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STRUCTURAL STABILITY RESULTS FOR THE THERMOELASTICITY OF TYPE III

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Abstract. The equations arising from the thermoelastic theory are analyzed in a linear approximation. First, we establish the convergence result on the coefficient c. Next, we establish that the solution depends continuously on changes in the coefficient c. The main tool used in this paper is the energy method.

1. Introduction

We consider the structural stability of a problem arising from the thermoelastic theory which is discussed in the work of Green and Naghdi [6, 7] and Quintanilla [18]. For other thermoelastic equations, one could refer to [4, 16, 17, 19]. The governing equations of linear theory of thermoelasticity of type III are

$$\rho\ddot{u}_i = \mu\triangle u_i + (\lambda+\mu)u_{j,ji} - \beta\theta_{,i}, \tag{1.1}$$

$$c\ddot{\alpha} = -\beta\dot{u}_{i,i} + k\triangle\alpha + b\triangle\theta. \tag{1.2}$$

Here u_i is the displacement, the constant ρ is the density of the considered medium, θ is the temperature, α is a variable which is typical of this theory and satisfies $\dot{\alpha} = \theta$, λ and μ are the *Lamé* constants and we assume that they satisfy $\mu > 0$ and $\mu + \lambda > 0$, β is the coupling parameter and is related to the thermal expansion coefficient, $b > 0$ is the thermal conductivity, $c > 0$ is the specific heat and $k > 0$ is a parameter which is typical on the theories of type II and III. On a macroscopic scale the scalar α is regarded as representing some "mean" thermal displacement magnitude, and for brevity is referred to as thermal displacement. Its presence in some sense introduces a "thermal memory" and enhances heat

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propagation as a thermal displacement wave. (Both types II and III theories for heat flow in a stationary rigid solid accommodate finite wave speed.)

Studies of the concept of structural stability have been gaining much impetus; see e.g., accounts in the books of Ames and Straughan [1] and the monograph of Straughan [21, 22], see also the papers [2, 3, 5, 10, 11, 12, 13, 14, 15, 20] and the papers cited therein. Structural stability which stresses the continuous dependence or convergence on changes on differential equations may be reflected physically by changes in constitutive parameters. We believe that the mathematical analysis of these equations will help to reveal the applicability of them in physics. On the other hand, continuous dependence (or convergence) results are important because of the inevitable error that arises in both numerical computation and the physical measurement of data. It is relevant to know the magnitude of the effect of such errors on the solutions.

In [18], Quintanilla studied the equations (1.1) and (1.2), he obtained some results on the convergence and structural stability. But he didn't touch with the structural stability on the coefficients ρ and c. Later, in [9], the authors studies the result of continuous dependence on the coefficient ρ. In the present paper, we also investigate the structural stability of the equations (1.1) and (1.2), unlike the above two papers, we obtain both the convergence and continuous dependence results on the coefficient c which can't follow from the methods of [9] and [18].

In this article we study how a solution to (1.1) and (1.2) on an arbitrary bounded spatial domain Ω behaves under changes in the parameter c. This is a singular problem, and so the convergence result is of interest. Often when a parameter tends to zero in a physical problem, this can lead to dramatic consequences including finite time blow up; see, e.g., the accounts in the book by Straughan [23]. Hence, we deem that the derivation of accurate a priori bounds that estimate the convergence rate in a suitable terms of c is of practical value. What's more, in deriving our convergence result as $c \to 0$, we change the type of equation (1.2) from a hyperbolic equation to a parabolic equation, while the previous papers don't change the style. So the argument to derive the result in this paper is more involved, the result established in this paper is more interesting.

In the present paper, the comma is used to indicate partial differentiation, the differentiation with respect to the direction x_k is denoted as, k, thus $u_{,i}$ denotes $\frac{\partial u}{\partial x_i}$, and $\dot{u}$ denotes $\frac{\partial u}{\partial t}$. The usual summation convection is employed with repeated Latin subscripts i summed from 1 to 3. Hence, $u_{i,i} = \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i}$.

