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UNIFORM MODERATE DEVIATION OF SAMPLE QUANTILES AND ORDER STATISTICS

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Abstract. In this article, we establish the $\mathscr{F}_p$-uniform moderate deviation principles of the sample quantiles and order statistics for a sequence of independent and identically distributed samples.

1. Introduction

First, suppose that we have an independent and identically distributed sample of size n from a distribution function $F(x)$ with a continuous probability density function $f(x)$. Let ξ_p denote the unique p-th quantile of $F(x)$, i.e.,

$$\xi_p = \inf\{x : F(x) \geq p\}, \quad p \in (0, 1).$$

Note that ξ_p satisfies

$$F(\xi_p-) \leq p \leq F(\xi_p).$$

Let us define the sample distribution function $F_n(x)$ by

$$F_n(x) = \frac{1}{n}\sum_{i=1}^{n} 1_{\{X_i \leq x\}}, \quad -\infty < x < \infty,$$

where 1_A denotes the indicator function of the set A. The sample p-th quantile is defined as the p-th quantile of the sample distribution function $F_n(x)$ and we denote it by $\hat{\xi}_{pn}$. Thus $\hat{\xi}_{pn}$ can be represented as

$$\hat{\xi}_{pn} = \inf\{x : F_n(x) \geq p\}, \quad p \in (0, 1).$$

The quantile not only can be used for describing some properties of random variables, but also there are not the restrictions of moment conditions. As a result, it is being widely employed in diverse problems in finance, such as, quantile-hedging, optimal portfolio allocation, risk management, and so on. In practice, the large sample theory which can give the asymptotic properties of sample estimator is an important method to analyze statistical problems. There are numerous literatures to study the sample quantiles. In [6], the strong

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consistency of the sample quantile is given, i.e., $\hat{\xi}_{pn} \xrightarrow{wp1} \xi_p$. In addition, if $F(x)$ possesses a continuous density function $f(x)$ in a neighborhood of ξ_p and $f(\xi_p) > 0$, then

$$\frac{n^{\frac{1}{2}} f(\xi_p)(\hat{\xi}_{pn} - \xi_p)}{[p(1-p)]^{\frac{1}{2}}} \to N(0,1) \quad \text{as } n \to \infty,$$

where $N(0,1)$ denotes the standard normal variable (see [6]). Suppose that $F(x)$ is twice differentiable at ξ_p, with $F'(\xi_p) = f(\xi_p) > 0$, then Bahadur [1] proved an elegant representation

$$\hat{\xi}_{pn} = \xi_p + \frac{p - F_n(\xi_p)}{f(\xi_p)} + \widetilde{R}_n \ \text{ a.e.},$$

where $\widetilde{R}_n = O(n^{-\frac{3}{4}}(\log n)^{\frac{3}{4}})$ a.e., as $n \to \infty$. Xu and Miao [7] obtained some asymptotic properties of the deviation between the sample quantiles $\hat{\xi}_{np}$ and the quantile ξ_p under some weak conditions. Miao et al. [5] gave the almost sure central limit theorem of the sample quantiles.

These results concentrated on the topic that the distribution function is fixed. Recently, Zielinski [8] introduced the definition of $\mathscr{F}_p$-uniformly strongly consistent estimator and proved that $\hat{\xi}_{pn}$ is an $\mathscr{F}_p$-uniformly strongly consistent estimator of ξ_p if and only if

$$\inf_{F \in \mathscr{F}_p} \min\{p - F(\xi_p - \epsilon), F(\xi_p + \epsilon) - p\} > 0 \ \text{ for every} \epsilon > 0,$$

where $\mathscr{F}_p$ denotes the family of all distribution functions with the unique p-th quantile ξ_p. Motivated by these works, we want to consider the uniform moderate deviation of the sample quantile. Under some assumptions for the function family $\mathscr{F}_p$, we show, in Section 2, that the sample quantile $\hat{\xi}_{pn}$ satisfies the moderate deviation principle.

