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CODIMENSION REDUCTION FOR SUBMANIFOLDS OF UNIT $(4m+3)$-SPHERE AND ITS APPLICATIONS

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Abstract. In this paper we establish codimension reduction theorem for submanifolds of a $(4m+3)$-dimensional unit sphere S^{4m+3} with Sasakian 3-structure and apply it to submanifolds of a quaternionic projective space.

1. Introduction

As is well-known, for a submanifold M of a Riemannian manifold $\widetilde{M}$, the codimension of M is said to be reduced if there exists a totally geodesic submanifold $\overline{M}$ of $\widetilde{M}$ such that $M \subset \overline{M}$.

In particular, when the ambient manifold is a complex manifold, the intermediate submanifold $\overline{M}$ is requested to be not only totally geodesic, but also complex submanifold.

The codimension reduction problem was investigated by Allendoerfer [1] in the case that the ambient manifold $\widetilde{M}$ is a Euclidean space and by Erbacher [14] in the case that $\widetilde{M}$ is a real space form. For submanifolds of a complex projective space, Cecil [2] proved a codimension reduction theorem for complex submanifolds. Okumura [14] extended Cecil's result to real submanifolds by using the standard submersion method established by Lawson [12] (for real submanifolds of a complex hyperbolic space, see [8]).

As a quaternionic analogue for real submanifolds of a quaternionic projective space, Kwon and the second author [11] provided a codimension reduction theorem which may correspond to Okumura's result in [14] (for real submanifolds of a quaternionic hyperbolic space, see [9]).

On the other hand, in 1982, Okumura [13] studied submanifolds M of an odd-dimensional sphere with the canonical Sasakian structure $\{\phi, \xi\}$ to which the structure vector field ξ is always tangent and proved that, under some

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additional conditions, if $\dim(T_xM \cap \phi T_xM^\perp)$ is less than that codimension, then there exists such a totally geodesic ϕ-invariant submanifold $\overline{M}$ that $M \subset \overline{M}$, where T_xM and $T_xM^\perp$ denote the tangent space and the normal space to M at $x \in M$, respectively. Using this theorem, in his paper [13], Okumura presented a codimension reduction theorem for real submanifolds of a complex projective space by means of the standard submersion method due to Lawson [12].

In this paper we first consider a $(4m+3)$-dimensional unit sphere with the canonical Sasakian 3-structure $\{\phi, \psi, \theta\}$ (for definition, see [7, 10, 17]). Let M be a real submanifold of the space to which the structure vector fields ξ, η, ζ are always tangent. If at each point $x \in M$ the tangent space T_xM satisfies

$$\phi T_xM \subset T_xM, \quad \psi T_xM \subset T_xM, \quad \theta T_xM \subset T_xM,$$

M is called an *invariant submanifold* under $\{\phi, \psi, \theta\}$. It is well known that an invariant submanifold is a manifold with Sasakian 3-structure. We consider the more general case that at each point $x \in M$ T_xM and $T_xM^\perp$ satisfy the condition that $\dim(T_xM \cap \phi T_xM^\perp \cap \psi T_xM^\perp \cap \theta T_xM^\perp)$ is independent of x. Such submanifolds involve invariant submanifolds as a special case.

The main purpose of the paper is to study relations between $\dim(T_xM \cap \phi T_xM^\perp \cap \psi T_xM^\perp \cap \theta T_xM^\perp)$ and the codimension of M, and to prove that, under some additional conditions, if $\dim(T_xM \cap \phi T_xM^\perp \cap \psi T_xM^\perp \cap \theta T_xM^\perp)$ is less than the codimension, then there exists a totally geodesic invariant submanifold M' such that $M \subset M'$, which will be used in codimension reducing for submanifolds of a quaternionic projective space by using the standard submersion method established by Lawson [12].

All manifolds, submanifolds and geometric objects will be assumed to be connected, differentiable and of class C^∞, and all maps also be of class C^∞ if not stated otherwise.

2. Submanifolds of a $(4m+3)$-dimensional unit sphere

Let us consider a $(4m+3)$-dimensional unit sphere S^{4m+3} as a real hypersurface of the real $4(m+1)$-dimensional quaternionic number space Q^{m+1}. For any point x in S^{4m+3}, we set

$$\xi = E_1x, \quad \eta = E_2x, \quad \zeta = E_3x,$$

where $\{E_1, E_2, E_3\}$ denotes the canonical quaternionic Kähler structure of Q^{m+1}. Then $\{\xi, \eta, \zeta\}$ becomes a Sasakian 3-structure, namely, ξ, η and ζ are mutually orthogonal unit Killing vector fields which satisfy

$$\begin{aligned} \bar\nabla_Y \bar\nabla_X \xi &= g(X, \xi)Y - g(Y, X)\xi, \\ \bar\nabla_Y \bar\nabla_X \eta &= g(X, \eta)Y - g(Y, X)\eta, \\ \bar\nabla_Y \bar\nabla_X \zeta &= g(X, \zeta)Y - g(Y, X)\zeta \end{aligned} \tag{2.1}$$

for any vector fields X, Y tangent to S^{4m+3}, where g denotes the canonical metric on S^{4m+3} induced from that of Q^{m+1} and $\bar\nabla$ the Riemannian connection with respect to g. In this case, putting

$$\phi X = \bar{\nabla}_X \xi, \quad \psi X = \bar{\nabla}_X \eta, \quad \theta X = \bar{\nabla}_X \zeta, \tag{2.2}$$

it follows that

$$\begin{aligned} &\phi\xi = 0,\ \psi\eta = 0,\ \theta\zeta = 0, \\ &\psi\zeta = -\theta\eta = \xi,\ \theta\xi = -\phi\zeta = \eta,\ \phi\eta = -\psi\xi = \zeta, \\ &[\eta, \zeta] = -2\xi,\ [\zeta, \xi] = -2\eta,\ [\xi, \eta] = -2\zeta, \end{aligned} \tag{2.3}$$

$$\begin{aligned} &\phi^2 = -I + f_\xi \otimes \xi, \quad \psi^2 = -I + f_\eta \otimes \eta, \quad \theta^2 = -I + f_\zeta \otimes \zeta, \\ &\psi\theta = \phi + f_\zeta \otimes \eta, \quad \theta\phi = \psi + f_\xi \otimes \zeta, \quad \phi\psi = \theta + f_\eta \otimes \xi, \\ &\theta\psi = -\phi + f_\eta \otimes \zeta, \quad \phi\theta = -\psi + f_\zeta \otimes \xi, \quad \psi\phi = -\theta + f_\xi \otimes \eta, \end{aligned} \tag{2.4}$$

where I denotes the identity transformation and

$$f_\xi(X) = g(\xi, X), \quad f_\eta(X) = g(\eta, X), \quad f_\zeta(X) = g(\zeta, X). \tag{2.5}$$

Moreover, from (2.1) and (2.2), we have

$$\begin{aligned} &(\bar{\nabla}_Y \phi)X = g(X, \xi)Y - g(Y, X)\xi, \quad (\bar{\nabla}_Y \psi)X = g(X, \eta)Y - g(Y, X)\eta, \\ &(\bar{\nabla}_Y \theta)X = g(X, \zeta)Y - g(Y, X)\zeta \end{aligned} \tag{2.6}$$

for any vector fields X, Y tangent to S^{4m+3} (cf. [7, 10, 15, 17]).

Let M be an $(n+3)$-dimensional submanifold isometrically immersed in S^{4m+3} and denote by TM and $TM^\perp$ the tangent and normal bundle of M, respectively. We shall delete the isometric immersion $\tilde{\iota} : M \to S^{4m+3}$ and its differential ι_* in our notations. Let ∇ and $\nabla^\perp$ denote the covariant differentiation in M and the normal connection of M in S^{4m+3}, respectively. To each $N_x \in T_x M^\perp$, we extend N_x to a normal vector field N defined in a neighborhood of x. Given an orthonormal basis $\{(N_1)_x, \ldots, (N_p)_x\}$ of $T_x M^\perp$, we denote by H_A the Weingarten map with respect to N_A, which will be called the *second fundamental tensor* associated to N_A. If the second fundamental tensors H_A $(A = 1, \ldots, p)$ vanish identically on M, M is called a *totally geodesic submanifold.* The first normal space N_x^1 is defined to be the orthogonal complement of $\{N_x \in T_x M^\perp \mid H_N = 0\}$ in $T_x M^\perp$ (cf. [4]). If $N_1, \ldots, N_p$ are orthonormal normal vector fields in a neighborhood of $x \in M$, they determine normal connection forms s_{AB} in a neighborhood of x by

$$\nabla_X^\perp N_A = \sum_{B=1}^{p} s_{AB}(X) N_B$$

for X tangent to M. Then we have the following Gauss and Weingarten formulas:

$$\bar{\nabla}_X Y = \nabla_X Y + \sum_{A=1}^{p} g(H_A X, Y) N_A, \quad g(H_A X, Y) = g(X, H_A Y), \tag{2.7}$$

$$\bar{\nabla}_X N_A = -H_A X + \sum_{B=1}^{p} s_{AB}(X) N_B, \quad s_{AB}(X) = -s_{BA}(X). \tag{2.8}$$

