega) = \|\sigma\|\tilde{r}(\omega)$ satisfies for $\mathbb{P}$-a.s $\omega \in \Omega$,

$$(3.3) \qquad e^{-\epsilon|t|}r(\omega) \leq r(\theta_t \omega) \leq e^{\epsilon|t|}r(\omega).$$

Lemma 3.2. *For any $\epsilon > 0$, there is a tempered random variable $r'(\omega) > 0$ such that*

$$\|Az(\theta_t\omega)\| \leq e^{\epsilon|t|}r'(\omega),$$

where $r'(\omega) = \|A\sigma\|\tilde{r}(\omega)$ satisfies for $\mathbb{P}$-a.s $\omega \in \Omega$,

$$(3.4) \qquad\qquad e^{-\epsilon|t|}r'(\omega) \leq r'(\theta_t\omega) \leq e^{\epsilon|t|}r'(\omega).$$

Let $y(t) = x(t) - z(\theta_t\omega)$. Then $y(t)$ satisfies the following evolution equation with a random variable:

$$(3.5) \qquad\qquad \frac{dy(t)}{dt} = Ay(t) + F(\theta_t\omega, y_t),$$

with initial function

$$y_0(s) = x_0(s) - z(\theta_s\omega), \quad s \leq 0,$$

where $F(\theta_t\omega, y_t) := f(y_t + z(\theta_{t+}.\omega)) + Az(\theta_t\omega) + z(\theta_t\omega)$. Therefore, in order to study the asymptotic behavior of x in C_γ, it suffices to investigate Eq. (3.5) with each initial function $y_0 \in C_\gamma$.

According to the first part in Section 2, if A satisfies (H1), then it generates an integrated semigroup $S(t)$, $t \geq 0$, and its part A_0 generates a C_0-semigroup $T_0(t)$, $t \geq 0$. Moreover, the author in [20] gives the relationship between $S(t)$ and $T_0(t)$:

$$(3.6) \qquad S(t)x = \lim_{\lambda \to +\infty} \int_0^t T_0(s)\lambda(\lambda I - A)^{-1}x\,ds \quad \text{for} \quad x \in E, \ t \geq 0.$$

On the other hand, if we denote $F^\omega(t, \xi) := F(\theta_t\omega, y_t)$, it is easy to see that $F^\omega : \mathbb{R}^+ \times C_\gamma \to C_\gamma$ is continuous in t and globally Lipschitz continuous in ξ for each $\omega \in \Omega$. By the classical theory concerning with the existence and uniqueness of the solutions, we obtain:

Proposition 3.3. *Assume that* (H1) *and* (H2) *are satisfied. For $\mathbb{P}$-a.e. $\omega \in \Omega$ and each $y_0 \in C_\gamma$, if $y_0(0) \in \overline{D(A)}$, Eq.* (3.5) *possesses a unique global integral solution $y(\cdot, \omega, y_0) \in C((-\infty, +\infty), E)$ with $y(0, \omega, y_0) = y_0$, which can be expressed as*
(3.7)

$$y_t(s, \omega, y_0) = \begin{cases} T_0(t+s)y_0(0) \\ \quad + \lim_{\lambda \to \infty} \int_0^{t+s} T_0(t+s-\tau)\lambda(\lambda I - A)^{-1}F(\theta_\tau\omega, y_\tau(\cdot, \omega, y_0))d\tau, \\ \quad -t < s \leq 0, \\ y_0(t+s), \qquad s \leq -t. \end{cases}$$

Here $y_0(0) \in \overline{D(A)}$ is a necessary condition for the existence of integral solutions, see Remark 2.2.

Denote by

$$X = \{\xi \in C_\gamma : \xi(0) \in \overline{D(A)}\},$$

which is also a separable Banach space. Then Eq. (3.5) generates a random dynamical system ϕ over $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$, where

$$(3.8) \qquad \phi(t, \omega, y_0) = y_t(\cdot, \omega, y_0), \quad \forall (t, \omega, y_0) \in \mathbb{R}^+ \times \Omega \times X.$$

Define $\varphi : \mathbb{R} \times \Omega \times X \to X$ by
$$(3.9)$$
$$\varphi(t, \omega, x_0) = x_t(\cdot, \omega, x_0) = y_t(\cdot, \omega, y_0) + z(\theta_{t+}.\omega), \quad \forall (t, \omega, x_0) \in \mathbb{R}^+ \times \Omega \times X.$$

Then φ is a continuous random dynamical system associated with Eq. (1.1) on X.

Note that the two random dynamical systems are equivalent. It is easy to check that φ has a random attractor provided ϕ possesses a random attractor. Then, we only need to consider the random dynamical system ϕ.