Another nature estimator of the quantile is the order statistics. Let

$$X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)}$$

denote the order statistics of the sample $\{X_1, \ldots, X_n\}$ of observations on $F(x)$. For more detail about order statistics, one can refer to David [2] and Serfling [6]. Assume that $F(x)$ is twice differentiable at ξ_p, with $F'(\xi_p) = f(\xi_p) > 0$, then as $k_n = np + o(\sqrt{n}(\log n)^{\delta})$ for some $\delta \ge \frac{1}{2}$, Bahadur [1] firstly established the following representation for order statistics

$$X_{(k)} = \xi_p + \frac{k_n/n - F_n(\xi_p)}{f(\xi_p)} + R_n \ \text{ a.e.},$$

where

$$R_n = O(n^{-3/4}(\log n)^{(1/2)(\delta+1)}) \ \text{ a.e. as } \ n \to \infty.$$

With respect to moderate deviation of order statistics, Miao et al. [4] obtained the asymptotic properties of the deviation between order statistics and p-quantile, which included large and moderate deviation and the Bahadur asymptotic efficiency.

In general, from the Bahadur representation, some properties of sample quantile and order statistics are usually consistent. However, in Section 3, we will show that the moderate deviation principle for order statistics is different from sample quantile (there are a little different in their assumptions and representation of moderate deviation). In Section 4, we will give two examples to show that these assumptions can be satisfied.

2. $\mathscr{F}_p$-uniformly moderate deviation for sample quantiles

Theorem 2.1. *For* $p \in (0,1)$*, let* $\mathscr{F}_p$ *be the family of all distribution functions with the unique p-th quantile* ξ_p*. Let* $X_1, \ldots, X_n$ *be independent identically distributed random variables with distribution function* $F(x) \in \mathscr{F}_p$*. Assume that* $F(x)$ *is twice differentiable at* ξ_p*, with* $F'(\xi_p) = f(\xi_p) > 0$*. Let* $\{b_n\}$ *be a positive sequence satisfying*

$$b_n \to \infty \quad \text{and} \quad \frac{b_n}{\sqrt{n}} \to 0 \quad \text{as} \quad n \to \infty.$$

(1) *If there exists a constant* $\gamma > 0$ *such that*

$$\sup_{F \in \mathscr{F}_p} \sup_{x \in (\xi_p, \xi_p+\gamma)} \frac{|F''(x)|}{(f(\xi_p))^2} < \infty, \tag{2.1}$$

then we have

$$\lim_{n\to\infty} \frac{1}{b_n^2} \log \sup_{F \in \mathscr{F}_p} P_F \left(\frac{\sqrt{n} f(\xi_p)}{b_n} (\hat{\xi}_{pn} - \xi_p) \geq r \right) = -\frac{r^2}{2p(1-p)}. \tag{2.2}$$

(2) *If there exists a constant* $\gamma > 0$ *such that*

$$\sup_{F \in \mathscr{F}_p} \sup_{x \in (\xi_p-\gamma, \xi_p)} \frac{|F''(x)|}{(f(\xi_p))^2} < \infty, \tag{2.3}$$

then we have

$$\lim_{n\to\infty} \frac{1}{b_n^2} \log \sup_{F \in \mathscr{F}_p} P_F \left(\frac{\sqrt{n} f(\xi_p)}{b_n} (\hat{\xi}_{pn} - \xi_p) \leq -r \right) = -\frac{r^2}{2p(1-p)}. \tag{2.4}$$

(3) *In particular, if there exists a neighborhood of* ξ_p*, denoted by* I*, such that*

$$\sup_{F \in \mathscr{F}_p} \sup_{x \in I} \frac{|F''(x)|}{(f(\xi_p))^2} < \infty, \tag{2.5}$$

then for any $r > 0$*, we have the following* $\mathscr{F}_p$*-uniform moderate deviation,*

$$\lim_{n\to\infty} \frac{1}{b_n^2} \log \sup_{F \in \mathscr{F}_p} P_F \left(\frac{\sqrt{n} f(\xi_p)}{b_n} |\hat{\xi}_{pn} - \xi_p| \geq r \right) = -\frac{r^2}{2p(1-p)}.$$

Remark 2.1. Here we need notice that the interval $(\xi_p - \gamma, \xi_p)$, $(\xi_p, \xi_p + \gamma)$ and the neighborhood I maybe dependent on the distribution function F.