The mean curvature vector field μ of M is defined by

$$\mu = \frac{1}{n+3} \sum_{A=1}^{p} (\mathrm{trace} H_A) N_A. \tag{2.9}$$

The submanifold M is said to be *minimal* if μ vanishes identically on M. Differentiating (2.9) covariantly, we have

$$(n+3)\nabla_X^{\perp}\mu = \sum_{A=1}^{p} \{(X \mathrm{trace} H_A) N_A + \sum_{B=1}^{p} (\mathrm{trace} H_A) s_{AB}(X) N_B\}.$$

Hence the mean curvature vector field is parallel with respect to the normal connection $\nabla^{\perp}$ if and only if

$$X(\mathrm{trace} H_A) = \sum_{B=1}^{p} (\mathrm{trace} H_B) s_{AB}(X). \tag{2.10}$$

Let us denote by R and R^N the curvature tensors for ∇ and $\nabla^{\perp}$, respectively. Since the curvature tensor $\bar{R}$ for $\bar{\nabla}$ on S^{4m+3} is given by

$$\bar{R}(X,Y)Z = g(Y,Z)X - g(X,Z)Y,$$

we have the following relations:

$$(\nabla_X H_A)Y - (\nabla_Y H_A)X = \sum_{B=1}^{p} \{s_{AB}(X) H_B Y - s_{AB}(Y) H_B X\}, \tag{2.11}$$

$$R^N(X,Y)N_A = \sum_{B=1}^{p} g((H_A H_B - H_B H_A)X, Y). \tag{2.12}$$

If R^N vanishes identically on M, the normal connection of M in S^{4m+3} is said to be *flat*. The normal connection of M is flat if and only if $H_A H_B = H_B H_A$ for all $A, B = 1, 2, \ldots, p$ (cf. [3]).

For any $X \in TM$ and for $N_A, A+1, 2, \ldots, p$, the transforms $\phi X, \psi X, \theta X$ and $\phi N_A, \psi N_A, \theta N_A$ are, respectively, written in the following forms:

$$\text{(i) } \phi X = FX + \sum_{A=1}^{p} u^A(X) N_A, \quad \text{(ii) } \psi X = GX + \sum_{A=1}^{p} v^A(X) N_A, \tag{2.13}$$

$$\text{(iii) } \theta X = HX + \sum_{A=1}^{p} w^A(X) N_A,$$

$$\text{(i) } \phi N_A = -U_A + \sum_{B=1}^{p} P_{AB}^{\phi} N_B, \quad \text{(ii) } \psi N_A = -V_A + \sum_{B=1}^{p} P_{AB}^{\psi} N_B, \tag{2.14}$$

(iii) $\theta N_A = -W_A + \sum_{A=1}^{p} P^{\theta}_{AB} N_B,$

where $\{F, G, H\}$ and $\{P^{\phi}, P^{\psi}, P^{\theta}\}$ define endomorphisms of TM and $TM^{\perp}$, respectively, and $\{U_A, V_A, W_A\}$ and $\{u^A, v^A, w^A\}$ are local tangent vector fields and local 1-forms on M. They satisfy

$$g(FX, Y) = -g(X, FY), \quad g(GX, Y) = -g(X, GY), \quad g(HX, Y) = -g(X, HY), \tag{2.15}$$

$$P^{\phi}_{AB} = -P^{\phi}_{BA}, \quad P^{\psi}_{AB} = -P^{\psi}_{BA}, \quad P^{\theta}_{AB} = -P^{\theta}_{BA}, \tag{2.16}$$

$$u^A(X) = g(U_A, X), \quad v^A(X) = g(V_A, X), \quad w^A(X) = g(W_A, X) \tag{2.17}$$

for tangent vectors X, Y to M. If $U_A = 0, V_A = 0, W_A = 0, A = 1, 2, \ldots, p$ identically, the submanifold is called an *invariant submanifold* under $\{\phi, \psi, \theta\}$.

In what follows we assume that *the Sasakian 3-structure vector fields* ξ, η, ζ *are always tangent to* M and use the same notations as appeared in the case of ambient manifold. Then, from (2.3), (2.4) and (2.13), we have

$$F\xi = 0, \quad G\eta = 0, \quad H\zeta = 0, \tag{2.18}$$

$$F\eta = \zeta, \quad F\zeta = -\eta, \quad G\zeta = \xi, \quad G\xi = -\zeta, \quad H\xi = \eta, \quad H\eta = -\xi, \tag{2.19}$$

$$u^A(\xi) = u^A(\eta) = u^A(\zeta) = 0, \quad v^A(\xi) = v^A(\eta) = v^A(\zeta) = 0, \quad w^A(\xi) = w^A(\eta) = w^A(\zeta) = 0, \quad A = 1, 2, \ldots, p. \tag{2.20}$$

Applying ϕ to both sides of $(2.13)_{(\mathrm{i})}$ and $(2.14)_{(\mathrm{i})}$, it follows from (2.4), (2.5), (2.13)-(2.14) and (2.16)-(2.17) that

$$F^2X = -X + \sum_{A=1}^{p} u^A(X) U_A + g(\xi, X)\xi, \quad FU_A = -\sum_{B=1}^{p} P^{\phi}_{AB} U_B, \quad g(U_A, U_B) = \delta_{AB} + \sum_{C=1}^{p} P^{\phi}_{AC} P^{\phi}_{CB} \tag{2.21}$$

because the structure vector field ξ is tangent to M. Similarly, from $(2.13)_{(\mathrm{ii})}$, $(2.13)_{(\mathrm{iii})}$, $(2.14)_{(\mathrm{ii})}$ and $(2.14)_{(\mathrm{iii})}$, we get

$$G^2X = -X + \sum_{A=1}^{p} v^A(X) V_A + g(\eta, X)\eta, \quad GV_A = -\sum_{B=1}^{p} P^{\psi}_{AB} V_B, \quad g(V_A, V_B) = \delta_{AB} + \sum_{C=1}^{p} P^{\psi}_{AC} P^{\psi}_{CB}, \tag{2.22}$$

$$H^2X = -X + \sum_{A=1}^{p} w^A(X) W_A + g(\zeta, X)\zeta, \quad HW_A = -\sum_{B=1}^{p} P^{\theta}_{AB} W_B, \quad g(W_A, W_B) = \delta_{AB} + \sum_{C=1}^{p} P^{\theta}_{AC} P^{\theta}_{CB}. \tag{2.23}$$

Applying ψ and θ to both sides of $(2.13)_{(i)}$, respectively, and using (2.3)-(2.5), (2.13)-(2.14) and (2.16)-(2.17), we have

$$GFX = -HX + \sum_{A=1}^{p} u^A(X)V_A + g(\xi, X)\eta, \tag{2.24}$$

$$v^A(FX) = -w^A(X) + \sum_{B=1}^{p} P^{\psi}_{AB} u^B(X),$$

$$HFX = GX + \sum_{A=1}^{p} u^A(X)W_A + g(\xi, X)\zeta, \tag{2.25}$$

$$w^A(FX) = v^A(X) + \sum_{B=1}^{p} P^{\theta}_{AB} u^B(X).$$

Similarly, it follows from $(2.13)_{(ii)}$ and $(2.13)_{(iii)}$ that

$$HGX = -FX + \sum_{A=1}^{p} v^A(X)W_A + g(\eta, X)\zeta, \tag{2.26}$$

$$w^A(GX) = -u^A(X) + \sum_{B=1}^{p} P^{\theta}_{AB} v^B(X),$$

$$FGX = HX + \sum_{A=1}^{p} v^A(X)U_A + g(\eta, X)\xi, \tag{2.27}$$

$$u^A(GX) = w^A(X) + \sum_{B=1}^{p} P^{\phi}_{AB} v^B(X),$$

$$FHX = -GX + \sum_{A=1}^{p} w^A(X)U_A + g(\zeta, X)\xi, \tag{2.28}$$

$$u^A(HX) = -v^A(X) + \sum_{B=1}^{p} P^{\phi}_{AB} w^B(X),$$

$$GHX = FX + \sum_{A=1}^{p} w^A(X)V_A + g(\zeta, X)\eta, \tag{2.29}$$

$$v^A(HX) = u^A(X) + \sum_{B=1}^{p} P^{\psi}_{AB} w^B(X).$$

Applying ψ and θ to both sides of $(2.14)_{(i)}$, respectively, and using (2.4)-(2.5), (2.13)-(2.14) and (2.17), we have

$$GU_A = -W_A - \sum_{B=1}^{p} P^{\phi}_{AB} V_B, \quad g(U_A, V_B) = P^{\theta}_{AB} + \sum_{C=1}^{p} P^{\phi}_{AC} P^{\psi}_{CB}, \tag{2.30}$$

$$HU_A = V_A - \sum_{B=1}^{p} P^{\phi}_{AB} W_B, \quad g(U_A, W_B) = -P^{\psi}_{AB} + \sum_{C=1}^{p} P^{\phi}_{AC} P^{\theta}_{CB}. \tag{2.31}$$