4. Existence of random attractors

In this section, we establish the existence of a random attractor by proving the existence of a random absorbing set and the asymptotic compactness for ϕ. To this end, we need the following assumption on the C_0-semigroup $T_0(t)$, $t \geq 0$:

(H3) $\|T_0\|_{\mathcal{L}} \leq e^{-\alpha t}$ for some $\alpha > 0$.

Lemma 4.1. *Suppose that* (H2) *holds. For* $0 \leq \tau \leq t$, *we have*
$$\|F(\theta_{\tau-t}\omega, 0)\| \leq (L+1)e^{\epsilon(t-\tau)}r(\omega) + \|f(0)\|.$$

Proof. Let $\epsilon < \gamma$, where ϵ is the one in (3.4). By the definition of F, we have the following estimation

$$\begin{aligned}
&\|F(\theta_{\tau-t}\omega, 0)\| \\
&= \|f(z(\theta_{\tau-t+}.\omega)) + Az(\theta_{\tau-t}\omega) + z(\theta_{\tau-t}\omega)\| \\
&\leq \|f(z(\theta_{\tau-t+}.\omega)) - f(0)\| + \|f(0)\| + \|Az(\theta_{\tau-t}\omega)\| + \|z(\theta_{\tau-t}\omega)\| \\
&\leq L\|z(\theta_{\tau-t+}.\omega)\|_\gamma + \|f(0)\| + \|Az(\theta_{\tau-t}\omega)\| + \|z(\theta_{\tau-t}\omega)\| \\
&\leq L \sup_{-\infty < s \leq 0} e^{\gamma s}\|z(\theta_{\tau-t+s}\omega)\| + \|f(0)\| + e^{\epsilon|\tau-t|}r'(\omega) + e^{\epsilon|\tau-t|}r(\omega) \\
&\leq (L+1)e^{\epsilon(t-\tau)}r(\omega) + e^{\epsilon(t-\tau)}r'(\omega) + \|f(0)\|. \qquad \square
\end{aligned}$$

Lemma 4.2. *Assume that* (H1)-(H3) *are satisfied. For* $\mathbb{P}$-*a.s* $\omega \in \Omega$, *we have*

$$\|y_t(\cdot, \theta_{-t}\omega, y_0(\theta_{-t}\omega))\|_\gamma$$
$$\leq \left(\|y_0\|_\gamma + \frac{L(L+1)}{(\epsilon-L)(\min\{\gamma,\alpha\}-L)}r(\omega) + \frac{L}{(\epsilon-L)(\min\{\gamma,\alpha\}-L)}r'(\omega) \right.$$
$$\left. - \frac{1}{\min\{\gamma,\alpha\}-L}\|f(0)\| \right) e^{(L-\min\{\gamma,\alpha\})t}$$
$$+ \left(\frac{\epsilon(L+1)}{(\min\{\gamma,\alpha\}-\epsilon)(L-\epsilon)}r(\omega) + \frac{\epsilon}{(\min\{\gamma,\alpha\}-\epsilon)(L-\epsilon)}r'(\omega) \right) e^{(\epsilon-\min\{\gamma,\alpha\})t}$$

$$+ \Big(\frac{\min\{\gamma, \alpha\}(L+1)}{(\min\{\gamma, \alpha\} - \epsilon)(\min\{\gamma, \alpha\} - L)} r(\omega) + \frac{\min\{\gamma, \alpha\}}{(\min\{\gamma, \alpha\} - \epsilon)(\min\{\gamma, \alpha\} - L)} r'(\omega)$$

$$+ \frac{1}{\min\{\gamma, \alpha\} - L} \|f(0)\| \Big),$$

where ϵ is the one in (3.4), $L \neq \epsilon$, $\min\{\gamma, \alpha\} \neq \epsilon$, $\min\{\gamma, \alpha\} \neq L$.

Proof. Suppose $\gamma \geq \alpha$. From (3.7) we get that

$$\|y_t(\cdot, \theta_{-t}\omega, y_0(\theta_{-t}\omega))\|_\gamma$$

$$= \sup_{-\infty < s \leq 0} e^{\gamma s} \|y(t+s, \theta_{-t}\omega, y_0(\theta_{-t}\omega))\|$$

$$\leq \max \Big\{ \sup_{-t < s \leq 0} e^{\gamma s} \Big(\|T_0(t+s) y_0(0)\|$$

$$+ \lim_{\lambda \to +\infty} \int_0^{t+s} \|T_0(t+s-\tau)\lambda(\lambda I - A) F(\theta_{\tau - t}\omega, y_\tau(\cdot, \theta_{-t}\omega, y_0(\theta_{-t}\omega))\| d\tau \Big),$$

$$\sup_{s \leq -t} e^{\gamma s} \|y_0(t+s)\| \Big\}.$$

In what follows, for simplicity, we take $M = 1$ in (H1), i.e.,

$$\|(\lambda I - A)^{-1}\|_{\mathcal{L}} \leq \frac{1}{\lambda - \omega} \quad \text{for any } \lambda > \omega.$$