2. Convergence result as $c \to 0$

Let Ω be a bounded domain in $\mathbb{R}^3$, with boundary $\partial\Omega$ smooth enough to allow applications of the divergence theorem. Let (u_i, α) and (v_i, z) be the solutions of the following equations as $c \to 0$.

$$\rho \ddot{u}_i = \mu \Delta u_i + (\lambda + \mu) u_{j,ji} - \beta \dot{\alpha}_{,i} \quad \text{in} \quad \Omega \times (0, t), \tag{2.1}$$

$$c\ddot{\alpha} = -\beta\dot{u}_{i,i} + k\Delta\alpha + b\Delta\dot{\alpha} \quad \text{in} \quad \Omega\times(0,t), \tag{2.2}$$

$$u_i = \tilde{u}_i(\mathbf{x}),\ \alpha = \tilde{\alpha}(\mathbf{x}) \quad \text{on} \quad \partial\Omega, \tag{2.3}$$

$$\alpha(\mathbf{x},0) = \alpha^0(\mathbf{x}),\ u_i(x,0) = u_i^0(\mathbf{x}),\ \dot{u}_i(\mathbf{x},0) = \nu_i^0(\mathbf{x}),\ \dot{\alpha}(\mathbf{x},0) = \theta^0(\mathbf{x}), \tag{2.4}$$

and

$$\rho\ddot{v}_i = \mu\Delta v_i + (\lambda+\mu)v_{j,ji} - \beta\dot{z}_{,i} \quad \text{in} \quad \Omega\times(0,t), \tag{2.5}$$

$$0 = -\beta\dot{v}_{i,i} + k\Delta z + b\Delta\dot{z} \quad \text{in} \quad \Omega\times(0,t), \tag{2.6}$$

$$u_i = \tilde{u}_i(\mathbf{x}), z = \tilde{\alpha}(x) \quad \text{on} \quad \partial\Omega, \tag{2.7}$$

$$z(\mathbf{x},0) = \alpha^0(\mathbf{x}),\ v_i(\mathbf{x},0) = u_i^0(\mathbf{x}),\ \dot{v}_i(\mathbf{x},0) = \nu_i^0(\mathbf{x}). \tag{2.8}$$

We now define w_i, π as

$$w_i = u_i - v_i,\ \pi = \alpha - z, \tag{2.9}$$

then (w_i, π) solves the following boundary initial problems:

$$\rho\ddot{w}_i = \mu\Delta w_i + (\lambda+\mu)w_{j,ji} - \beta\dot{\pi}_{,i} \quad \text{in} \quad \Omega\times(0,t), \tag{2.10}$$

$$c\ddot{\alpha} = -\beta\dot{w}_{i,i} + k\Delta\pi + b\Delta\dot{\pi} \quad \text{in} \quad \Omega\times(0,t), \tag{2.11}$$

$$w_i = \pi = 0 \quad \text{on} \quad \partial\Omega, \tag{2.12}$$

$$w_i(\mathbf{x},0) = \dot{w}_i(\mathbf{x},0) = \pi(\mathbf{x},0) = 0,\ \dot{\pi}(\mathbf{x},0) = \theta^0(\mathbf{x}) - \dot{z}_i(\mathbf{x},0) \quad \text{in} \quad \Omega. \tag{2.13}$$

Let us rearrange (2.11) and then form

$$\int_0^t \int_\Omega \dot{\pi}(c\ddot{\pi} + c\ddot{z} + \beta\dot{w}_{i,i} - k\Delta\pi - b\Delta\dot{\pi})dxd\eta = 0.$$

After some integrations, we obtain from the identity

$$\begin{aligned}\frac{c}{2}\int_\Omega(\dot{\pi})^2dx + \beta\int_0^t\int_\Omega\dot{\pi}\dot{w}_{i,i}dxd\eta + \frac{k}{2}\int_\Omega\pi_{,i}\pi_{,i}dx + b\int_0^t\int_\Omega\dot{\pi}_{,i}\dot{\pi}_{,i}dxd\eta \\ = \frac{c}{2}\int_{\Omega(0)}(\theta^0-\dot{z})^2dx + \frac{c}{2}\int_\Omega(\dot{z})^2dx - \frac{c}{2}\int_{\Omega(0)}(\dot{z})^2dx - c\int_0^t\int_\Omega\dot{\alpha}\ddot{z}dxd\eta,\end{aligned} \tag{2.14}$$

where Ω_0 denotes $\Omega\times\{t=0\}$.

Multiplying (2.10) by $\dot{w}_i$ and integrating over $\Omega\times(0,t)$, we obtain

$$\frac{\rho}{2}\int_\Omega\dot{w}_i\dot{w}_idx + \frac{\mu}{2}\int_\Omega w_{i,j}w_{i,j}dx + \frac{\lambda+\mu}{2}\int_\Omega(w_{j,j})^2 + \beta\int_0^t\int_\Omega\dot{w}_i\dot{\pi}_{,i}dxd\eta = 0. \tag{2.15}$$