Similarly, it follows from $(2.14)_{\text{(ii)}}$ and $(2.14)_{\text{(iii)}}$ that

$$HV_A = -U_A - \sum_{B=1}^{p} P^{\psi}_{AB} W_B, \quad g(V_A, W_B) = P^{\phi}_{AB} + \sum_{C=1}^{p} P^{\psi}_{AC} P^{\theta}_{CB}, \tag{2.32}$$

$$FV_A = W_A - \sum_{B=1}^{p} P^{\psi}_{AB} U_B, \quad g(V_A, U_B) = -P^{\theta}_{AB} + \sum_{C=1}^{p} P^{\psi}_{AC} P^{\phi}_{CB}, \tag{2.33}$$

$$FW_A = -V_A - \sum_{B=1}^{p} P^{\theta}_{AB} U_B, \quad g(W_A, U_B) = P^{\psi}_{AB} + \sum_{C=1}^{p} P^{\theta}_{AC} P^{\phi}_{CB}, \tag{2.34}$$

$$GW_A = U_A - \sum_{B=1}^{p} P^{\theta}_{AB} V_B, \quad g(W_A, V_B) = -P^{\phi}_{AB} + \sum_{C=1}^{p} P^{\theta}_{AC} P^{\psi}_{CB}. \tag{2.35}$$

Differentiating $(2.13)_{\text{(i)}}$ covariantly and making use of (2.6)-(2.8), (2.13)-(2.14) and (2.16), we obtain

$$(\nabla_Y F)X = g(X,\xi)Y - g(X,Y)\xi - \sum_{A=1}^{p} g(H_A X, Y) U_A + \sum_{A=1}^{p} u^A(X) H_A Y, \tag{2.36}$$

$$(\nabla_Y u^A)X = -g(H_A F X, Y) - \sum_{B=1}^{p} P^{\phi}_{AB} g(H_B X, Y) + \sum_{B=1}^{p} s_{AB}(Y) u^B(X). \tag{2.37}$$

Similarly, from $(2.13)_{\text{(ii)}}$ and $(2.13)_{\text{(iii)}}$, we also get

$$(\nabla_Y G)X = g(X,\eta)Y - g(X,Y)\eta - \sum_{A=1}^{p} g(H_A X, Y) V_A + \sum_{A=1}^{p} v^A(X) H_A Y, \tag{2.38}$$

$$(\nabla_Y v^A)X = -g(H_A G X, Y) - \sum_{B=1}^{p} P^{\psi}_{AB} g(H_B X, Y) + \sum_{B=1}^{p} s_{AB}(Y) v^B(X), \tag{2.39}$$

$$(2.40)\qquad (\nabla_Y H)X = g(X,\zeta)Y - g(X,Y)\zeta - \sum_{A=1}^{p} g(H_A X, Y)W_A + \sum_{A=1}^{p} w^A(X)H_A Y,$$

$$(2.41)\qquad (\nabla_Y w^A)X = -g(H_A H X, Y) - \sum_{B=1}^{p} P^{\theta}_{AB}\, g(H_B X, Y) + \sum_{B=1}^{p} s_{AB}(Y)w^B(X).$$

Differentiating $(2.14)_{(i)}$ covariantly and taking account of (2.6)-(2.8), (2.13)-(2.14) and (2.16), we obtain

$$(2.42)\qquad \nabla_X U_A = F H_A X - \sum_{B=1}^{p} P^{\phi}_{AB} H_B X + \sum_{B=1}^{p} s_{AB}(X)U_B,$$

$$(2.43)\qquad \nabla_X P^{\phi}_{AB} = g(U_A, H_B X) - u^B(H_A X) - \sum_{C=1}^{p} P^{\phi}_{AC} s_{CB}(X) + \sum_{C=1}^{p} P^{\phi}_{BC} s_{CA}(X).$$

Similarly, from $(2.13)_{(ii)}$ and $(2.13)_{(iii)}$, we also get

$$(2.44)\qquad \nabla_X V_A = G H_A X - \sum_{B=1}^{p} P^{\psi}_{AB} H_B X + \sum_{B=1}^{p} s_{AB}(X)V_B,$$

$$(2.45)\qquad \nabla_X P^{\psi}_{AB} = g(V_A, H_B X) - v^B(H_A X) - \sum_{C=1}^{p} P^{\psi}_{AC} s_{CB}(X) + \sum_{C=1}^{p} P^{\psi}_{BC} s_{CA}(X),$$

$$(2.46)\qquad \nabla_X W_A = H H_A X - \sum_{B=1}^{p} P^{\theta}_{AB} H_B X + \sum_{B=1}^{p} s_{AB}(X)W_B,$$

$$(2.47)\qquad \nabla_X P^{\theta}_{AB} = g(W_A, H_B X) - w^B(H_A X) - \sum_{C=1}^{p} P^{\theta}_{AC} s_{CB}(X) + \sum_{C=1}^{p} P^{\theta}_{BC} s_{CA}(X).$$

Moreover, it is clear from (2.2) that

$$(2.48)\qquad \nabla_X \xi = FX, \quad \nabla_X \eta = GX, \quad \nabla_X \zeta = HX,$$

$$H_A\xi = U_A, \quad H_A\eta = V_A, \quad H_A\zeta = W_A. \tag{2.49}$$

3. Laplacian for a global function defined on M

We define a function f on M by

$$f = \sum_{A=1}^{p}\{u^A(U_A) + v^A(V_A) + w^A(W_A)\}.$$

Then, since ξ, η, ζ are mutually orthogonal unit vector fields, (2.21)-(2.23) yield

$$f = \text{tr } F^2 + \text{tr } G^2 + \text{tr } H^2 + 3(n-1), \quad (\text{tr} := \text{trace}) \tag{3.1}$$

which means that f is independent of the choice of $N_A{}'s$ and thus f is a global function defined on M. f vanishes identically on M if and only if M is an invariant submanifold under $\{\phi, \psi, \theta\}$.

From now on we compute the Laplacian Δf. For any vector field X on M it follows from (2.15), (2.17)-(2.18), (2.36), (2.38) and (2.40) that

$$\begin{aligned}
\frac{1}{2}Xf &= \frac{1}{2}X(\text{tr } F^2 + \text{tr } G^2 + \text{tr } H^2)\\
&= \text{tr } (\nabla_X F)F + \text{tr } (\nabla_X G)G + \text{tr } (\nabla_X H)H\\
&= 2\sum_{A=1}^{p}\{g(FH_AX, U_A) + g(GH_AX, V_A) + g(HH_AX, W_A)\},
\end{aligned}$$

from which together with (2.20)-(2.23), (2.42), (2.44), (2.46) and (2.49), we get

$$\begin{aligned}
&\frac{1}{4}(\nabla_Y\nabla_X f - \nabla_{\nabla_Y X} f) = \frac{1}{4}\{\nabla_Y(Xf) - (\nabla_Y X)f\} \\
&= \sum_{A=1}^{p}\{g((\nabla_Y F)H_AX, U_A) + g(F(\nabla_Y H_A)X, U_A) + g(FH_AX, \nabla_Y U_A)\\
&\quad + g((\nabla_Y G)H_AX, V_A) + g(G(\nabla_Y H_A)X, V_A) + g(GH_AX, \nabla_Y V_A)\\
&\quad + g((\nabla_Y H)H_AX, W_A) + g(H(\nabla_Y H_A)X, W_A) + g(HH_AX, \nabla_Y W_A)\\
&= \sum_{A=1}^{p}[g(U_A, X)g(U_A, Y) + g(V_A, X)g(V_A, Y) + g(W_A, X)g(W_A, Y)\\
&\quad - g((\nabla_Y H_A)FU_A, X) - g((\nabla_Y H_A)GV_A, X) - g((\nabla_Y H_A)HW_A, X)\\
&\quad - g(H_AF^2H_AX, Y) - g(H_AG^2H_AX, Y) - g(H_AH^2H_AX, Y)\\
&\quad + \sum_{B=1}^{p}\{g(H_AU_B, X)g(H_BU_A, Y) + g(H_AV_B, X)g(H_BV_A, Y)\\
&\quad + g(H_AW_B, X)g(H_BW_A, Y) - g(H_BH_AX, Y)g(U_B, U_A)\\
&\quad - g(H_BH_AX, Y)g(V_B, V_A) - g(H_BH_AX, Y)g(W_B, W_A)\\
&\quad - P^{\phi}_{AB}g(H_BFH_AX, Y) - P^{\psi}_{AB}g(H_BGH_AX, Y)
\end{aligned} \tag{3.2}$$

$$- P^{\theta}_{AB} g(H_B H H_A X, Y) + s_{AB}(Y) g(F H_A X, U_B)$$
$$+ s_{AB}(Y) g(G H_A X, V_B) + s_{AB}(Y) g(H H_A X, W_B)\}].$$