In fact, this can be done if we employ the renorming lemma in [17, Page 17] to introduce an equivalent norm in E. Therefore,

$$\|y_t(\cdot, \theta_{-t}\omega, y_0(\theta_{-t}\omega))\|_\gamma$$

$$\leq \max \Big\{ \sup_{-t < s \leq 0} e^{\gamma s} e^{-\alpha(t+s)} \|y_0(0)\|$$

$$+ \sup_{-t < s \leq 0} e^{\gamma s} e^{-\alpha(t+s)} \int_0^{t+s} e^{\alpha \tau} (L \|y_\tau(\cdot, \theta_{-t}\omega, y_0(\theta_{-t}\omega))\|_\gamma + \|F(\theta_{\tau - t}\omega, 0)\|) d\tau,$$

$$e^{-\gamma t} \|y_0\|_\gamma \Big\}$$

$$\leq \max \Big\{ e^{-\alpha t} \|y_0(0)\| + L e^{-\alpha t} \int_0^t e^{\alpha \tau} \|y_\tau(\cdot, \theta_{-t}\omega, y_0(\theta_{-t}\omega))\|_\gamma d\tau$$

$$+ e^{-\alpha t} \int_0^t e^{\alpha \tau} \Big((L+1) e^{\epsilon(t-\tau)} r(\omega) + e^{\epsilon(t-\tau)} r'(\omega) + \|f(0)\| \Big) d\tau, e^{-\gamma t} \|y_0\|_\gamma \Big\}$$

$$\leq e^{-\alpha t} \|y_0\|_\gamma + L e^{-\alpha t} \int_0^t e^{\alpha \tau} \|y_\tau(\cdot, \theta_{-t}\omega, y_0(\theta_{-t}\omega))\|_\gamma d\tau$$

$$+ (L+1) e^{-\alpha t} \frac{e^{\alpha t} - e^{\epsilon t}}{\alpha - \epsilon} r(\omega) + e^{-\alpha t} \frac{e^{\alpha t} - e^{\epsilon t}}{\alpha - \epsilon} r'(\omega) + e^{-\alpha t} \frac{e^{\alpha t} - 1}{\alpha} \|f(0)\|.$$

Then, it deduces that

$$e^{\alpha t} \|y_t(\cdot, \theta_{-t}\omega, y_0(\theta_{-t}\omega))\|_\gamma$$

$$\leq \|y_0\|_\gamma + (L+1)\frac{e^{\alpha t}-e^{\epsilon t}}{\alpha-\epsilon}r(\omega) + \frac{e^{\alpha t}-e^{\epsilon t}}{\alpha-\epsilon}r'(\omega) + \frac{e^{\alpha t}-1}{\alpha}\|f(0)\|$$

$$+ L\int_0^t e^{\alpha\tau}\|y_\tau(\cdot,\theta_{-t}\omega,y_0(\theta_{-t}\omega))\|_\gamma d\tau.$$

In view of the generalized Gronwall inequality, we obtain that

$$e^{\alpha t}\|y_t(\cdot,\theta_{-t}\omega,y_0(\theta_{-t}\omega))\|_\gamma$$

$$\leq \|y_0\|_\gamma + (L+1)\frac{e^{\alpha t}-e^{\epsilon t}}{\alpha-\epsilon}r(\omega) + \frac{e^{\alpha t}-e^{\epsilon t}}{\alpha-\epsilon}r'(\omega) + \frac{e^{\alpha t}-1}{\alpha}\|f(0)\|$$

$$+ L\int_0^t \Big(\|y_0\|_\gamma + (L+1)\frac{e^{\alpha s}-e^{\epsilon s}}{\alpha-\epsilon}r(\omega) + \frac{e^{\alpha s}-e^{\epsilon s}}{\alpha-\epsilon}r'(\omega)$$

$$+ \frac{e^{\alpha s}-1}{\alpha}\|f(0)\|\Big)e^{L(t-s)}ds$$

$$\leq \Big(\|y_0\|_\gamma + \frac{L(L+1)}{(\epsilon-L)(\alpha-L)}r(\omega) + \frac{L}{(\epsilon-L)(\alpha-L)}r'(\omega) - \frac{1}{\alpha-L}\|f(0)\|\Big)e^{Lt}$$