Combining (2.14) and (2.15), we get

$$\begin{aligned}\frac{c}{2}\int_\Omega(\dot{\pi})^2dx + \frac{k}{2}\int_\Omega\pi_{,i}\pi_{,i}dx + b\int_0^t\int_\Omega\dot{\pi}_{,i}\dot{\pi}_{,i}dxd\eta \\ + \frac{\rho}{2}\int_\Omega\dot{w}_i\dot{w}_idx + \frac{\mu}{2}\int_\Omega w_{i,j}w_{i,j}dx + \frac{\lambda+\mu}{2}\int_\Omega(w_{j,j})^2\end{aligned} \tag{2.16}$$

$$= \frac{c}{2}\int_{\Omega(0)}(\theta^0-\dot z)^2dx+\frac{c}{2}\int_{\Omega}(\dot z)^2dx-\frac{c}{2}\int_{\Omega(0)}(\dot z)^2dx-c\int_0^t\int_\Omega \dot\alpha\ddot z dxd\eta.$$

Using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\frac{c}{2}\int_\Omega(\dot\pi)^2dx+\frac{k}{2}\int_\Omega \pi_{,i}\pi_{,i}dx+b\int_0^t\int_\Omega\dot\pi_{,i}\dot\pi_{,i}dxd\eta\\
&\quad+\frac{\rho}{2}\int_\Omega \dot w_i\dot w_i dx+\frac{\mu}{2}\int_\Omega w_{i,j}w_{i,j}dx+\frac{\lambda+\mu}{2}\int_\Omega (w_{j,j})^2\\
&\le c\int_{\Omega(0)}(\theta^0)^2dx+\frac{c}{2}\int_\Omega(\dot z)^2dx+c\int_{\Omega(0)}(\dot z)^2dx\\
&\quad+c\sqrt{\int_0^t\int_\Omega(\dot\alpha)^2dxd\eta}\sqrt{\int_0^t\int_\Omega(\ddot z)^2dxd\eta}.
\end{aligned}\tag{2.17}$$

To proceed, we multiply (2.2) by $\dot\alpha$ and integrate over $\Omega\times(0,t)$ to find

$$\begin{aligned}
&\frac{c}{2}\int_\Omega(\dot\alpha)^2dx+\beta\int_0^t\int_\Omega \dot u_{i,i}\dot\alpha dxd\eta+\frac{k}{2}\int_\Omega\alpha_{,j}\alpha_{,j}dx+b\int_0^t\int_\Omega\dot\alpha_{,j}\dot\alpha_{,j}dxd\eta\\
&=\frac{c}{2}\int_{\Omega(0)}(\theta^0)^2dx+\frac{k}{2}\int_{\Omega(0)}\alpha^0_{,j}\alpha^0_{,j}dx.
\end{aligned}\tag{2.18}$$

Multiplying (2.1) by $\dot u_i$ and integrating over $\Omega\times(0,t)$, we obtain

$$\begin{aligned}
&\frac{\rho}{2}\int_\Omega \dot u_i\dot u_i dx+\frac{\mu}{2}\int_\Omega u_{i,j}u_{i,j}dx+\frac{\lambda+\mu}{2}\int_\Omega(u_{j,j})^2dx+\beta\int_0^t\int_\Omega\dot\alpha_{,i}\dot u_i dxd\eta\\
&=\frac{\rho}{2}\int_{\Omega(0)}\nu_i^0\nu_i^0dx+\frac{\mu}{2}\int_{\Omega(0)}u^0_{i,j}u^0_{i,j}dx+\frac{\lambda+\mu}{2}\int_{\Omega(0)}(u^0_{j,j})^2dx.
\end{aligned}\tag{2.19}$$

We combine (2.18) and (2.19) to obtain

$$\begin{aligned}
&\frac{c}{2}\int_\Omega(\dot\alpha)^2dx+\frac{k}{2}\int_\Omega\alpha_{,j}\alpha_{,j}dx+b\int_0^t\int_\Omega\dot\alpha_{,j}\dot\alpha_{,j}dxd\eta+\frac{\rho}{2}\int_\Omega\dot u_i\dot u_i dx\\
&\quad+\frac{\mu}{2}\int_\Omega u_{i,j}u_{i,j}dx+\frac{\lambda+\mu}{2}\int_\Omega(u_{j,j})^2dx\\
&=\frac{c}{2}\int_{\Omega(0)}(\theta^0)^2dx+\frac{k}{2}\int_{\Omega(0)}\alpha^0_{,j}\alpha^0_{,j}dx+\frac{\rho}{2}\int_{\Omega(0)}\nu_i^0\nu_i^0dx+\frac{\mu}{2}\int_{\Omega(0)}u^0_{i,j}u^0_{i,j}dx\\
&\quad+\frac{\lambda+\mu}{2}\int_{\Omega(0)}(u^0_{j,j})^2dx\\
&=k_1(x).
\end{aligned}$$