On the other hand, substituting FU_A, GV_A and HW_A for X into (2.11), respectively, we have

$$(\nabla_Y H_A) FU_A = (\nabla_{FU_A} H_A) Y + \sum_{B=1}^{p} \{s_{AB}(Y) H_B FU_A - s_{AB}(FU_A) H_B Y\},$$
$$(\nabla_Y H_A) GV_A = (\nabla_{GV_A} H_A) Y + \sum_{B=1}^{p} \{s_{AB}(Y) H_B GV_A - s_{AB}(GV_A) H_B Y\},$$
$$(\nabla_Y H_A) HW_A = (\nabla_{HW_A} H_A) Y + \sum_{B=1}^{p} \{s_{AB}(Y) H_B HW_A - s_{AB}(HW_A) H_B Y\},$$

which together with (3.2) yield

$$\begin{aligned}
&\frac{1}{4}(\nabla_Y \nabla_X f - \nabla_{\nabla_Y X} f)\\
&= \sum_{A=1}^{p} [g(U_A, X) g(U_A, Y) + g(V_A, X) g(V_A, Y) + g(W_A, X) g(W_A, Y)\\
&\quad - g((\nabla_{FU_A} H_A) Y, X) - g((\nabla_{GV_A} H_A) Y, X) - g((\nabla_{HW_A} H_A) Y, X)\\
&\quad - g(H_A F^2 H_A X, Y) - g(H_A G^2 H_A X, Y) - g(H_A H^2 H_A X, Y)\\
&\quad + \sum_{B=1}^{p} \{s_{AB}(FU_A) g(H_B Y, X) + s_{AB}(GV_A) g(H_B Y, X)\\
&\quad + s_{AB}(HW_A) g(H_B Y, X) + g(H_A U_B, X) g(H_B U_A, Y)\\
&\quad + g(H_A V_B, X) g(H_B V_A, Y) + g(H_A W_B, X) g(H_B W_A, Y)\\
&\quad - g(H_B H_A X, Y) g(U_B, U_A) - g(H_B H_A X, Y) g(V_B, V_A)\\
&\quad - g(H_B H_A X, Y) g(W_B, W_A) - P^{\phi}_{AB} g(H_B F H_A X, Y)\\
&\quad - P^{\psi}_{AB} g(H_B G H_A X, Y) - P^{\theta}_{AB} g(H_B H H_A X, Y)\}].
\end{aligned}$$

Hence we have

(3.3)

$$\begin{aligned}
\frac{1}{4}\Delta f = &\sum_{A=1}^{p} [g(U_A, U_A) + g(V_A, V_A) + g(W_A, W_A) - \mathrm{tr}\ F^2 H_A^2 - \mathrm{tr}\ G^2 H_A^2\\
&- \mathrm{tr}\ H^2 H_A^2 - \nabla_{FU_A}(\mathrm{tr}\ H_A) - \nabla_{GV_A}(\mathrm{tr}\ H_A) - \nabla_{HW_A}(\mathrm{tr}\ H_A)\\
&+ \sum_{B=1}^{p} \{s_{AB}(FU_A) \mathrm{tr}\ H_B + s_{AB}(GV_A) \mathrm{tr}\ H_B + s_{AB}(HW_A) \mathrm{tr}\ H_B\\
&+ g(H_A U_B, H_B U_A) + g(H_A V_B, H_B V_A) + g(H_A W_B, H_B W_A)\\
&- (\mathrm{tr}\ H_B H_A) g(U_B, U_A) - (\mathrm{tr}\ H_B H_A) g(V_B, V_A)
\end{aligned}$$

$$- (\text{tr } H_B H_A) g(W_B, W_A) - P^{\phi}_{AB}(\text{tr } FH_A H_B) - P^{\psi}_{AB}(\text{tr } GH_A H_B) - P^{\theta}_{AB}(\text{tr } HH_A H_B)\}].$$

On the other hand, (2.21)-(2.23) and (2.49) imply

$$\text{tr } F^2 H_A^2 = -\text{tr } H_A^2 + g(U_A, U_A) + \sum_{B=1}^{p} g(H_A U_B, H_A U_B),$$

$$\text{tr } G^2 H_A^2 = -\text{tr } H_A^2 + g(V_A, V_A) + \sum_{B=1}^{p} g(H_A V_B, H_A V_B),$$

$$\text{tr } H^2 H_A^2 = -\text{tr } H_A^2 + g(W_A, W_A) + \sum_{B=1}^{p} g(H_A W_B, H_A W_B),$$

from which combined with (3.3) it follows that

(3.4)

$$\begin{aligned}\frac{1}{4}\Delta f = & \sum_{A=1}^{p}[3\text{tr } H_A^2 - (FU_A)\text{tr } H_A - (GV_A)\text{tr } H_A - (HW_A)\text{tr } H_A \\ & + \sum_{B=1}^{p}\{s_{AB}(FU_A)\text{tr } H_B + s_{AB}(GV_A)\text{tr } H_B + s_{AB}(HW_A)\text{tr } H_B \\ & + g(H_A U_B, H_B U_A - H_A U_B) + g(H_A V_B, H_B V_A - H_A V_B) \\ & + g(H_A W_B, H_B W_A - H_A W_B) - (\text{tr } H_B H_A) g(U_B, U_A) \\ & - (\text{tr } H_B H_A) g(V_B, V_A) - (\text{tr } H_B H_A) g(W_B, W_A) \\ & - P^{\phi}_{AB}(\text{tr } FH_A H_B) - P^{\psi}_{AB}(\text{tr } GH_A H_B) - P^{\theta}_{AB}(\text{tr } HH_A H_B)\}].\end{aligned}$$

Now we prepare some lemmas for later use.

Lemma 3.1. *Let M be a submanifold of a unit $(4m+3)$-sphere S^{4m+3} to which the Sasakian 3-structure vector fields ξ, η, ζ are always tangent. If the normal connection of M in S^{4m+3} is flat, then*

$$\sum_{A=1}^{p} u^A(U_A), \quad \sum_{A=1}^{p} v^A(V_A), \quad \sum_{A=1}^{p} w^A(W_A)$$

are constant and consequently the function f is also constant.

Proof. For any vector field X tangent to M, it follows from (2.7), (2.15), (2.21) and (2.42) that

$$\begin{aligned}\frac{1}{2}X(\sum_{A=1}^{p} u^A(U_A)) &= \sum_{A=1}^{p} g(\nabla_X U_A, U_A) \\ &= \sum_{A=1}^{p}[\{g(FH_A X, U_A) - \sum_{B=1}^{p} P^{\phi}_{AB} g(X, H_B U_A)]\end{aligned}$$

$$= \sum_{A,B=1}^{p} P^{\phi}_{AB} g(X, H_A U_B - H_B U_A).$$

On the other hand, if the normal connection is flat, then by means of (2.49) we obtain

$$\begin{aligned} H_A U_B - H_B U_A &= (H_A H_B - H_B H_A)\xi = 0, \\ H_A V_B - H_B V_A &= (H_A H_B - H_B H_A)\eta = 0, \\ H_A W_B - H_B W_A &= (H_A H_B - H_B H_A)\zeta = 0, \end{aligned} \tag{3.5}$$

which together with the above equation yield $X(\sum_{A=1}^{p} u^A(U_A)) = 0$, namely $\sum_{A=1}^{p} u^A(U_A)$ is constant. Similarly we can prove that $\sum_{A=1}^{p} v^A(V_A)$ and $\sum_{A=1}^{p} w^A(W_A)$ are also constant. □

Lemma 3.2. *Let M be as in Lemma* 3.1. *If the normal connection of M in S^{4m+3} is flat and the mean curvature vector field μ is parallel with respect to the normal connection, then*

$$3\sum_{A=1}^{p} \text{tr } H_A^2 = \sum_{A,B=1}^{p} \{(\text{tr } H_A H_B) g(U_A, U_B) + (\text{tr } H_A H_B) g(V_A, V_B) + (\text{tr } H_A H_B) g(W_A, W_B)\}. \tag{3.6}$$

Proof. Owing to Lemma 3.1, it follows from (2.10), (3.4) and (3.5) that

$$\begin{aligned} 3\sum_{A=1}^{p} \text{tr } H_A^2 = \sum_{A,B=1}^{p} \{&(\text{tr } H_B H_A) g(U_B, U_A) + (\text{tr } H_B H_A) g(V_B, V_A) \\ &+ (\text{tr } H_B H_A) g(W_B, W_A) + P^{\phi}_{AB}(\text{tr } F H_A H_B) \\ &+ P^{\psi}_{AB}(\text{tr } G H_A H_B) + P^{\theta}_{AB}(\text{tr } H H_A H_B)\}, \end{aligned}$$

from which combined with (2.16) and $H_A H_B = H_B H_A$, we get (3.6). □

4. Submanifolds with $\dim(TM \cap \phi TM^{\perp} \cap \psi TM^{\perp} \cap \theta TM^{\perp}) < p$

Suppose that at a point $x \in M$

$$\dim(T_x M \cap \phi T_x M^{\perp} \cap \psi T_x M^{\perp} \cap \theta T_x M^{\perp}) = q.$$

Then we can choose in $TM^{\perp}$ $3q$ orthonormal normal vector fields $N_\alpha (\alpha = 1, \dots, 3q)$ in such a way that

$$\phi_x(N_\alpha)_x, \psi_x(N_\alpha)_x, \theta_x(N_\alpha)_x \in T_x M \oplus \text{Span}\{N_\alpha\}_{\alpha=1,\dots,3q},$$

and further

$$\phi_x(N_1)_x = \psi_x(N_{q+1})_x = \theta_x(N_{2q+1})_x, \dots, \phi_x(N_q)_x = \psi_x(N_{2q})_x = \theta_x(N_{3q})_x. \tag{4.1}$$