The following lemma will be applied in our proof.

Lemma 2.1 ([6]). *Let F be a distribution function. Assume that the function $F^{-1}(t)$, $0 < t < 1$, is nondecreasing and continuous, and satisfies*

$$F^{-1}(F(x)) \le x, \quad x \in (-\infty, +\infty),$$

and

$$F(F^{-1}(t)) \ge t, \quad t \in (0,1).$$

Hence we have

$$F(x) \ge t \Leftrightarrow x \ge F^{-1}(t).$$

Proof of Theorem 2.1. For any $r > 0$, we give the following result firstly

$$\lim_{n\to\infty} \frac{1}{b_n^2} \log \sup_{F\in\mathscr{F}_p} P_F\left(\frac{\sqrt{n}f(\xi_p)}{b_n}(\hat{\xi}_{pn} - \xi_p) \ge r\right) = -\frac{r^2}{2p(1-p)}. \tag{2.6}$$

By Lemma 2.1, we have

$$\begin{aligned} &P_F\left(\frac{\sqrt{n}f(\xi_p)}{b_n}(\hat{\xi}_{pn} - \xi_p) \ge r\right) \\ &= P_F\left(\frac{1}{n}\sum_{i=1}^{n} I_{\{X_i \le \frac{b_n r}{\sqrt{n}f(\xi_p)} + \xi_p\}} \le p\right) \\ &= P_F\left(\sum_{i=1}^{n} I_{\{X_i \ge \frac{b_n r}{\sqrt{n}f(\xi_p)} + \xi_p\}} \ge n(1-p)\right) \\ &= P_F\left(\sum_{i=1}^{n} W_{ni} \ge b_n\sqrt{n}\delta_n\right), \end{aligned} \tag{2.7}$$

where

$$W_{ni} = I_{\{X_i \ge \frac{b_n r}{\sqrt{n}f(\xi_p)} + \xi_p\}} - E_F I_{\{X_i \ge \frac{b_n r}{\sqrt{n}f(\xi_p)} + \xi_p\}} \tag{2.8}$$

and

$$\delta_n = \frac{n(1-p) - nE_F I_{\{X_i \ge \frac{b_n r}{\sqrt{n}f(\xi_p)} + \xi_p\}}}{b_n\sqrt{n}}. \tag{2.9}$$

It is easy to check

$$E_F I_{\{X_i \ge \frac{b_n r}{\sqrt{n}f(\xi_p)} + \xi_p\}} = 1 - F\left(\frac{b_n r}{\sqrt{n}f(\xi_p)} + \xi_p\right) \tag{2.10}$$

and by utilizing Taylor's formula we have

$$F\left(\frac{b_n r}{\sqrt{n}f(\xi_p)} + \xi_p\right) = F(\xi_p) + F'(\xi_p)\frac{b_n r}{\sqrt{n}f(\xi_p)} + \frac{1}{2}F''(\eta)\left(\frac{r b_n}{\sqrt{n}f(\xi_p)}\right)^2, \tag{2.11}$$

where $\eta \in \left(\xi_p, \xi_p + \frac{b_n r}{\sqrt{n}f(\xi_p)}\right)$. Thus from the condition (2.1), we get

$$F\left(\frac{b_n r}{\sqrt{n}f(\xi_p)} + \xi_p\right) = p + \frac{b_n r}{\sqrt{n}} + R_n, \tag{2.12}$$

where $R_n = o\left(\frac{b_n}{\sqrt{n}}\right)$ is independent of F. From (2.9), (2.10), (2.12), we have

$$\delta_n = \frac{n(1-p) - n(1-p) + b_n\sqrt{n}r + o\left(b_n\sqrt{n}\right)}{b_n\sqrt{n}} = r + o(1). \tag{2.13}$$