$$+ \Big(\frac{\epsilon(L+1)}{(\alpha-\epsilon)(L-\epsilon)}r(\omega) + \frac{\epsilon}{(\alpha-\epsilon)(L-\epsilon)}r'(\omega)\Big)e^{\epsilon t}$$

$$+ \Big(\frac{\alpha(L+1)}{(\alpha-\epsilon)(\alpha-L)}r(\omega) + \frac{\alpha}{(\alpha-\epsilon)(\alpha-L)}r'(\omega) + \frac{1}{\alpha-L}\|f(0)\|\Big)e^{\alpha t},$$

which yields that

$$\|y_t(\cdot,\theta_{-t}\omega,y_0(\theta_{-t}\omega))\|_\gamma$$

$$\leq \Big(\|y_0\|_\gamma + \frac{L(L+1)}{(\epsilon-L)(\alpha-L)}r(\omega) + \frac{L}{(\epsilon-L)(\alpha-L)}r'(\omega) - \frac{1}{\alpha-L}\|f(0)\|\Big)e^{(L-\alpha)t}$$

$$+ \Big(\frac{\epsilon(L+1)}{(\alpha-\epsilon)(L-\epsilon)}r(\omega) + \frac{\epsilon}{(\alpha-\epsilon)(L-\epsilon)}r'(\omega)\Big)e^{(\epsilon-\alpha)t}$$

$$+ \Big(\frac{\alpha(L+1)}{(\alpha-\epsilon)(\alpha-L)}r(\omega) + \frac{\alpha}{(\alpha-\epsilon)(\alpha-L)}r'(\omega) + \frac{1}{\alpha-L}\|f(0)\|\Big).$$

Similarly, for the case that $\gamma < \alpha$, we can get

$$\|y_t(\cdot,\theta_{-t}\omega,y_0(\theta_{-t}\omega))\|_\gamma$$

$$\leq \Big(\|y_0\|_\gamma + \frac{L(L+1)}{(\epsilon-L)(\gamma-L)}r(\omega) + \frac{L}{(\epsilon-L)(\gamma-L)}r'(\omega) - \frac{1}{\gamma-L}\|f(0)\|\Big)e^{(L-\gamma)t}$$

$$+ \Big(\frac{\epsilon(L+1)}{(\gamma-\epsilon)(L-\epsilon)}r(\omega) + \frac{\epsilon}{(\gamma-\epsilon)(L-\epsilon)}r'(\omega)\Big)e^{(\epsilon-\gamma)t}$$

$$+ \Big(\frac{\gamma(L+1)}{(\gamma-\epsilon)(\gamma-L)}r(\omega) + \frac{\gamma}{(\gamma-\epsilon)(\gamma-L)}r'(\omega) + \frac{1}{\gamma-L}\|f(0)\|\Big).$$

Therefore, the conclusion holds. $\square$

Lemma 4.3. *Under the conditions of* (H1)-(H3) *and* $L < \min\{\gamma,\alpha\}$, *there exists a tempered random set* $\omega \mapsto K(\omega)$ *attracting any tempered random set*

$\omega \mapsto B(\omega)$, *that is, for* $\mathbb{P}$-*a.e* $\omega \in \Omega$, *there is* $T_B(\omega) > 0$ *such that*

$$\phi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)) \subset K(\omega), \quad \forall t \geq T_B(\omega).$$

Proof. For $y_0(\theta_{-t}\omega) \in B(\theta_{-t}\omega)$, we have

$$\|\phi(t, \theta_{-t}\omega, y_0(\theta_{-t}\omega))\|_\gamma$$
$$= \|y_t(\cdot, \theta_{-t}\omega, y_0(\theta_{-t}\omega))\|_\gamma$$
$$\leq \left(\|y_0\|_\gamma + \frac{L(L+1)}{(\epsilon - L)(\min\{\gamma, \alpha\} - L)} r(\omega) + \frac{L}{(\epsilon - L)(\min\{\gamma, \alpha\} - L)} r'(\omega) \right.$$
$$\left. - \frac{1}{\min\{\gamma, \alpha\} - L} \|f(0)\| \right) e^{(L - \min\{\gamma, \alpha\})t}$$
$$+ \left(\frac{\epsilon(L+1)}{(\min\{\gamma, \alpha\} - \epsilon)(L - \epsilon)} r(\omega) + \frac{\epsilon}{(\min\{\gamma, \alpha\} - \epsilon)(L - \epsilon)} r'(\omega) \right) e^{(\epsilon - \min\{\gamma, \alpha\})t}$$
$$+ \left(\frac{\min\{\gamma, \alpha\}(L+1)}{(\min\{\gamma, \alpha\} - \epsilon)(\min\{\gamma, \alpha\} - L)} r(\omega) + \frac{\min\{\gamma, \alpha\}}{(\min\{\gamma, \alpha\} - \epsilon)(\min\{\gamma, \alpha\} - L)} r'(\omega) \right.$$
$$\left. + \frac{1}{\min\{\gamma, \alpha\} - L} \|f(0)\| \right).$$