In fact, if $\{(X_1)_x, \dots, (X_q)_x\}$ is an orthonormal basis of $T_xM \cap \phi T_xM^\perp \cap \psi T_xM^\perp \cap \theta T_xM^\perp$, then there exist $3q$ normal vector fields N_α such that

$$(4.2)\qquad \begin{aligned} &(X_1)_x = -\phi_x(N_1)_x = -\psi_x(N_{q+1})_x = -\theta_x(N_{2q+1})_x, \dots, \\ &(X_q)_x = -\phi_x(N_q)_x = -\psi_x(N_{2q})_x = -\theta_x(N_{3q})_x \end{aligned}$$

and consequently all of $(X_i)_x$ are mutually orthogonal to ξ, η and ζ because of (2.3). With such a choice of $N_\alpha (\alpha = 1, \dots, 3q)$, it follows from (2.14) that

$$(4.3a)\qquad \begin{aligned} &(X_1)_x = (U_1)_x = (V_{q+1})_x = (W_{2q+1})_x, \\ &\qquad\vdots \\ &(X_q)_x = (U_q)_x = (V_{2q})_x = (W_{3q})_x, \\ &(U_{q+1})_x = \cdots = (U_{3q})_x = 0, \\ &(V_1)_x = \cdots = (V_q)_x = (V_{2q+1})_x = \cdots = (V_{3q})_x = 0, \\ &(W_1)_x = \cdots = (W_{2q})_x = 0, \end{aligned}$$

$$(4.3b)\qquad \begin{aligned} &P^\phi_{(q+1)(2q+1)} = -P^\phi_{(2q+1)(q+1)} = 1, \dots, P^\phi_{(2q)(3q)} = -P^\phi_{(3q)(2q)} = 1, \\ &P^\psi_{(1)(2q+1)} = -P^\psi_{(2q+1)(1)} = -1, \dots, P^\psi_{(q)(3q)} = -P^\psi_{(3q)(q)} = -1, \\ &P^\theta_{(1)(q+1)} = -P^\psi_{(q+1)(1)} = 1, \dots, P^\theta_{(q)(2q)} = -P^\psi_{(2q)(q)} = 1, \\ &P^\phi_{\alpha\nu} = 0, \quad P^\psi_{\alpha\nu} = 0, \quad P^\theta_{\alpha\nu} = 0, \ (\alpha = 1, \dots, 3q, \ \nu = 3q+1, \dots, p), \end{aligned}$$

$$(4.3c)\qquad \begin{aligned} &\phi_x(N_\nu)_x = -(U_\nu)_x + \sum_{\delta=3q+1}^{p} P^\phi_{\nu\delta}(x)(N_\delta)_x, \\ &\psi_x(N_\nu)_x = -(V_\nu)_x + \sum_{\delta=3q+1}^{p} P^\psi_{\nu\delta}(x)(N_\delta)_x, \\ &\theta_x(N_\nu)_x = -(W_\nu)_x + \sum_{\delta=3q+1}^{p} P^\theta_{\nu\delta}(x)(N_\delta)_x, \end{aligned}$$

where we have used (2.4) and (4.2). Furthermore, it is clear from (2.4), (4.1) and (4.2) that

$$(4.4)\qquad \begin{aligned} &g_x((X_i)_x, (U_\nu)_x) = 0, \quad g_x((X_i)_x, (V_\nu)_x) = 0, \quad g_x((X_i)_x, (W_\nu)_x) = 0, \\ &\qquad i = 1, \dots, q, \quad \nu = 3q+1, \dots, p. \end{aligned}$$

Lemma 4.1. *If the normal connection is flat, q is constant over M.*

Proof. We put

$$f_1 = \sum_{A=1}^{p} u^A(U_A), \quad f_2 = \sum_{A=1}^{p} v^A(V_A), \quad f_3 = \sum_{A=1}^{p} w^A(W_A).$$

Assume that at $y \in M$

$$\dim(T_yM \cap \phi T_yM^\perp \cap \psi T_yM^\perp \cap \theta T_yM^\perp) = q'$$

and let say $q < q'$. At x and y the function f_1 can be rewritten as the following:

$$(4.5)\qquad \begin{aligned} f_1(x) &= \sum_{\alpha=1}^{3q} u^\alpha(U_\alpha)(x) + \sum_{\nu=3q+1}^{3q'} u^\nu(U_\nu)(x) + \sum_{\nu=3q'+1}^{p} u^\nu(U_\nu)(x), \\ f_1(y) &= \sum_{\alpha=1}^{3q'} u^\alpha(U_\alpha)(y) + \sum_{\nu=3q'+1}^{p} u^\nu(U_\nu)(y). \end{aligned}$$

By means of Lemma 3.1, the function f_1 is constant and consequently (4.3) and (4.5) imply

$$3q + \sum_{\nu=3q+1}^{3q'} u^\nu(U_\nu)(x) + \sum_{\nu=3q'+1}^{p} u^\nu(U_\nu)(x) = 3q' + \sum_{\nu=3q'+1}^{p} u^\nu(U_\nu)(y),$$

or equivalently,

$$(4.6)\quad 3(q-q') + \sum_{\nu=3q+1}^{3q'} u^\nu(U_\nu)(x) + \sum_{\nu=3q'+1}^{p} \{u^\nu(U_\nu)(x) - u^\nu(U_\nu)(y)\} = 0.$$

On the other hand, it follows from (2.21) that $u^\nu(U_\nu) = 1 - \sum_{A=1}^{p}(P^\phi_{\nu A})^2$ and thus

$$\sum_{\nu=3q+1}^{3q'} u^A(U_A)(x) = 3(q'-q) - \sum_{\nu=3q+1}^{3q'} \sum_{A=1}^{p} (P^\phi_{\nu A})^2(x),$$

from which, inserting back into (4.6), we have

$$(4.7)\qquad -\sum_{\nu=3q+1}^{3q'} \sum_{A=1}^{p} (P^\phi_{\nu A})^2(x) + \sum_{\nu=3q'+1}^{p} \{u^\nu(U_\nu)(x) - u^\nu(U_\nu)(y)\} = 0.$$

Since $u^\nu(U_\nu)$ and $P^\phi_{\nu A}$ are differentiable functions, we obtain

$$\lim_{x\to y}\{u^\nu(U_\nu)(x) - u^\nu(U_\nu)(y)\} = 0.$$

Hence it is clear from (4.7) that

$$\sum_{A=1}^{p} (P^\phi_{\nu A})^2(y) = 0, \text{ i.e., } P^\phi_{\nu A}(y) = 0,\ \nu = q+1, \ldots, q',$$

which is a contradiction because of (4.3b). By using the functions f_2 or f_3 we can derive the same conclusion. □

In the following we assume that $3q < p$ and that the mean curvature vector field μ is parallel with respect to the normal connection. Then (2.21)-(2.23), (3.6), (4.3) and (4.4) yield

$$\sum_{\nu=3q+1}^{p}(\text{tr } H_\nu^2)[\{1-g(U_\nu,U_\nu)\}+\{1-g(V_\nu,V_\nu)\}+\{1-g(W_\nu,W_\nu)\}]$$
$$=\sum_{\nu=3q+1}^{p}(\text{tr } H_\nu^2)\sum_{A=1}^{p}\{(P_{\nu A}^{\phi})^2+(P_{\nu A}^{\psi})^2+(P_{\nu A}^{\theta})^2\}=0,$$

which implies tr $H_\nu^2=0$ for $\nu=3q+1,\dots,p$. Thus $H_\nu=0, \nu=3q+1,\dots,p$ and $U_\nu=V_\nu=W_\nu=0, \nu=3q+1,\dots,p$ by means of (2.49). Particularly, when $q=0$, we have the following.

Theorem 4.2. *Let M be an $(n+3)$-dimensional complete submanifold isometrically immersed in a unit $(4m+3)$-sphere S^{4m+3} to which the structure vector fields ξ,η,ζ are always tangent. Suppose that the normal connection of M in S^{4m+3} is flat and that the mean curvature vector field is parallel with respect to the normal connection. If $\dim(T_xM\cap\phi T_xM^\perp\cap\psi T_xM^\perp\cap\theta T_xM^\perp)=0$ at some point $x\in M$, then M is a totally geodesic, invariant submanifold and consequently a great sphere.*

Corollary 4.3. *Let M be an $(n+3)$-dimensional complete, minimal submanifold isometrically immersed in a unit $(4m+3)$-sphere S^{4m+3} to which the structure vector fields ξ,η,ζ are always tangent. Suppose that the normal connection of M in S^{4m+3} is flat and that the mean curvature vector field is parallel with respect to the normal connection. If $\dim(T_xM\cap\phi T_xM^\perp\cap\psi T_xM^\perp\cap\theta T_xM^\perp)=0$ at some point $x\in M$, then M is a totally geodesic, invariant submanifold and consequently a great sphere.*

On the other side, in order to consider the case where $0<3q<p$, we will prepare the following two Lemmas.