Furthermore, since

$$W_{ni} = I_{\{X_i \geq \xi_p + \frac{b_n r}{\sqrt{n}f(\xi_p)}\}} - 1 + F\left(\xi_p + \frac{b_n r}{\sqrt{n}f(\xi_p)}\right), \tag{2.14}$$

it is easy to have

$$E_F(W_{ni}) = 0, \quad Var_F(W_{ni}) = p(1-p) + O(b_n/\sqrt{n}). \tag{2.15}$$

Through the above discussions, the equation (2.7) can be rewritten as follows

$$P_F\left(\hat{\xi}_{pn} \geq \xi_p + \frac{b_n r}{\sqrt{n}f(\xi_p)}\right) = P_F\left(\frac{1}{b_n\sqrt{n}}\sum_{i=1}^n W_{ni} \geq r + o(1)\right).$$

Next, we prove the following Cramér functional holds: for any $\lambda \in \mathbb{R}$,

$$\lim_{n\to\infty}\frac{1}{b_n^2}\log\sup_{F\in\mathscr{F}_p} E_F\exp\left\{\frac{\lambda b_n}{\sqrt{n}}\sum_{i=1}^n W_{ni}\right\} = \frac{\lambda^2 p(1-p)}{2}. \tag{2.16}$$

By Triangle inequality, we have

(2.17)

$$\begin{aligned}
&\left|\frac{1}{b_n^2}\log\sup_{F\in\mathscr{F}_p} E_F\exp\left\{\frac{\lambda b_n}{\sqrt{n}}\sum_{i=1}^n W_{ni}\right\} - \frac{\lambda^2 p(1-p)}{2}\right| \\
\leq\ & \frac{n}{b_n^2}\left|\log\sup_{F\in\mathscr{F}_p} E_F\exp\left\{\frac{\lambda b_n}{\sqrt{n}}W_{n1}\right\} - \left(\sup_{F\in\mathscr{F}_p} E_F\exp\left\{\frac{\lambda b_n}{\sqrt{n}}W_{n1}\right\} - 1\right)\right| \\
&+ \frac{n}{b_n^2}\left|\left(\sup_{F\in\mathscr{F}_p} E_F\exp\left\{\frac{\lambda b_n}{\sqrt{n}}W_{n1}\right\} - 1\right) - \frac{\lambda^2 p(1-p)b_n^2}{2n}\right|.
\end{aligned}$$

By the elementary inequality:

$$\left|e^x - 1 - x - \frac{1}{2}x^2\right| \leq |x|^3 e^{|x|}, \quad \forall\ x \in \mathbb{R},$$

we have

$$\begin{aligned}
&\frac{n}{b_n^2}\left|\left(\sup_{F\in\mathscr{F}_p} E_F\exp\left\{\frac{\lambda b_n}{\sqrt{n}}W_{n1}\right\} - 1\right) - \frac{\lambda^2 p(1-p)b_n^2}{2n}\right| \\
\leq\ & \frac{n}{b_n^2}\sup_{F\in\mathscr{F}_p}\left|E_F\left(\exp\left\{\frac{\lambda b_n}{\sqrt{n}}W_{n1}\right\} - 1 - \frac{1}{2}\left(\frac{\lambda b_n}{\sqrt{n}}W_{n1}\right)^2\right)\right| \\
&+ \frac{n}{b_n^2}\sup_{F\in\mathscr{F}_p}\left|\frac{1}{2}E_F\left(\frac{\lambda b_n}{\sqrt{n}}W_{n1}\right)^2 - \frac{\lambda^2 p(1-p)b_n^2}{2n}\right|
\end{aligned} \tag{2.18}$$

$$\le \frac{n}{b_n^2} \sup_{F\in\mathscr{F}_p} E_F\left(\left|\frac{\lambda b_n}{\sqrt{n}}W_{n1}\right|^3 \exp\left\{\frac{\lambda b_n}{\sqrt{n}}|W_{n1}|\right\}\right)$$
$$+ \frac{n}{b_n^2} \sup_{F\in\mathscr{F}_p}\left|\frac{1}{2}E_F\left(\frac{\lambda b_n}{\sqrt{n}}W_{n1}\right)^2 - \frac{\lambda^2 p(1-p)b_n^2}{2n}\right|.$$