Take $\epsilon < \min\{\gamma, \alpha\}$, then there exists $T_B(\omega) > 0$ such that for all $t \geq T_B(\omega)$,

$$\|\phi(t, \theta_{-t}\omega, y_0(\theta_{-t}\omega))\|_\gamma \leq c_1 r(\omega) + c_2 r'(\omega) + c_3,$$

where

$$c_1 = \frac{\min\{\gamma, \alpha\}(L+1)}{(\min\{\gamma, \alpha\} - \epsilon)(\min\{\gamma, \alpha\} - L)} + 1,$$
$$c_2 = \frac{\min\{\gamma, \alpha\}}{(\min\{\gamma, \alpha\} - \epsilon)(\min\{\gamma, \alpha\} - L)} + 1,$$
$$c_3 = \frac{1}{\min\{\gamma, \alpha\} - L} \|f(0)\| + 1.$$

For any given $\omega \in \Omega$, we denote by

$$K(\omega) = \{\xi \in C_\gamma : \|\xi\|_\gamma \leq c_1 r(\omega) + c_2 r'(\omega) + c_3\}.$$

Then $\omega \mapsto K(\omega)$ is a tempered random set because $r(\omega)$ and $r'(\omega)$ are tempered. Moreover, it is absorbing. $\square$

Lemma 4.4. *Assume that* (H1)-(H3) *are satisfied. For any* $y_{01}, y_{02} \in B(\omega)$, *where* $\omega \mapsto B(\omega)$ *is a tempered random set, we have*
(4.1)
$$\|\phi(t, \omega, y_{01}) - \phi(t, \omega, y_{02})\|_\gamma \leq e^{-(\min\{\gamma, \alpha\} - L)t} \|y_{01} - y_{02}\|_\gamma, \quad \forall t \geq 0, \ \omega \in \Omega.$$

Proof. According to (3.7) and (3.8), we have

$$\|\phi(t, \omega, y_{01}) - \phi(t, \omega, y_{01})\|_\gamma = \|y_t(\cdot, \omega, y_{01}) - y_t(\cdot, \omega, y_{02})\|_\gamma$$
$$\leq \max\left\{ \sup_{-t < s \leq 0} e^{\gamma s} \|T_0(t+s)(y_{01}(0) - y_{02}(0))\| \right.$$

$$+ \sup_{-t < s \leq 0} e^{\gamma s} \lim_{\lambda \to +\infty} \int_0^{t+s} \|T_0(t+s-\tau)\lambda(\lambda I - A)^{-1}(F(\theta_\tau \omega, y_\tau(\cdot, \omega, y_{01}))$$

$$- F(\theta_\tau \omega, y_\tau(\cdot, \omega, y_{02})))\|d\tau, \sup_{s \leq -t} e^{\gamma s} \|y_{01}(t+s) - y_{02}(t+s)\|\Big\}$$

$$\leq e^{-\min\{\gamma,\alpha\}t} \|y_{01} - y_{02}\|_\gamma$$

$$+ L e^{-\min\{\gamma,\alpha\}t} \int_0^t e^{\min\{\gamma,\alpha\}\tau} \|y_\tau(\cdot, \omega, y_{01}) - y_\tau(\cdot, \omega, y_{02})\|_\gamma.$$

Then it is easy to obtain that

$$e^{\min\{\gamma,\alpha\}t} \|y_t(\cdot, \omega, y_{01}) - y_t(\cdot, \omega, y_{02})\|_\gamma$$

$$\leq \|y_{01} - y_{02}\|_\gamma + L \int_0^t e^{\min\{\gamma,\alpha\}\tau} \|y_\tau(\cdot, \omega, y_{01}) - y_\tau(\cdot, \omega, y_{02})\|_\gamma.$$