Lemma 4.4. *For $\alpha=1,\dots,3q$ and $\nu=3q+1,\dots,p$, $s_{\nu\alpha}=0$.*

Proof. Since $U_\nu=V_\nu=W_\nu=0$ and $H_\nu=0$, (2.42), (2.44) and (2.46) give

$$\sum_{B=1}^{p}s_{\nu B}U_B=\sum P_{\nu B}^{\phi}H_BX,\quad \sum_{B=1}^{p}s_{\nu B}V_B=\sum P_{\nu B}^{\psi}H_BX,$$
$$\sum_{B=1}^{p}s_{\nu B}W_B=\sum P_{\nu B}^{\theta}H_BX,$$

from which together with $P_{\alpha\nu}^{\phi}=P_{\alpha\nu}^{\psi}=P_{\alpha\nu}^{\theta}=0$, it follows that

$$\sum_{\alpha=1}^{q}s_{\nu\alpha}U_\alpha=0,\quad \sum_{\alpha=q+1}^{2q}s_{\nu\alpha}V_\alpha=0,\quad \sum_{\alpha=2q+1}^{3q}s_{\nu\alpha}W_\alpha=0.$$

Hence it is clear from (4.3) that $s_{\nu\alpha}=0$ for $\alpha=1,\dots,3q;\nu=3q+1,\dots,p$. □

Lemma 4.5. *The first normal space of M in S^{4m+3} is invariant under parallel translation with respect to the normal connection.*

Proof. Since $X_i \neq 0 (i = 1, \dots, q), U_\nu = V_\nu = W_\nu = 0$ and $H_\nu = 0 (\nu = 3q + 1, \dots, p)$, we can see that (2.49) and (4.3) imply that the first normal space is spanned by $N_\alpha (\alpha = 1, \dots, 3q)$. For any vector field X tangent to M, by means of Lemma 4.4 we have

$$\nabla^\perp_X N_\alpha = \sum_{A=1}^{p} s_{\alpha A}(X) N_A = \sum_{\beta=1}^{3q} s_{\alpha\beta}(X) N_\beta,$$

which show that the first normal space is invariant under parallel translation with respect to the normal connection. □

Combining Lemma 4.4 with the results due to Allendoerfer [1] and Erbacher [4] yields that there exists a totally geodesic submanifold M' of S^{4m+3} of dimension $(n + 3 + 3q)$ such that $M \subset M'$. By means of (4.2) and (4.3) with $U_\nu = V_\nu = W_\nu = 0 (\nu = 3q+1, \dots, p)$, we can easily see that M' is an invariant submanifold of S^{4m+3} and consequently a $(4m' + 3)$-dimensional sphere for an integer m'.

Summing up, we may conclude:

Theorem 4.6. *Let M be an $(n + 3)$-dimensional submanifold isometrically immersed in a unit $(4m + 3)$-sphere S^{4m+3} to which the structure vector fields ξ, η, ζ are always tangent. Suppose that the normal connection of M in S^{4m+3} is flat and that the mean curvature vector field is parallel with respect to the normal connection. If $\dim(T_xM \cap \phi T_xM^\perp \cap \psi T_xM^\perp \cap \theta T_xM^\perp) = q (3q < p)$ at some point $x \in M$, then either M is a totally geodesic, invariant submanifold of S^{4m+3}, or there exists a totally geodesic, invariant submanifold S^{n+3+3q} of S^{4m+3} such that $M \subset S^{n+3+3q}$.*

5. Submanifolds with *L*-flat normal connection

In this section we try to apply the results which are obtained in the previous sections to submanifolds of a quaternionic projective space.

Let QP^m be a real $4m$-dimensional quaternionic projective space with quaternionic Kählerian structure $\{J, K, L\}$ and let $\widetilde{g}$ be the Fubini-Study metric which satisfies the Hermitian conditions

$$\widetilde{g}(J\widetilde{X}, J\widetilde{Y}) = \widetilde{g}(\widetilde{X}, \widetilde{Y}),\ \widetilde{g}(K\widetilde{X}, K\widetilde{Y}) = \widetilde{g}(\widetilde{X}, \widetilde{Y}),\ \widetilde{g}(L\widetilde{X}, L\widetilde{Y}) = \widetilde{g}(\widetilde{X}, \widetilde{Y}). \tag{5.1}$$

Then we have

$$\begin{aligned} &J^2 = -I, \quad K^2 = -I, \quad L^2 = -I, \\ &J = KL = -LK, \quad K = LJ = -JL, \quad L = JK = -KJ \end{aligned} \tag{5.2}$$

and

$$\begin{aligned}
\widetilde{\nabla}_{\widetilde{X}} J &= \qquad\qquad r(\widetilde{X})K - q(\widetilde{X})L, \\
\widetilde{\nabla}_{\widetilde{X}} K &= -r(\widetilde{X})J \qquad\qquad + p(\widetilde{X})L, \\
\widetilde{\nabla}_{\widetilde{X}} L &= \quad q(\widetilde{X})J - p(\widetilde{X})K
\end{aligned} \tag{5.3}$$

for any vector field $\widetilde{X}$ in QP^m, where $\widetilde{\nabla}$ denotes the Riemannian connection with respect to $\widetilde{g}$, and p, q and r are certain local 1-forms (cf. [5]). It is well known (cf. [6, 15]) that the quaternionic Kählerian structure $\{J, K, L\}$ is induced from the Sasakian 3-structure $\{\phi, \psi, \theta\}$ of a unit $(4m+3)$-sphere S^{4m+3} by the Hopf fibration $\tilde{\pi} : S^{4m+3} \to QP^m$. Relations between these structures are given by

$$\begin{aligned}
&\phi = J^*, \quad \psi = K^*, \quad \theta = L^* \\
&g(X, Y) = \widetilde{g}^*(X, Y) + f_\xi(X)f_\xi(Y) + f_\eta(X)f_\eta(Y) + f_\zeta(X)f_\zeta(Y),
\end{aligned} \tag{5.4}$$

where $*$ denotes the horizontal lift of indicated quantities. We notice that the structure vector fields ξ, η and ζ are the unit vertical vector fields for the fibration.

Let M be an n-dimensional real submanifold of QP^m and construct a S^3-bundle $\tilde{\pi}^{-1}(M)$ over M in such a way that the following diagram is commutative :

$$\begin{array}{ccc}
\tilde{\pi}^{-1}(M) & \xrightarrow{\ \tilde{\iota}\ } & S^{4m+3} \\
{\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle \tilde{\pi}} \\
M & \xrightarrow{\ \iota\ } & QP^m
\end{array}$$

where $\tilde{\iota} : \tilde{\pi}^{-1}(M) \to S^{4m+3}$ and $\iota : M \to QP^m$ are isometric immersions. Then $\widetilde{\pi}^{-1}(M)$ is an $(n+3)$-dimensional submanifold of S^{4m+3} to which the structure vector fields ξ, η and ζ are tangent. Given an orthonormal basis $N_1, \ldots, N_p$ in $TM^\perp$, horizontal lifts $N_1^*, \ldots, N_p^*$ are mutually orthonormal normal vector fields to $\tilde{\pi}^{-1}(M)$ with respect to the Riemannian metric g of $\tilde{\pi}^{-1}(M)$ which is induced from that of S^{4m+3}. The transforms for $X \in TM$ and for N_A by $\{J, K, L\}$ are, respectively, written by

$$\begin{aligned}
&JX = \grave{F}X + \sum_{A=1}^{p} \grave{u}^A(X)N_A, \quad KX = \grave{G}X + \sum_{A=1}^{p} \grave{v}^A(X)N_A, \\
&LX = \grave{H}X + \sum_{A=1}^{p} \grave{w}^A(X)N_A,
\end{aligned} \tag{5.5}$$

$$\begin{aligned}
&JN_A = -\grave{U}_A + \sum_{B=1}^{p} P_{AB}^J N_B, \quad KN_A = -\grave{V}_A + \sum_{B=1}^{p} P_{AB}^K N_B, \\
&LN_A = -\grave{W}_A + \sum_{B=1}^{p} P_{AB}^L N_B,
\end{aligned} \tag{5.6}$$

where $\{\grave{F}, \grave{G}, \grave{H}\}$ and $\{P^J, P^K, P^L\}$ define endomorphisms of TM and of $TM^{\perp}$, respectively, and $\{\grave{U}_A, \grave{V}_A, \grave{W}_A\}$ and $\{\grave{u}^A, \grave{v}^A, \grave{w}^A\}$ are local tangent vector fields and local 1-forms on M. Denoting by $\grave{g}$ the Riemannian metric induced on M from that of QP^m, we have

$$(5.7)\qquad \grave{g}(\grave{F}X, Y) = -\grave{g}(X, \grave{F}Y), \quad \grave{g}(\grave{G}X, Y) = -\grave{g}(X, \grave{G}Y),$$
$$\grave{g}(\grave{H}X, Y) = -\grave{g}(X, \grave{H}Y),$$

$$(5.8)\qquad P^J_{AB} = -P^J_{BA}, \quad P^K_{AB} = -P^K_{BA}, \quad P^L_{AB} = -P^L_{BA},$$