Since W_{n1} is a bounded random variable, then from (2.15) and (2.18), we have

$$\frac{n}{b_n^2}\left|\left(\sup_{F\in\mathscr{F}_p} E_F \exp\left\{\frac{\lambda b_n}{\sqrt{n}}W_{n1}\right\}-1\right) - \frac{\lambda^2 p(1-p)b_n^2}{2n}\right| \to 0 \quad \text{as } n\to\infty. \tag{2.19}$$

Furthermore, by the fact that W_{n1} is a bounded random variable again, for enough large n, we have

$$\left|\sup_{F\in\mathscr{F}_p} E_F \exp\left\{\frac{\lambda b_n}{\sqrt{n}}W_{n1}\right\}-1\right| \le \frac{1}{2}.$$

From this estimate and the following inequalities:

$$|\log(1+x)-x| \le 2x^2 \quad \text{for all } |x| \le \frac{1}{2}$$

and

$$|e^x - 1 - x| \le x^2 e^{|x|} \quad \text{for all} \quad x\in\mathbb{R},$$

we get

(2.20)

$$\frac{n}{b_n^2}\left|\log \sup_{F\in\mathscr{F}_p} E_F \exp\left\{\frac{\lambda b_n}{\sqrt{n}}W_{n1}\right\} - \left(\sup_{F\in\mathscr{F}_p} E_F \exp\left\{\frac{\lambda b_n}{\sqrt{n}}W_{n1}\right\}-1\right)\right|$$
$$\le \frac{2n}{b_n^2}\left(\sup_{F\in\mathscr{F}_p} E_F \exp\left\{\frac{\lambda b_n}{\sqrt{n}}W_{n1}\right\}-1\right)^2$$
$$\le \frac{2n}{b_n^2}\sup_{F\in\mathscr{F}_p}\left|E_F \exp\left\{\frac{\lambda b_n}{\sqrt{n}}W_{n1}\right\}-1\right|^2$$
$$\le \frac{2n}{b_n^2}\sup_{F\in\mathscr{F}_p}\left(E_F\left(\left(\frac{\lambda b_n}{\sqrt{n}}W_{n1}\right)^2 \exp\left\{\frac{\lambda b_n}{\sqrt{n}}|W_{n1}|\right\}\right)\right)^2 \to 0.$$

Hence from (2.17), (2.19) and (2.20), the limit (2.16) holds. By the Gärtner-Ellis theorem (see [3]), we have

$$\lim_{n\to\infty}\frac{1}{b_n^2}\log\sup_{F\in\mathscr{F}_p} P_F\left(\frac{\sqrt{n}f(\xi_p)}{b_n}(\hat{\xi}_{pn}-\xi_p)\ge r\right) = -\frac{r^2}{2p(1-p)}. \tag{2.21}$$

Likewise, by Lemma 2.1, we have

$$\begin{aligned} &P_F\left(\frac{\sqrt{n}f(\xi_p)}{b_n}(\hat{\xi}_{pn}-\xi_p)\le -r\right)\\ &= P_F\left(\frac{1}{n}\sum_{i=1}^{n} I_{\{X_i\le \xi_p-\frac{b_n r}{\sqrt{n}f(\xi_p)}\}}\ge p\right) \qquad (2.22)\\ &= P_F\left(\sum_{i=1}^{n} V_{ni}\ge b_n\sqrt{n}\delta_n^{'}\right), \end{aligned}$$

where

$$V_{ni}=I_{\{X_i\le \xi_p-\frac{b_n r}{\sqrt{n}f(\xi_p)}\}}-E_F I_{\{X_i\le \xi_p-\frac{b_n r}{\sqrt{n}f(\xi_p)}\}} \qquad (2.23)$$

and

$$\delta_n^{'}=\frac{np-nEI_{\{X_i\le \xi_p-\frac{b_n r}{\sqrt{n}f(\xi_p)}\}}}{b_n\sqrt{n}}.$$