By the classical Gronwall inequality, we arrive at

$$e^{\min\{\gamma,\alpha\}t} \|y_t(\cdot, \omega, y_{01}) - y_t(\cdot, \omega, y_{02})\|_\gamma \leq e^{Lt} \|y_{01} - y_{02}\|_\gamma,$$

which yields the conclusion. $\qquad \square$

Lemma 4.5. *Let* (H1)-(H3) *hold and* $L < \min\{\gamma, \alpha\}$. *Then* ϕ *is pullback asymptotically compact.*

Proof. We need to prove that for every sequence $t_n \to +\infty$ and $\mathbb{P}$-a.e $\omega \in \Omega$, the sequence $\{\phi(t_n, \theta_{-t_n}\omega, y_0(\theta_{-t_n}\omega))\}_{n=1}^{+\infty}$ has a convergent subsequence as $t_n \to +\infty$, where $y_0(\theta_{-t_n}\omega) \in B(\theta_{-t_n}\omega)$ with $\omega \mapsto B(\omega)$ tempered. We do this by proving the following limit of the Kuratowski's measure of non-compactness:

$$\kappa\Big(\phi(t_n, \theta_{-t_n}\omega, B(\theta_{-t_n}\omega))\Big) \to 0, \quad t_n \to +\infty.$$

Replacing t by t_n and ω by $\theta_{-t_n}\omega$ in (4.1), for any $y_{01}(\theta_{-t_n}\omega), y_{02}(\theta_{-t_n}\omega) \in B(\theta_{-t_n}\omega)$, we have

$$\|\phi(t_n, \theta_{-t_n}\omega, y_{01}(\theta_{-t_n}\omega)) - \phi(t_n, \theta_{-t_n}\omega, y_{02}(\theta_{-t_n}\omega))\|_\gamma$$

$$\leq e^{-(\min\{\gamma,\alpha\}-L)t_n} \|y_{01}(\theta_{-t_n}\omega)) - y_{02}(\theta_{-t_n}\omega))\|_\gamma.$$

Since $\omega \mapsto B(\omega)$ is tempered, for any $\epsilon > 0$ and each $\omega \in \Omega$, there exist tempered random sets $B_i(\theta_{-t_n}\omega)$, $i = 1, 2, \ldots, m$, such that $B(\theta_{-t_n}\omega) \subset \bigcup_{i=1}^m B_i(\theta_{-t_n}\omega)$ and the diameter of $B_i(\theta_{-t_n}\omega)$ satisfies

$$\text{diam}(B_i(\theta_{-t_n}\omega)) \leq \kappa(B(\theta_{-t_n}\omega)) + \epsilon, \quad i = 1, 2, \ldots, m.$$

For any $u, v \in \phi(t_n, \theta_{-t_n}\omega, B_i(\theta_{-t_n}\omega))$, there exist $u_0, v_0 \in B_i(\theta_{-t_n}\omega)$ such that $u = \phi(t_n, \theta_{-t_n}\omega, u_0)$, $v = \phi(t_n, \theta_{-t_n}\omega, v_0)$. Thus,

$$\|u - v\|_\gamma = \|\phi(t_n, \theta_{-t_n}\omega, u_0) - \phi(t_n, \theta_{-t_n}\omega, v_0)\|_\gamma$$

$$\leq e^{-(\min\{\gamma,\alpha\}-L)t_n} \|u_0 - v_0\|_\gamma$$

$$\leq e^{-(\min\{\gamma,\alpha\}-L)t_n} \text{diam}(B_i(\theta_{-t_n}\omega))$$

$$\leq e^{-(\min\{\gamma,\alpha\}-L)t_n}\kappa(B(\theta_{-t_n}\omega)) + \epsilon,$$

which implies that

$$\mathrm{diam}(\phi(t_n,\theta_{-t_n}\omega, B_i(\theta_{-t_n}\omega)) \leq e^{-(\min\{\gamma,\alpha\}-L)t_n}\kappa(B(\theta_{-t_n}\omega)) + \epsilon.$$

Therefore,

$$\kappa\Big(\phi(t_n,\theta_{-t_n}\omega, B_i(\theta_{-t_n}\omega))\Big) \leq e^{-(\min\{\gamma,\alpha\}-L)t_n}\kappa(B(\theta_{-t_n}\omega)) + \epsilon,$$

and hence

$$\kappa\Big(\phi(t_n,\theta_{-t_n}\omega, B(\theta_{-t_n}\omega))\Big) \leq e^{-(\min\{\gamma,\alpha\}-L)t_n}\kappa(B(\theta_{-t_n}\omega)) + \epsilon.$$

By the arbitrary of ϵ, we obtain that

$$\kappa\Big(\phi(t_n,\theta_{-t_n}\omega, B(\theta_{-t_n}\omega))\Big) \leq e^{-(\min\{\gamma,\alpha\}-L)t_n}\kappa(B(\theta_{-t_n}\omega)) \to 0, \ t_n \to +\infty.$$

$\square$

As a consequence of Theorem 2.10, Lemmas 4.3 and 4.5, we have already proved the main result of this paper.