$$(5.9)\qquad \grave{u}^A(X) = \grave{g}(\grave{U}_A, X), \quad \grave{v}^A(X) = \grave{g}(\grave{V}_A, X), \quad \grave{w}^A(X) = \grave{g}(\grave{W}_A, X)$$

for vector fields X, Y tangent to M. Applying J, K and L to (5.5) and making use of (5.2), we can easily obtain the following relations (5.10) and (5.11):

$$(5.10)\qquad \grave{F}^2X = -X + \sum_{A=1}^{p} \grave{u}^A(X)\grave{U}_A, \quad \grave{G}^2X = -X + \sum_{A=1}^{p} \grave{v}^A(X)\grave{V}_A,$$
$$\grave{H}^2X = -X + \sum_{A=1}^{p} \grave{w}^A(X)\grave{W}_A,$$

$$(5.11)\qquad \grave{G}\grave{H}X = \grave{F}X + \sum_{A=1}^{p} \grave{w}^A(X)\grave{V}_A, \quad \grave{H}\grave{G}X = -\grave{F}X + \sum_{A=1}^{p} \grave{v}^A(X)\grave{W}_A,$$
$$\grave{H}\grave{F}X = \grave{G}X + + \sum_{A=1}^{p} \grave{u}^A(X)\grave{W}_A, \quad \grave{F}\grave{H}X = -\grave{G}X + \sum_{A=1}^{p} \grave{w}^A(X)\grave{U}_A,$$
$$\grave{F}\grave{G}X = \grave{H}X + \sum_{A=1}^{p} \grave{v}^A(X)\grave{U}_A, \quad \grave{G}\grave{F}X = -\grave{H}X + \sum_{A=1}^{p} \grave{u}^A(X)\grave{V}_A.$$

Next, applying J, K and L to (5.6) and taking account of (5.2), we have the following relations (5.12)-(5.15):

$$(5.12)\qquad \grave{F}\grave{U}_A = -\sum_{B=1}^{p} P^J_{AB}\grave{U}_B, \ \grave{G}\grave{V}_A = -\sum_{B=1}^{p} P^K_{AB}\grave{V}_B, \ \grave{H}\grave{W}_A = -\sum_{B=1}^{p} P^L_{AB}\grave{W}_B,$$

$$(5.13)\qquad \grave{G}\grave{U}_A = -\grave{W}_A - \sum_{B=1}^{p} P^J_{AB}\grave{V}_B, \quad \grave{H}\grave{U}_A = \grave{V}_A - \sum_{B=1}^{p} P^J_{AB}\grave{W}_B,$$
$$\grave{H}\grave{V}_A = -\grave{U}_A - \sum_{B=1}^{p} P^K_{AB}\grave{W}_B, \quad \grave{F}\grave{V}_A = \grave{W}_A - \sum_{B=1}^{p} P^K_{AB}\grave{U}_B,$$
$$\grave{F}\grave{W}_A = -\grave{V}_A - \sum_{B=1}^{p} P^L_{AB}\grave{U}_B, \quad \grave{G}\grave{W}_A = \grave{U}_A - \sum_{B=1}^{p} P^L_{AB}\grave{V}_B,$$

$$(5.14)\qquad \grave{g}(\grave{U}_A, \grave{U}_B) = \delta_{AB} + \sum_{C=1}^{p} P^J_{AC}P^J_{CB}, \ \grave{g}(\grave{V}_A, \grave{V}_B) = \delta_{AB} + \sum_{C=1}^{p} P^K_{AC}P^K_{CB},$$

$$\grave{g}(\grave{W}_A, \grave{W}_B) = \delta_{AB} + \sum_{C=1}^{p} P^L_{AC} P^L_{CB},$$

$$\grave{g}(\grave{U}_A, \grave{V}_B) = P^L_{AB} + \sum_{C=1}^{p} P^J_{AC} P^K_{CB}, \ \grave{g}(\grave{V}_A, \grave{W}_B) = P^J_{AB} + \sum_{C=1}^{p} P^K_{AC} P^L_{CB}, \tag{5.15}$$

$$\grave{g}(\grave{W}_A, \grave{U}_B) = P^K_{AB} + \sum_{C=1}^{p} P^L_{AC} P^J_{CB}.$$

Let $\grave{\nabla}$ and $\grave{\nabla}^{\perp}$ denote the Riemannian connection induced in M and the normal connection of M in QP^m, respectively. Denoting by $\grave{H}_A$ and $\grave{s}_{AB}$ the Weingarten maps with respect to N_A and the connection forms of $\grave{\nabla}^{\perp}$, respectively, we have Gauss and Weingarten formulas for $\widetilde{\nabla}, \grave{\nabla}$ and $\grave{\nabla}^{\perp}$ which are similar to (2.7). Differentiating (5.5) covariantly and using (5.3), we can easily obtain

(5.16)

$$(\grave{\nabla}_Y \grave{F})X = r(Y)\grave{G}X - q(Y)\grave{H}X - \sum_{A=1}^{p} \grave{g}(\grave{H}_A X, Y)\grave{U}_A + \sum_{A=1}^{p} \grave{u}^A(X)\grave{H}_A Y,$$

$$(\grave{\nabla}_Y \grave{G})X = -r(Y)\grave{F}X + p(Y)\grave{H}X - \sum_{A=1}^{p} \grave{g}(\grave{H}_A X, Y)\grave{V}_A + \sum_{A=1}^{p} \grave{v}^A(X)\grave{H}_A Y,$$

$$(\grave{\nabla}_Y \grave{H})X = q(Y)\grave{F}X - p(Y)\grave{G}X - \sum_{A=1}^{p} \grave{g}(\grave{H}_A X, Y)\grave{W}_A + \sum_{A=1}^{p} \grave{w}^A(X)\grave{H}_A Y.$$

Differentiating (5.6) covariantly and using (5.3), we have the following relations (5.17) and (5.18):

(5.17)

$$\grave{\nabla}_X \grave{U}_A = r(X)\grave{V}_A - q(X)\grave{W}_A + \grave{F}\grave{H}_A X - \sum_{B=1}^{p} P^J_{AB}\grave{H}_B X + \sum_{B=1}^{p} \grave{s}_{AB}(X)\grave{U}_B,$$

$$\grave{\nabla}_X \grave{V}_A = -r(X)\grave{U}_A + p(X)\grave{W}_A + \grave{G}\grave{H}_A X - \sum_{B=1}^{p} P^K_{AB}\grave{H}_B X + \sum_{B=1}^{p} \grave{s}_{AB}(X)\grave{V}_B,$$

$$\grave{\nabla}_X \grave{W}_A = q(X)\grave{U}_A - p(X)\grave{V}_A + \grave{H}\grave{H}_A X - \sum_{B=1}^{p} P^L_{AB}\grave{H}_B X + \sum_{B=1}^{p} \grave{s}_{AB}(X)\grave{W}_B,$$

(5.18)

$$\begin{aligned}\grave{\nabla}^{\perp}_X P^J_{AB} &:= \nabla_X P^J_{AB} + \sum_{C=1}^{p} P^J_{CB}\grave{s}_{CA}(X) + \sum_{C=1}^{p} P^J_{AC}\grave{s}_{CB}(X) \\ &= r(X)P^K_{AB} - q(X)P^L_{AB} + \grave{g}(\grave{U}_A, \grave{H}_B X) - \grave{u}^B(\grave{H}_A X),\end{aligned}$$

$$\grave{\nabla}^{\perp}_X P^K_{AB} := \nabla_X P^K_{AB} + \sum_{C=1}^{p} P^K_{CB}\grave{s}_{CA}(X) + \sum_{C=1}^{p} P^K_{AC}\grave{s}_{CB}(X)$$

$$= -r(X)P^J_{AB} + p(X)P^L_{AB} + \grave{g}(\grave{V}_A, \grave{H}_B X) - \grave{v}^B(\grave{H}_A X),$$

$$\grave{\nabla}^{\perp}_X P^L_{AB} := \nabla_X P^L_{AB} + \sum_{C=1}^{p} P^L_{CB}\grave{s}_{CA}(X) + \sum_{C=1}^{p} P^L_{AC}\grave{s}_{CB}(X)$$
$$= q(X)P^J_{AB} - p(X)P^K_{AB} + \grave{g}(\grave{W}_A, \grave{H}_B X) - \grave{w}^B(\grave{H}_A X).$$

On the other hand, QP^m is of constant Q-sectional curvature 4 and so the curvature tensor $\widetilde{R}$ of QP^m has the following form (cf. [5]):

$$\begin{aligned}\widetilde{R}(\widetilde{X},\widetilde{Y})\widetilde{Z} = {} & \tilde{g}(\widetilde{Y},\widetilde{Z})\widetilde{X} - \tilde{g}(\widetilde{X},\widetilde{Z})\widetilde{Y} \\ & + \tilde{g}(J\widetilde{Y},\widetilde{Z})J\widetilde{X} - \tilde{g}(J\widetilde{X},\widetilde{Z})J\widetilde{Y} - 2\tilde{g}(J\widetilde{X},\widetilde{Y})J\widetilde{Z} \\ & + \tilde{g}(K\widetilde{Y},\widetilde{Z})K\widetilde{X} - \tilde{g}(K\widetilde{X},\widetilde{Z})K\widetilde{Y} - 2\tilde{g}(K\widetilde{X},\widetilde{Y})K\widetilde{Z} \\ & + \tilde{g}(L\widetilde{Y},\widetilde{Z})L\widetilde{X} - \tilde{g}(L\widetilde{X},\widetilde{Z})L\widetilde{Y} - 2\tilde{g}(L\widetilde{X},\widetilde{Y})L\widetilde{Z}.\end{aligned}$$