By the same proof as the term $P_F\left(\hat{\xi}_{pn}-\xi_p\ge \frac{b_n r}{\sqrt{n}f(\xi_p)}\right)$, it follows that

$$\lim_{n\to\infty}\frac{1}{b_n^2}\log\sup_{F\in\mathscr{F}_p}P_F\left(\frac{\sqrt{n}f(\xi_p)}{b_n}(\hat{\xi}_{pn}-\xi_p)\le -r\right)=-\frac{r^2}{2p(1-p)}.$$

This limit and (2.21) yield the desired results. □

3. $\mathscr{F}_p$-uniformly moderate deviation for order statistics

In this section, we obtain the $\mathscr{F}_p$-uniformly moderate deviation principle of order statistics by utilizing the method to deal with the sample quantile.

Theorem 3.1. *For $p\in(0,1)$, let $\mathscr{F}_p$ be the family of all distribution functions with the unique p-th quantile ξ_p. Let $X_1,\ldots,X_n$ be independent identically distributed random variables with distribution function $F(x)\in\mathscr{F}_p$. Assume that $F(x)$ is twice differentiable at ξ_p, with $F'(\xi_p)=f(\xi_p)>0$. Let $\{b_n\}$ be a positive sequence satisfying*

$$b_n\to\infty \quad \text{and} \quad \frac{b_n}{\sqrt{n}}\to 0 \quad \text{as} \quad n\to\infty.$$

Then for any $r>0$, we have

(1) as $k_n=n(1-p)+o(b_n\sqrt{n})$, if there exists a constant $\gamma>0$ such that

$$\sup_{F\in\mathscr{F}_p}\ \sup_{x\in(\xi_p,\xi_p+\gamma)}\frac{|F^{''}(x)|}{(f(\xi_p))^2}<\infty, \qquad (3.1)$$

then

$$\lim_{n\to\infty}\frac{1}{b_n^2}\log\sup_{F\in\mathscr{F}_p}P_F\left(\frac{\sqrt{n}f(\xi_p)}{b_n}(X_{(k_n)}-\xi_p)\ge r\right)=-\frac{r^2}{2p(1-p)}, \qquad (3.2)$$

(2) *as* $k_n = np + o(b_n\sqrt{n})$, *if there exists a constant* $\gamma > 0$ *such that*

$$\sup_{F\in\mathscr{F}_p}\sup_{x\in(\xi_p-\gamma,\xi_p)}\frac{|F^{''}(x)|}{(f(\xi_p))^2} < \infty, \tag{3.3}$$

then

$$\lim_{n\to\infty}\frac{1}{b_n^2}\log\sup_{F\in\mathscr{F}_p}P_F\left(\frac{\sqrt{n}f(\xi_p)}{b_n}(X_{(k_n)}-\xi_p)\le -r\right) = -\frac{r^2}{2p(1-p)}. \tag{3.4}$$

Remark 3.1. From the proof of Theorem 3.1, we know that the two assumptions $k_n = n(1-p)+o(b_n\sqrt{n}), k_n = np+o(b_n\sqrt{n})$ are technical conditions in some sense, so the following standard moderate deviation can not be obtained.

$$\lim_{n\to\infty}\frac{1}{b_n^2}\log\sup_{F\in\mathscr{F}_p}P_F\left(\frac{\sqrt{n}f(\xi_p)}{b_n}|X_{(k_n)}-\xi_p|\ge r\right) = -\frac{r^2}{2p(1-p)}. \tag{3.5}$$