Using the Poincare inequality, we have

$$\int_0^t\int_\Omega(\dot\alpha)^2dxd\eta\le\frac{1}{\lambda^*}\int_0^t\int_\Omega\dot\alpha_{,j}\dot\alpha_{,j}dxd\eta,$$

where λ^* is the smallest eigenvalue in the problem

$$\begin{aligned} \triangle\psi + \lambda\psi &= 0, \quad \text{in } \Omega, \\ \psi &= 0, \quad \text{on } \partial\Omega, \end{aligned}$$

lower bounds for λ^* are well known, see e.g., [8].

We can easily obtain

$$\int_0^t \int_\Omega (\dot{\alpha})^2 dx d\eta \leq \frac{k_1(x)}{b\lambda^*}. \tag{2.20}$$

Then differentiate (2.6) with respect to t, multiply the result by $\ddot{z}$ and integrate over $\Omega \times (0, t)$ to see that

(2.21)
$$\beta \int_0^t \int_\Omega \ddot{v}_{i,i}\ddot{z} dx d\eta + \frac{k}{2}\int_\Omega \dot{z}_{,j}\dot{z}_{,j} dx + b\int_0^t \int_\Omega \ddot{z}_{,j}\ddot{z}_{,j} dx d\eta = \frac{k}{2}\int_{\Omega(0)} \dot{z}_{,j}\dot{z}_{,j} dx.$$

We also differentiate (2.5) with respect to t, multiply the result by $\ddot{v}_i$ and integrate over $\Omega \times (0, t)$ to see that

(2.22)
$$\frac{\rho}{2}\int_\Omega \ddot{v}_i\ddot{v}_i dx + \frac{\mu}{2}\int_\Omega \dot{v}_{i,j}\dot{v}_{i,j} dx + \frac{\lambda+\mu}{2}\int_\Omega \dot{v}_{i,i}\dot{v}_{j,j} dx + \beta\int_0^t \int_\Omega \ddot{v}_i\ddot{z}_{,i} dx d\eta$$

(2.23)
$$= \frac{\rho}{2}\int_{\Omega(0)} \ddot{v}_i\ddot{v}_i dx + \frac{\mu}{2}\int_{\Omega(0)} \nu^0_{i,j}\nu^0_{i,j} dx + \frac{\lambda+\mu}{2}\int_{\Omega(0)} \dot{v}_{i,i}\dot{v}_{j,j} dx.$$

We assume that the system of equations is satisfied at $t = 0$, thus, we obtain

$$X_i^0 = \ddot{v}_i|_{t=0} = \rho^{-1}(\mu\triangle u_i^0 + (\lambda+\mu)u^0_{j,j} - \beta\theta^0_{,j}). \tag{2.24}$$

Thus, combining (2.21), (2.22) and (2.23), we obtain

$$\begin{aligned} \frac{k}{2}\int_\Omega \dot{z}_{,j}\dot{z}_{,j} dx + b\int_0^t \int_\Omega \ddot{z}_{,j}\ddot{z}_{,j} dx d\eta + \frac{\rho}{2}\int_\Omega \ddot{v}_i\ddot{v}_i dx + \frac{\mu}{2}\int_\Omega \dot{v}_{i,j}\dot{v}_{i,j} dx \\ = \frac{k}{2}\int_{\Omega(0)} \dot{z}_{,j}\dot{z}_{,j} dx + \frac{\rho}{2}\int_{\Omega(0)} X_i^0 X_i^0 dx + \frac{\mu}{2}\int_{\Omega(0)} \nu^0_{i,j}\nu^0_{i,j} dx. \end{aligned} \tag{2.25}$$

To handle the term on the right side, we have

$$\begin{aligned} \int_{\Omega(0)} \dot{z}_{,j}\dot{z}_{,j} dx &= -\int_{\Omega(0)} \dot{z}_{,jj}\dot{z} dx \\ &= \frac{1}{b}\int_{\Omega(0)} (-\beta\dot{v}_{i,i} + kz_{,jj})\dot{z} dx \\ &= \frac{\beta}{b}\int_{\Omega(0)} \dot{v}_i\dot{z}_{,i} dx - \frac{k}{b}\int_{\Omega(0)} z_{,j}\dot{z}_{,j} dx \\ &= \frac{\beta}{b}\int_{\Omega(0)} \nu_i^0\dot{z}_{,i} dx - \frac{k}{b}\int_{\Omega(0)} \alpha^0_{,j}\dot{z}_{,j} dx. \end{aligned}$$

Using the Schwarz inequality, we get

$$\int_{\Omega(0)} \dot{z}_{,j}\dot{z}_{,j}dx \le k_2 \int_{\Omega(0)} \nu_i^0\nu_i^0 dx + k_3 \int_{\Omega(0)} \alpha_{,j}^0\alpha_{,j}^0 dx, \tag{2.26}$$

where $k_2 = \frac{2\beta^2}{b^2}$, $k_3 = \frac{2k^2}{b^2}$.