Thus, using (5.5) and (5.6), we have the following Codazzi and Ricci equations (5.19) and (5.20), respectively:

$$\begin{aligned}(5.19)\qquad & (\grave{\nabla}_X \grave{H}_A)Y - (\grave{\nabla}_Y \grave{H}_A)X \\ = {} & \sum_{B=1}^{p}\{\grave{s}_{AB}(X)\grave{H}_B Y - \grave{s}_{AB}(Y)\grave{H}_B X\} \\ & - \grave{g}(\grave{U}_A, Y)\grave{F}X + \grave{g}(\grave{U}_A, X)\grave{F}Y - 2\grave{g}(\grave{F}X, Y)\grave{U}_A \\ & - \grave{g}(\grave{V}_A, Y)\grave{G}X + \grave{g}(\grave{V}_A, X)\grave{G}Y - 2\grave{g}(\grave{G}X, Y)\grave{V}_A \\ & - \grave{g}(\grave{W}_A, Y)\grave{H}X + \grave{g}(\grave{W}_A, X)\grave{H}Y - 2\grave{g}(\grave{H}X, Y)\grave{W}_A,\end{aligned}$$

$$\begin{aligned}(5.20)\qquad & \grave{R}^{\perp}(X,Y)N_A \\ = {} & \sum_{B=1}^{p}\{\grave{g}((\grave{H}_A\grave{H}_B - \grave{H}_B\grave{H}_A)X, Y) \\ & + \grave{g}(\grave{U}_A, Y)\grave{g}(\grave{U}_B, X) - \grave{g}(\grave{U}_A, X)\grave{g}(\grave{U}_B, Y) - 2\grave{g}(\grave{F}X, Y)P^J_{AB} \\ & + \grave{g}(\grave{V}_A, Y)\grave{g}(\grave{V}_B, X) - \grave{g}(\grave{V}_A, X)\grave{g}(\grave{V}_B, Y) - 2\grave{g}(\grave{G}X, Y)P^K_{AB} \\ & + \grave{g}(\grave{W}_A, Y)\grave{g}(\grave{W}_B, X) - \grave{g}(\grave{W}_A, X)\grave{g}(\grave{W}_B, Y) - 2\grave{g}(\grave{H}X, Y)P^L_{AB}\}N_B,\end{aligned}$$

where $\grave{R}^{\perp}$ denotes the curvature tensor of the normal connection $\grave{\nabla}^{\perp}$. Here we notice that if M is an invariant submanifold of QP^m, then M is totally geodesic (cf. [6]) and $\grave{U}_A = \grave{V}_A = \grave{W}_A = 0$ $(A = 1, \ldots, p)$.

If $R^{\perp}$ satisfies

$$\begin{aligned}(5.21)\qquad & \grave{R}^{\perp}(X,Y)N_A \\ = {} & \sum_{B=1}^{p}\{-2\grave{g}(\grave{F}X, Y)P^J_{AB} - 2\grave{g}(\grave{G}X, Y)P^K_{AB} - 2\grave{g}(\grave{H}X, Y)P^L_{AB}\}N_B\end{aligned}$$

and

$$(5.22)\qquad \begin{aligned} \grave{\nabla}^{\perp}_X P^J_{AB} &= r(X)P^K_{AB} - q(X)P^L_{AB},\\ \grave{\nabla}^{\perp}_X P^K_{AB} &= -r(X)P^J_{AB} + p(X)P^L_{AB},\\ \grave{\nabla}^{\perp}_X P^L_{AB} &= q(X)P^J_{AB} - p(X)P^K_{AB}, \end{aligned}$$

then the normal connection of M is said to be *lift-flat* or briefly *L-flat.* It is well known ([17, Theorem 3.5, p. 431]) that the normal connection of M is L-flat if and only if the normal connection of $\widetilde{\pi}^{-1}(M)$ is flat. In [17], when (5.22) is satisfied, the structure induced in the normal bundle of M in QP^m is said to be *parallel.*

Let H_A, μ and $\grave{\mu}$ be the Weingarten map with respect to N^*_A, the mean curvature vector field of $\widetilde{\pi}^{-1}(M)$ and of M, respectively. Then the following relations are known (cf. [16]):

$$(5.23)\qquad H_A X^* = (\grave{H}_A X)^* + \grave{g}(\grave{U}_A, X)^*\xi + \grave{g}(\grave{V}_A, X)^*\eta + \grave{g}(\grave{W}_A, X)^*\zeta,$$

$$(5.24)\qquad \mathrm{tr}\ H_A = (\mathrm{tr}\ \grave{H}_A)^*,\ (A = 1, \ldots, p)$$

$$(5.25)\qquad \nabla^{\perp}_{X^*}\mu = \frac{n}{n+3}(\grave{\nabla}^{\perp}_X \grave{\mu})^*,$$

$$(5.26)\qquad {P^J_{AB}}^* = s_{AB}(\xi), \quad {P^K_{AB}}^* = s_{AB}(\eta), \quad {P^L_{AB}}^* = s_{AB}(\zeta).$$

It is clear from (5.23) that M is minimal if and only if $\widetilde{\pi}^{-1}(M)$ is minimal (cf. [16]). Finally we verify

Theorem 5.1. *Let M be an n-dimensional real minimal submanifold of QP^m. If the normal connection of M in QP^m is L-flat and at some point of $x \in M$, $\dim(T_xM \cap JT_xM^{\perp} \cap KT_xM^{\perp} \cap LT_xM^{\perp}) = q(3q < p := 4m - n)$, then either M is a totally geodesic, invariant submanifold of QP^m or there exist a real $(n+3q)$-dimensional totally geodesic, invariant submanifold $QP^{(n+3q)/4}$ of QP^m such that $M \subset QP^{(n+3q)/4}$.*

Proof. Since $\dim(T_xM\cap JT_xM^{\perp}\cap KT_xM^{\perp}\cap LT_xM^{\perp}) = q$ and the Riemannian metric $\widetilde{g}$ satisfies the Hermitian conditions, there exist mutually orthonormal normal vectors $n_1, \ldots, n_{3q}$ such that

$$J_x n_1 = K_x n_{q+1} = L_x n_{2q+1}, \ldots, J_x n_q = K_x n_{2q} = L_x n_{3q}$$

constitute an orthonormal basis for $T_xM \cap JT_xM^{\perp} \cap KT_xM^{\perp} \cap LT_xM^{\perp}$. We extend $n_1, \ldots, n_{3q}$ to local fields $N_1, \ldots, N_{3q}$ in $TM^{\perp}$ and choose $N_{3q+1}, \ldots, N_p$ in $TM^{\perp}$ so that $N_1, \ldots, N_{3q}, N_{3q+1}, \ldots, N_p$ are mutually orthonormal. Then $N^*_1, \ldots, N^*_{3q}, N^*_{3q+1}, \ldots, N^*_p$ are orthonormal vector fields in $T\widetilde{\pi}^{-1}(M)^{\perp}$. Let $y \in \widetilde{\pi}^{-1}(x)$, then

$$\dim(T_y\widetilde{\pi}^{-1}(M) \cap \phi_y T_y\widetilde{\pi}^{-1}(M)^{\perp} \cap \psi_y T_y\widetilde{\pi}^{-1}(M)^{\perp} \cap \theta_y T_y\widetilde{\pi}^{-1}(M)^{\perp}) = q$$

because of (5.4). Furthermore, $\widetilde{\pi}^{-1}(M)$ is minimal in S^{4m+3} because of (5.24) and the normal connection of $\widetilde{\pi}^{-1}(M)$ is flat. Thus, by means of Theorem 4.6, $\widetilde{\pi}^{-1}(M)$ is a totally geodesic invariant submanifold S^{n+3} of S^{4m+3}, or there exists a totally geodesic invariant submanifold S^{n+3+3q} such that $\widetilde{\pi}^{-1}(M) \subset S^{n+3+3q}$. S^{n+3+3q} is a S^3-bundle over a quaternionic projective space $QP^{(n+3q)/4}$ of a real $(n+3q)$-dimension and $\{\xi, \eta, \zeta\}$ are the unit vertical vector fields of the S^3-bundle. Thus the immersion : $QP^{(n+3q)/4} \to QP^m$ is compatible with the Hopf fibration $\widetilde{\pi} : S^{4m+3} \to QP^m$. Since S^{n+3+3q} is a totally geodesic submanifold in S^{4m+3}, (5.23) implies that $QP^{(n+3q)/4}$ is a totally geodesic, invariant submanifold of QP^m. This completes the proof. □

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