In particular, if $p=\frac{1}{2}$, then (3.5) holds, i.e.,

$$\lim_{n\to\infty}\frac{1}{b_n^2}\log\sup_{F\in\mathscr{F}_p}P_F\left(\frac{\sqrt{n}f(\xi_{\frac{1}{2}})}{b_n}|X_{(k_n)}-\xi_{\frac{1}{2}}|\ge r\right) = -2r^2. \tag{3.6}$$

Corollary 3.1. *Under the assumptions of Theorem 3.1, for* $p > \frac{1}{2}$, *if there exists a neighborhood of* ξ_p, *denoted by* I, *such that*

$$\sup_{F\in\mathscr{F}_p}\sup_{x\in I}\frac{|F^{''}(x)|}{(f(\xi_p))^2} < \infty, \tag{3.7}$$

then for any $r>0$, *we have*

$$\lim_{n\to\infty}\frac{1}{b_n^2}\log\sup_{F\in\mathscr{F}_p}P_F\left(\frac{\sqrt{n}f(\xi_p)}{b_n}|X_{(k_n)}-\xi_p|\ge r\right) = -\frac{r^2}{2p(1-p)}, \tag{3.8}$$

where k_n *satisfies* $k_n = np+o(b_n\sqrt{n})$.

Proof of Theorem 3.1. Firstly we give the proof of (3.2). It is easy to see that

$$\begin{aligned}
&P_F\left(\frac{\sqrt{n}f(\xi_p)}{b_n}(X_{(k_n)}-\xi_p)\ge r\right)\\
&= P_F\left(X_{(k_n)}\ge\frac{b_n r}{\sqrt{n}f(\xi_p)}+\xi_p\right)\\
&= P_F\left(\sum_{i=1}^n I_{\{X_i\ge\frac{b_n r}{\sqrt{n}f(\xi_p)}+\xi_p\}}\ge k_n\right)\\
&= P_F\left(\sum_{i=1}^n W_{ni}\ge b_n\sqrt{n}t_n\right),
\end{aligned}\tag{3.9}$$

where the sequence $\{W_{ni}, 1\le i\le n\}$ is defined in (2.8) and

$$t_n = \frac{k_n - nE_F I_{\{X_i\ge\frac{b_n r}{\sqrt{n}f(\xi_p)}+\xi_p\}}}{b_n\sqrt{n}} = r+o(1)$$

as $k_n = n(1-p) + o(b_n\sqrt{n})$. Hence by the same proof of Theorem 2.1, we can obtain the desired result. Next we show (3.4). Similarly, we have

$$
\begin{aligned}
&P_F\left(\frac{\sqrt{n}f(\xi_p)}{b_n}(X_{(k_n)} - \xi_p) \le -r\right)\\
&= P_F\left(\sum_{i=1}^{n} I_{\{X_i \le \xi_p - \frac{b_n r}{\sqrt{n}f(\xi_p)}\}} \ge k_n\right)\\
&= P_F\left(\sum_{i=1}^{n} V_{ni} \ge b_n\sqrt{n}t_n^{'}\right),
\end{aligned}
\tag{3.10}
$$

where the sequence $\{V_{ni}, 1 \le i \le n\}$ is defined in (2.23) and

$$
t_n^{'} = \frac{k_n - nE_F I_{\{X_i \le \xi_p - \frac{b_n r}{\sqrt{n}f(\xi_p)}\}}}{b_n\sqrt{n}} = r + o(1),
$$

as $k_n = np + o(b_n\sqrt{n})$. Then by the similar proof of Theorem 2.1, we can obtain the relation (3.4). So the proof of the theorem is completed. □

Proof of Corollary 3.1. As the same as the proof of Theorem 3.1, it follows that if $p > \frac{1}{2}$, then $t_n = O(\sqrt{n}/b_n)$. For this case, the probability

$$
P_F\left(\sum_{i=1}^{n} W_{ni} \ge b_n\sqrt{n}t_n\right)
$$

could be neglected in the sense of moderate deviation. Hence, the remainder of the proof is easy. □

4. Further discussions

In this section, we discuss the conditions (2.1), (2.3) and (2.5) by two examples.