Theorem 4.6. *Suppose that* (H1)-(H3) *hold. If* $L < \min\{\gamma,\alpha\}$, *the continuous random dynamical system* ϕ *defined in* (3.8) *possesses a unique random attractor* $\omega \mapsto A(\omega) \subset X$, *where*

$$(4.2) \qquad A(\omega) = \bigcap_{\tau \geq 0} \overline{\bigcup_{t \geq \tau} \phi(t,\theta_{-t}\omega, K(\theta_{-t}\omega))}, \quad \omega \in \Omega,$$

with $K(\omega)$ *given in Lemma 4.3.*

Corollary 4.7. *If* (H1)-(H3) *are satisfied and* $L < \min\{\gamma,\alpha\}$, *then the continuous random dynamical system* ψ *associated with* (1.1) *possesses a unique random attractor* $\omega \mapsto A(\omega) + z(\theta.\omega) \subset X$, *where* $A(\omega)$ *is given in* (4.2), $z(\theta_s\omega) = \sigma\tilde{z}(\theta_s\omega)$, $s \leq 0$, *with* $\tilde{z}$ *given in* (3.1)

5. Example

As an application of Theorem 4.6, we consider the following partial differential equation with white noise

(5.1)
$$\begin{cases} \partial_t u + \partial_a u = -\mu(a)u(t,a) \\ \quad + \displaystyle\int_{-\infty}^{0} k(s)\int_{0}^{+\infty} f(a,b,u(t+s,b))dbds + \delta(a)\dot{W}(t), & t > 0, \ a > 0, \\ u(t,0) = \beta \displaystyle\int_{0}^{+\infty} u(t,a)da, & t > 0, \\ u(s,a) = u_0(s,a), & a > 0, \end{cases}$$

where $u(t,\cdot) \in L^1(0,+\infty)$, the space of Lebesgue integrable functions with values in $\mathbb{R}$, $\mu \in L^1(0,+\infty)$ with nonnegative values, $\delta \in H^1(0,+\infty)$, $\delta(0) = 0$,

$\beta \geq 0$ and $W(t)$ is the white noise. For more information about Eq. (5.1) without the white noise, we refer the reader to [24].

Let
$$E = \mathbb{R} \times L^1(0, +\infty)$$

with the usual product norm of $\mathbb{R} \times L^1(0, +\infty)$. Define $A : D(A) \subset E \to E$ as follows

$$(5.2) \qquad A \begin{pmatrix} 0 \\ \phi \end{pmatrix} = \begin{pmatrix} -\phi(0) \\ -\phi' - \mu\phi \end{pmatrix}, \quad \begin{pmatrix} 0 \\ \phi \end{pmatrix} \in D(A),$$

where

$$D(A) = \{0\}_{\mathbb{R}} \times \{\phi \in L^1(0, +\infty) : \phi' \in L^1(0, +\infty), \phi(0) = 0\}.$$

It is clear that $E_0 = \overline{D(A)} = \{0\}_{\mathbb{R}} \times L^1(0, +\infty) \neq E$.

Suppose that there exists a constant $\gamma > 0$, such that

$$(5.3) \qquad \mu(a) > \gamma, \quad \forall a \geq 0.$$

Then denote

$$C_\gamma = \left\{ \begin{pmatrix} a \\ \phi \end{pmatrix} : a \in \mathbb{R}, \phi \in C((-\infty, 0], L^1(0, +\infty)) \text{ and } \lim_{s \to -\infty} e^{\gamma s}\phi(s) \text{ exists} \right\}$$

with the norm

$$\left\| \begin{pmatrix} a \\ \phi \end{pmatrix} \right\|_\gamma = |a| + \sup_{-\infty < s \leq 0} e^{\gamma s} \|\phi\|_{L^1}$$

and define the nonlinear term $F : C_\gamma \to E$ as follows

$$(5.4) \qquad F\left(\begin{pmatrix} 0 \\ \phi \end{pmatrix} \right) = \begin{pmatrix} \beta \int_0^{+\infty} \phi(0)(a)da \\ \int_{-\infty}^0 k(s) \int_0^{+\infty} f(a, b, \phi(s)(b))dsdb \end{pmatrix}.$$