Combining (2.24) and (2.25), we obtain

(2.27)

$$\begin{aligned}
&\frac{k}{2}\int_\Omega \dot{z}_{,j}\dot{z}_{,j}dx + b\int_0^t\int_\Omega \ddot{z}_{,j}\ddot{z}_{,j}dxd\eta + \frac{\rho}{2}\int_\Omega \ddot{v}_i\ddot{v}_i dx + \frac{\mu}{2}\int_\Omega \dot{v}_{i,j}\dot{v}_{i,j}dx \\
&\le \frac{kk_2}{2}\int_{\Omega(0)} \nu_i^0\nu_i^0 dx + \frac{kk_3}{2}\int_{\Omega(0)} \alpha_{,j}^0\alpha_{,j}^0 dx + \frac{\rho}{2}\int_{\Omega(0)} X_i^0X_i^0 dx + \frac{\mu}{2}\int_{\Omega(0)} \nu_{i,j}^0\nu_{i,j}^0 dx \\
&= k_4(x).
\end{aligned}$$

Using the Poincare inequality, we obtain

$$\int_{\Omega(0)} (\dot{z})^2 dx \le \frac{1}{\lambda^*}\int_{\Omega(0)} \dot{z}_{,j}\dot{z}_{,j}dx, \tag{2.28}$$

$$\int_\Omega (\dot{z})^2 dx \le \frac{1}{\lambda^*}\int_\Omega \dot{z}_{,j}\dot{z}_{,j}dx, \tag{2.29}$$

and

$$\frac{1}{\lambda^*}\int_\Omega \ddot{z}_{,j}\ddot{z}_{,j}dx \ge \int_\Omega (\ddot{z})^2 dx, \tag{2.30}$$

where λ^* is the smallest eigenvalue in the membrane problem for Ω.

From (2.25) and (2.27), we obtain

$$\int_{\Omega(0)} (\dot{z})^2 dx \le \frac{k_2}{\lambda^*}\int_{\Omega(0)} \nu_i^0\nu_i^0 dx + \frac{k_3}{\lambda^*}\int_{\Omega(0)} \alpha_{,j}^0\alpha_{,j}^0 dx. \tag{2.31}$$

Combining (2.26), (2.28) and (2.29), we get

$$\int_0^t\int_\Omega (\ddot{z})^2 dxd\eta \le \frac{1}{\lambda^* b}k_4(x) \tag{2.32}$$

and

$$\int_\Omega (\dot{z})^2 dx \le \frac{2}{\lambda^* k}k_4(x). \tag{2.33}$$

On combining (2.17), (2.20), (2.30), (2.31) and (2.32), we get

$$\begin{aligned}
&\frac{c}{2}\int_\Omega (\dot{\pi})^2 dx + \frac{k}{2}\int_\Omega \pi_{,i}\pi_{,i}dx + b\int_0^t\int_\Omega \dot{\pi}_{,i}\dot{\pi}_{,i}dxd\eta \\
&\quad + \frac{\rho}{2}\int_\Omega \dot{w}_i\dot{w}_i dx + \frac{\mu}{2}\int_\Omega w_{i,j}w_{i,j}dx + \frac{\lambda+\mu}{2}\int_\Omega (w_{j,j})^2 \\
&\le c\int_{\Omega(0)} (\theta^0)^2 dx + \frac{ck_2}{2\lambda^*}\int_{\Omega(0)} \nu_i^0\nu_i^0 dx + \frac{ck_3}{2\lambda^*}\int_{\Omega(0)} \alpha_{,j}^0\alpha_{,j}^0 dx
\end{aligned} \tag{2.34}$$

$$+\frac{c}{\lambda^* k}k_4(x)+c\sqrt{\frac{k_1(x)t}{\beta\lambda^*}}\sqrt{\frac{k_4(x)}{\lambda^* b}}.$$

Inequality (2.33) is an a priori bound that demonstrates convergence in the measure indicated. The rate of convergence is $O(c)$.