Example 4.1. Let $X_1, \ldots, X_n$ be a sequence of independent identically distributed random variables with following distribution function

$$
F_\lambda(x) = \begin{cases} 1 - e^{-\lambda x}, & x > 0, \\ 0, & x \le 0, \end{cases}
$$

where $\lambda > 0$. For $p \in (0,1)$, let ξ_p be the p-th quantile of the sample distribution $F_\lambda(\cdot)$, then

$$
\xi_p = -\frac{1}{\lambda}\log(1-p).
$$

By some simple calculating, we can see that there can exist a positive constant γ such that

$$
\sup_{x\in(\xi_p,\xi_p+\gamma)} \frac{|F_\lambda^{''}(x)|}{(f(\xi_p))^2} = \sup_{x\in(\xi_p,\xi_p+\gamma)} \frac{e^{-\lambda x}}{(1-p)^2} = \frac{e^{-\lambda\xi_p}}{(1-p)^2} = \frac{1}{1-p},
$$

and

$$\sup_{x\in(\xi_p-\gamma,\xi_p)}\frac{|F_\lambda^{''}(x)|}{(f(\xi_p))^2}=\sup_{x\in(\xi_p-\gamma,\xi_p)}\frac{e^{-\lambda x}}{(1-p)^2}=\frac{e^{-\lambda(\xi_p-\gamma)}}{(1-p)^2}=\frac{1}{1-p}e^{\lambda\gamma}.$$

So if and only if $\mathscr{F}_p=\{F_\lambda(\cdot),\lambda\in(a,b)\subset(0,\infty)\}$, the conditions (2.1) and (2.3) hold. If we take $\gamma>0$, $\mathscr{F}_p=\{F_\lambda(\cdot),\lambda\in(a,b)\subset(0,\infty)\}$ and $I=(\xi_p-\gamma,\xi_p+\gamma)$, then

$$\sup_{F_\lambda\in\mathscr{F}_p}\sup_{x\in I}\frac{|F_\lambda^{''}(x)|}{(f(\xi_p))^2}=\sup_{\lambda\in(a,b)}\frac{1}{1-p}e^{\lambda\gamma}<\infty$$

which implies the condition (2.5).

Example 4.2. Let $X_1,\ldots,X_n$ be a sequence of independent identically distributed normal random variables $N(a,\sigma^2)$, where $a\in\mathbb{R}$, $\sigma^2>0$. Let $p=\frac{1}{2}$, then $\xi_{\frac{1}{2}}=a$. Furthermore, we have $f_{a,\sigma}(\xi_{\frac{1}{2}})=\frac{1}{\sqrt{2\pi}\sigma}$ and

$$\frac{|F_{a,\sigma}^{''}(x)|}{(f_{a,\sigma}(\xi_{\frac{1}{2}}))^2}=\frac{\sqrt{2\pi}|x-a|}{\sigma}e^{-\frac{(x-a)^2}{2\sigma^2}},$$

where $F_{a,\sigma}(x)$ and $f_{a,\sigma}(x)$ denote the distribution function and density function of the normal random variable $N(a,\sigma^2)$. If we take the function set

$$\mathscr{F}_{\frac{1}{2}}=\{F_{a,\sigma}(x);a\in\mathbb{R},\sigma^2>c\},$$

where c is a positive constant, then it is easy to see that there exists a neighborhood I (for example, $I=(a-\gamma,a+\gamma)$ for any $\gamma>0$) such that

$$\sup_{F_{a,\sigma}\in\mathscr{F}_{\frac{1}{2}}}\sup_{x\in I}\frac{|F_{a,\sigma}^{''}(x)|}{(f_{a,\sigma}(\xi_{\frac{1}{2}}))^2}<\infty.$$

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