Moreover, set $v(t) = \begin{pmatrix} 0 \\ u(t, \cdot) \end{pmatrix} \in E_0$, $v_t = \begin{pmatrix} 0 \\ u_t \end{pmatrix} \in C_\gamma$ and $\begin{pmatrix} 0 \\ \xi \end{pmatrix} = v_0 \in C_\gamma$, where $\xi(s)(a) = u_0(s, a)$, and $\sigma = \begin{pmatrix} 0 \\ \delta \end{pmatrix} \in E$, then Eq. (5.1) can be written as

$$(5.5) \qquad \begin{cases} dv(t) = Av(t)dt + F(v_t)dt + \sigma dW(t), & t > 0, \\ v(0) = v_0 \in C_\gamma. \end{cases}$$

Proposition 5.1. (i) *The operator A defined in* (5.2) *is a Hille-Yosida operator with* $(-\gamma, +\infty) \subset \rho(A)$ *and*

$$\|(\lambda I - A)^{-1}\|_{\mathcal{L}} \leq \frac{1}{\lambda + \gamma}, \quad \forall \lambda > -\gamma;$$

(ii) *the C_0-semigroup $T_0(t)$, generated by A_0 on X_0, satisfies that*

$$\|T_0(t)\|_{\mathcal{L}} \leq e^{-\gamma t}, \quad \forall t \geq 0.$$

Proof. (i) From (5.2), we know that

$$(\lambda I - A)\begin{pmatrix} 0 \\ \phi \end{pmatrix} = \begin{pmatrix} \phi(0) \\ \phi' + (\lambda + \mu)\phi \end{pmatrix}.$$

Set $y = \phi(0)$ and $\psi = \phi' + (\lambda + \mu)\phi$. Then

$$(5.6) \qquad \phi(a) = e^{-\lambda a - \int_0^a \mu(s)ds} y + \int_0^a e^{-\lambda(a-s) - \int_s^a \mu(\tau-s)d\tau} \psi(s) ds.$$

By (5.3), $\phi \in L^1(0, +\infty)$ provided that $\lambda > -\gamma$. Therefore, for any $\lambda > -\gamma$,

$$(\lambda I - A)^{-1}\begin{pmatrix} y \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ \phi \end{pmatrix}$$

if and only if (5.6) holds. A simple calculation shows that

$$\|(\lambda I - A)^{-1}\|_{\mathcal{L}} \leq \frac{1}{\lambda + \gamma}, \quad \forall \lambda > -\gamma.$$

(ii) The C_0-semigroup $T_0(t)$, generated by A_0 on E_0, possesses the following form

$$(5.7) \qquad\qquad T_0(t)\begin{pmatrix} 0 \\ \phi \end{pmatrix} = \begin{pmatrix} 0 \\ \widetilde{T}_0(t)\phi \end{pmatrix},$$

where

$$(5.8) \qquad\qquad \widetilde{T}_0(t)\phi = \begin{cases} 0, & a < t, \\ \phi(a-t)e^{-\int_{a-t}^a \mu(\tau)d\tau}, & a \geq t. \end{cases}$$

Then

$$\left\| T_0(t)\begin{pmatrix} 0 \\ \phi \end{pmatrix}\right\| = \|\widetilde{T}_0(t)\phi\|_{L^1}$$

$$= \int_t^{+\infty} |\phi(a-t)e^{-\int_{a-t}^a \mu(\tau)d\tau}| da$$

$$= \int_0^{+\infty} |\phi(a)|e^{-\int_a^{a+t} \mu(\tau)d\tau} da$$

$$\leq e^{-\gamma t}\|\phi\|_{L^1},$$

which implies that $\|T_0(t)\|_{\mathcal{L}} \leq e^{-\gamma t}, \forall t \geq 0$. $\qquad\qquad\square$

In order to obtain the existence of a random attractor of Eq. (5.1), we need the following assumption on f:

(H$_\gamma$) there exists a nonnegative function $L(\cdot) \in L^1(0, \infty)$ such that

$$\int_0^{+\infty} |f(a, b, \phi_1(s)(b)) - f(a, b, \phi_2(s)(b))| db \leq L(a)\|\phi_1(s) - \phi_2(s)\|_{L^1}, \quad \forall a \geq 0.$$

Then F is globally Lipschitz continuous with Lipschitzian constant

$$\beta + \|L\|_{L^1}\int_{-\infty}^0 e^{\gamma s}k(s)ds.$$

Theorem 5.2. *Suppose that* (H_γ) *and* (5.3) *hold true. Moreover, if*

$$\gamma > \beta + \|L\|_{L^1} \int_{-\infty}^{0} e^{\gamma s} k(s) ds,$$

then Eq. (5.1) *has a random attractor.*

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