Summarizing all the above discussions, we can establish the following theorem:

Theorem 1. *Let (u_i, α) and (v_i, z) be the classical solutions of the thermoelasticity of type III for different values of $c > 0$ and $c = 0$, respectively, (w_i, π) be the difference of (u_i, α) and (v_i, z), the estimate* (2.33) *is satisfied.*

3. Continuous dependence on the coefficient c

Let (u_i, α) and (v_i, z) be the solutions of the following equations for different values c_1 and c_2, respectively

$$\rho\ddot{u}_i=\mu\Delta u_i+(\lambda+\mu)u_{j,ji}-\beta\dot{\alpha}_{,i} \quad \text{in} \quad \Omega\times(0,t), \tag{3.1}$$

$$c_1\ddot{\alpha}=-\beta\dot{u}_{i,i}+k\Delta\alpha+b\Delta\dot{\alpha} \quad \text{in} \quad \Omega\times(0,t), \tag{3.2}$$

$$u_i=\tilde{u}_i(\mathbf{x}),\ z=\tilde{\alpha}(\mathbf{x}) \quad \text{on} \quad \partial\Omega, \tag{3.3}$$

$$\alpha(\mathbf{x},0)=\alpha^0(\mathbf{x}),\ u_i(x,0)=u_i^0(\mathbf{x}),\ \dot{u}_i(\mathbf{x},0)=\nu_i^0(\mathbf{x}),\ \dot{\alpha}(\mathbf{x},0)=\theta^0(\mathbf{x}), \tag{3.4}$$

and

$$\rho\ddot{v}_i=\mu\Delta v_i+(\lambda+\mu)v_{j,ji}-\beta\dot{z}_{,i} \quad \text{in} \quad \Omega\times(0,t), \tag{3.5}$$

$$c_2\ddot{z}=-\beta\dot{v}_{i,i}+k\Delta z+b\Delta\dot{z} \quad \text{in} \quad \Omega\times(0,t), \tag{3.6}$$

$$v_i=\tilde{u}_i(\mathbf{x}),\ \alpha=\tilde{\alpha}(\mathbf{x}) \quad \text{on} \quad \partial\Omega, \tag{3.7}$$

$$z(\mathbf{x},0)=\alpha^0(\mathbf{x}),\ v_i(\mathbf{x},0)=u_i^0(\mathbf{x}),\ \dot{v}_i(\mathbf{x},0)=\nu_i^0(\mathbf{x}),\ \dot{z}(\mathbf{x},0)=\theta^0(\mathbf{x}). \tag{3.8}$$

We now define w_i, π, c as

$$w_i=u_i-v_i,\ \pi=\alpha-z,\ c=c_1-c_2 \tag{3.9}$$

then (w_i, π) solves the boundary initial problems:

$$\rho\ddot{w}_i=\mu\Delta w_i+(\lambda+\mu)w_{j,ji}-\beta\dot{\pi}_{,i} \quad \text{in} \quad \Omega\times(0,t), \tag{3.10}$$

$$c\ddot{\alpha}+c_2\ddot{\pi}=-\beta\dot{w}_{i,i}+k\Delta\pi+b\Delta\dot{\pi} \quad \text{in} \quad \Omega\times(0,t), \tag{3.11}$$

$$w_i=\pi=0 \quad \text{on} \quad \partial\Omega, \tag{3.12}$$

$$w_i(\mathbf{x},0)=\dot{w}_i(\mathbf{x},0)=\pi(\mathbf{x},0)=\dot{\pi}(\mathbf{x},0)=0 \quad \text{in} \quad \Omega. \tag{3.13}$$

We want to use a time-integrated norm to obtain our results. To do this, we begin with the identity

$$\int_0^t\int_\Omega (t-\eta)\dot{w}_i(\rho\ddot{w}_i-\mu\Delta w_i-(\lambda+\mu)w_{j,ji}+\beta\dot{\pi}_{,i})dxd\eta=0. \tag{3.14}$$

Upon integration in x_i and t, we may deduce from this that

(3.15)
$$\frac{\rho}{2}\int_0^t\int_\Omega \dot{w}_i\dot{w}_i dxd\eta+\frac{\mu}{2}\int_0^t\int_\Omega w_{i,j}w_{i,j}dxd\eta+\frac{\lambda+\mu}{2}\int_0^t\int_\Omega (w_{j,j})^2dxd\eta$$
$$=\ -\beta\int_0^t\int_\Omega (t-\eta)\dot{w}_i\dot{\pi}_{,i}dxd\eta.$$

From (3.11), we also have the identity

$$\int_0^t\int_\Omega (t-\eta)\dot{\pi}(c\ddot{\alpha}+c_2\ddot{\pi}+\beta\dot{w}_{i,i}-k\Delta\pi-b\Delta\dot{\pi})dxd\eta=0. \tag{3.16}$$

We can also obtain by integration by parts

$$\frac{c_2}{2}\int_0^t\int_\Omega \dot{\pi}^2dxd\eta+\frac{k}{2}\int_0^t\int_\Omega \pi_{,j}\pi_{,j}dxd\eta+b\int_0^t\int_\Omega (t-\eta)\dot{\pi}_{,j}\dot{\pi}_{,j}dxd\eta \tag{3.17}$$
$$=\ -c\int_0^t\int_\Omega (t-\eta)\dot{\pi}\ddot{\alpha}dxd\eta-\beta\int_0^t\int_\Omega (t-\eta)\dot{w}_{i,i}\dot{\pi}dxd\eta.$$

On combining (3.15) and (3.17), we obtain

(3.18)
$$\frac{\rho}{2}\int_0^t\int_\Omega \dot{w}_i\dot{w}_i dxd\eta+\frac{\mu}{2}\int_0^t\int_\Omega w_{i,j}w_{i,j}dxd\eta+\frac{\lambda+\mu}{2}\int_0^t\int_\Omega (w_{j,j})^2dxd\eta$$
$$+\frac{c_2}{2}\int_0^t\int_\Omega \dot{\pi}^2dxd\eta+\frac{k}{2}\int_0^t\int_\Omega \pi_{,j}\pi_{,j}dxd\eta+b\int_0^t\int_\Omega (t-\eta)\dot{\pi}_{,j}\dot{\pi}_{,j}dxd\eta$$
$$=\ -c\int_0^t\int_\Omega (t-\eta)\dot{\pi}\ddot{\alpha}dxd\eta.$$

We use the arithmetic-geometric mean inequality on (3.18) to deduce that

(3.19)
$$\frac{\rho}{2}\int_0^t\int_\Omega \dot{w}_i\dot{w}_i dxd\eta+\frac{\mu}{2}\int_0^t\int_\Omega w_{i,j}w_{i,j}dxd\eta+\frac{\lambda+\mu}{2}\int_0^t\int_\Omega (w_{j,j})^2dxd\eta$$
$$+\frac{c_2}{4}\int_0^t\int_\Omega \dot{\pi}^2dxd\eta+\frac{k}{2}\int_0^t\int_\Omega \pi_{,j}\pi_{,j}dxd\eta+b\int_0^t\int_\Omega (t-\eta)\dot{\pi}_{,j}\dot{\pi}_{,j}dxd\eta$$
$$=\frac{c^2t}{c_2}\int_0^t\int_\Omega (t-\eta)(\ddot{\alpha})^2dxd\eta.$$

To estimate the right side of (3.19), we differentiate (3.2) to obtain

$$c_1\ddot{\alpha}_{,t}=-\beta\ddot{u}_{i,i}+k\Delta\dot{\alpha}+b\Delta\ddot{\alpha}. \tag{3.20}$$

Now, multiplying this equation by $(t-\eta)\ddot{\alpha}$, and integrate over $\Omega\times(0,t)$ to see that

$$\frac{c_1}{2}\int_0^t\int_\Omega (\ddot{\alpha})^2dxd\eta+\beta\int_0^t\int_\Omega (t-\eta)\ddot{\alpha}\ddot{u}_{i,i}dxd\eta+\frac{k}{2}\int_0^t\int_\Omega \dot{\alpha}_{,j}\dot{\alpha}_{,j}dxd\eta \tag{3.21}$$

$$
\begin{aligned}
&+ b\int_0^t\int_\Omega (t-\eta)\ddot{\alpha}_{,j}\ddot{\alpha}_{,j}dxd\eta \\
&= \frac{c_1 t}{2}\int_{\Omega(0)}\ddot{\alpha}^2dx + \frac{kt}{2}\int_{\Omega(0)}\theta^0_{,j}\theta^0_{,j}dx.
\end{aligned}
$$

Following the same procedure, from (3.1), we can also get

$$
\begin{aligned}
(3.22)\quad &\frac{\rho}{2}\int_0^t\int_\Omega \ddot{u}_i\ddot{u}_i dxd\eta + \frac{\mu}{2}\int_0^t\int_\Omega \dot{u}_{i,j}\dot{u}_{i,j}dxd\eta + \frac{\lambda+\mu}{2}\int_0^t\int_\Omega(\dot{u}_{j,j})^2dxd\eta \\
&+ \beta\int_0^t\int_\Omega (t-\eta)\ddot{\alpha}_{,i}\ddot{u}_i dxd\eta \\
&= \frac{\rho t}{2}\int_{\Omega(0)}\ddot{u}_i\ddot{u}_i dx + \frac{\mu t}{2}\int_{\Omega(0)}\nu^0_{i,j}\nu^0_{i,j}dx + \frac{(\lambda+\mu)t}{2}\int_{\Omega(0)}(\nu^0_{j,j})^2dx.
\end{aligned}
$$

On combining (3.21) and (3.22), we obtain

(3.23)

$$
\begin{aligned}
&\frac{c_1}{2}\int_0^t\int_\Omega(\ddot{\alpha})^2dxd\eta + \frac{k}{2}\int_0^t\int_\Omega \dot{\alpha}_{,j}\dot{\alpha}_{,j}dxd\eta + b\int_0^t\int_\Omega(t-\eta)\ddot{\alpha}_{,j}\ddot{\alpha}_{,j}dxd\eta \\
&+ \frac{\rho}{2}\int_0^t\int_\Omega \ddot{u}_i\ddot{u}_i dxd\eta + \frac{\mu}{2}\int_0^t\int_\Omega \dot{u}_{i,j}\dot{u}_{i,j}dxd\eta + \frac{\lambda+\mu}{2}\int_0^t\int_\Omega(\dot{u}_{j,j})^2dxd\eta \\
= &\frac{c_1 t}{2}\int_{\Omega(0)}\ddot{\alpha}^2dx + \frac{kt}{2}\int_{\Omega(0)}\theta^0_{,j}\theta^0_{,j}dx + \frac{\rho t}{2}\int_{\Omega(0)}\ddot{u}_i\ddot{u}_i dx + \frac{\mu t}{2}\int_{\Omega(0)}\nu^0_{i,j}\nu^0_{i,j}dx \\
&+ \frac{(\lambda+\mu)t}{2}\int_{\Omega(0)}(\nu^0_{j,j})^2dx.
\end{aligned}
$$

If we assume that the system of equations are satisfied at $t=0$, thus, we obtain

$$
(3.24)\qquad X_i^0 = \ddot{u}_i|_{t=0} = \rho^{-1}(\mu\triangle u_i^0 + (\lambda+\mu)u^0_{j,j} - \beta\theta^0_{,j}),
$$

$$
(3.25)\qquad Y_i^0 = \ddot{\alpha}|_{t=0} = c_1^{-1}(-\beta\nu^0_{i,i} + k\alpha^0_{,jj} + b\theta^0_{,jj}).
$$

On combining (3.19), (3.23), (3.24) and (3.25), we obtain

(3.26)

$$
\begin{aligned}
&\frac{\rho}{2}\int_0^t\int_\Omega \dot{w}_i\dot{w}_i dxd\eta + \frac{\mu}{2}\int_0^t\int_\Omega w_{i,j}w_{i,j}dxd\eta + \frac{\lambda+\mu}{2}\int_0^t\int_\Omega(w_{j,j})^2dxd\eta \\
&+ \frac{c_2}{4}\int_0^t\int_\Omega \dot{\pi}\dot{\pi}dxd\eta + \frac{k}{2}\int_0^t\int_\Omega \pi_{,j}\pi_{,j}dxd\eta + b\int_0^t\int_\Omega(t-\eta)\dot{\pi}_{,j}\dot{\pi}_{,j}dxd\eta \\
\le &\frac{2c^2t^2}{c_2c_1}\Big[\frac{c_1 t}{2}\int_{\Omega(0)}Y_i^0Y_i^0dx + \frac{kt}{2}\int_{\Omega(0)}\theta^0_{,j}\theta^0_{,j}dx + \frac{\rho t}{2}\int_{\Omega(0)}X_i^0X_i^0dx \\
&+ \frac{\mu t}{2}\int_{\Omega(0)}\nu^0_{i,j}\nu^0_{i,j}dx + \frac{(\lambda+\mu)t}{2}\int_{\Omega(0)}(\nu^0_{j,j})^2dx\Big].
\end{aligned}
$$

Estimate (3.26) establishes continuous dependence on c provided c_1 and c_2 are not small. The rate of the continuous is $O(c^2)$. The case of small c_1 is discussed in Section 2.

Following the discussions above, we can establish the following theorem:

Theorem 2. *Let (u_i, α) and (v_i, z) be the classical solutions of the thermoelasticity of type III for different values of c_1 and c_2 respectively, (w_i, π) be the difference of (u_i, α) and (v_i, z), and $c = c_1 - c_2$, then the estimate (3.26) is satisfied.*

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