Bull. Korean Math. Soc. **51** (2014), No. 5, pp. 1241–1257
http://dx.doi.org/10.4134/BKMS.2014.51.5.1241

GLOBAL WEAK MORREY ESTIMATES FOR SOME ULTRAPARABOLIC OPERATORS OF KOLMOGOROV-FOKKER-PLANCK TYPE

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Abstract. We consider a class of hypoelliptic operators of the following type

$$L=\sum_{i,j=1}^{p_0} a_{ij}\partial^2_{x_ix_j}+\sum_{i,j=1}^{N} b_{ij}x_i\partial_{x_j}-\partial_t,$$

where (a_{ij}), (b_{ij}) are constant matrices and (a_{ij}) is symmetric positive definite on $\mathbb{R}^{p_0}$ $(p_0\leqslant N)$. By establishing global Morrey estimates of singular integral on the homogenous space and the relation between Morrey space and weak Morrey space, we obtain the global weak Morrey estimates of the operator L on the whole space $\mathbb{R}^{N+1}$.

1. Introduction and main results

Let us concern a class of ultraparabolic operators of Kolmogorov-Fokker-Planck type in $\mathbb{R}^{N+1}$:

$$L_0=div(A\nabla)+\langle x,B\nabla\rangle-\partial_t=\sum_{i,j=1}^{N} a_{ij}\partial^2_{x_ix_j}+\sum_{i,j=1}^{N} b_{ij}x_i\partial_{x_j}-\partial_t, \tag{1.1}$$

where $1\leqslant p_0\leqslant N$, $A=(a_{ij})$ and $B=(b_{ij})$ are $N\times N$ matrices with constant real entries, $\nabla=(\partial_{x_1},\partial_{x_2},\dots,\partial_{x_N})$, div and $\langle\cdot,\cdot\rangle$ denote the gradient, the divergence and the inner product in $\mathbb{R}^N$, separately. The matrix A is supposed to be symmetric and positive semidefinite. We also assume that the following condition holds:

(H_0) $\mathrm{Ker}(A)$ does not contain nontrivial subspaces which are invariant for B.

Received March 28, 2011; Revised November 21, 2013.

2010 *Mathematics Subject Classification.* 35R05, 35B45, 42B20.

Key words and phrases. ultraparabolic operators, weak Morrey estimates, homogeneous type space.

This work was supported by the National Natural Science Foundation of China (Grant Nos. 11271299 and 11001221), the Mathematical Tianyuan Foundation of China (No. 11126027), Natural Science Foundation of Shanxi Province (2014021009-1) and the Shanxi University Yinjin Rencai Jianshe Xiangmu (No. 010951801004).

Hörmander in [12] pointed out that (H_0) implies (actually, is equivalent to) the hypoellipticity of (1.1). By introducing the matrix

$$C(t) = \int_0^t E(s)AE^T(s)ds, \tag{1.2}$$

where $E(s) = \exp(-sB^T)$, the authors in [14] showed that (H_0) is equivalent to the condition

$$C(t) > 0 \quad \text{for every } t > 0. \tag{1.3}$$

It is interesting to remark that the condition (1.3) can also be expressed in geometric-differential terms. In fact, setting

$$X_i = \sum_{j=1}^N a_{ij}\partial_{x_j},\ i = 1,\dots,N,\ Y = \langle x, B\nabla\rangle,$$

then (1.3) is equivalent to the following Hörmander's condition

$$\operatorname{rank}\mathcal{L}(X_1, X_2, \dots, X_N, Y)(x) = N,\ x \in \mathbb{R}^N, \tag{1.4}$$

where $\mathcal{L}(X_1, X_2, \dots, X_N, Y)$ denotes the Lie algebra generated by $X_1, X_2, \dots,$ X_N, Y. The proof of the equivalence between (H_0) and (1.4) is implicitly contained in the introduction of [12], and Kuptsov in [13] gave an explicit proof of the equivalence between (1.3) and (1.4).

The authors in [14] also proved that (1.4) implies that, for some basis on $\mathbb{R}^N$, the matrices A and B take the form:

$$A = \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix} \tag{1.5}$$

and

$$B = \begin{pmatrix} * & B_1 & 0 & \cdots & 0 \\ * & * & B_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & B_r \\ * & * & * & \cdots & * \end{pmatrix}, \tag{1.6}$$

respectively, where $A_0 = (a_{ij})_{i,j=1}^{p_0}$ is a $p_0 \times p_0$ constant matrix ($p_0 \leqslant N$) with rank p_0; B_j is a $p_{j-1} \times p_j$ block with rank p_j, $j = 1, 2, \dots, r$. Moreover $p_0 \geqslant p_1 \geqslant \cdots \geqslant p_r \geqslant 1$ and $p_0 + p_1 + \cdots + p_r = N$.

Specially, if we denote by B_0 the matrix obtained by annihilating all the $*$ blocks of the matrix written as (1.6), then the operator L_0 becomes

$$L = \operatorname{div}(A\nabla) + \langle x, B_0\nabla\rangle - \partial_t = \sum_{i,j=1}^{p_0} a_{ij}\partial^2_{x_i x_j} + \sum_{i,j=1}^{N} b_{ij}x_i\partial_{x_j} - \partial_t,$$

which is the principal part of L_0. In this paper, we will consider the operator L and make the following assumption:

(H_1) $A_0 = (a_{ij})_{i,j=1}^{p_0}$ is symmetric and positive definite, and there exists positive constant ν such that

$$\nu|\xi|^2 \leqslant \sum_{i,j=1}^{p_0} a_{ij}\xi_i\xi_j \leqslant \frac{1}{\nu}|\xi|^2 \tag{1.7}$$

for all $\xi \in \mathbb{R}^{p_0}$.

It is known that L is hypoelliptic (see [14]). On the other hand, L is a heat operator when $p_0 = N,\ B = 0$ and the degenerate operators (i.e., with $p_0 < N$) appear in many research fields. For instance, the Kolmogorov equation

$$\partial_{x_1}^2 u + x_1\partial_{x_2} u = \partial_t u,\ (x,t) \in \mathbb{R}^3$$

occurs in the financial problem (see [1, 10]), in the kinetic theory (see [6, 16]) as well as in the visual perception problem (see [18]).

We know that L is a class of Kolmogorov-Fokker-Planck ultraparabolic operator. Owing to its importance in physics and in mathematical finance, it has been extensively studied (see [3, 4, 11, 14, 19, 20]). The authors in [11, 14, 19, 20] proved an invariant Harnack inequality for the non-negative solutions of the equation $Lu = 0$. The local L^p estimates have been studied by the authors in [3] and [4].

There are also some authors studied the Morrey estimates for some operators (see [7, 15, 22]). The local Morrey estimates for second-order nondivergence elliptic operators in Euclidean spaces were established by G. Fazio and M. Ragusa in [7]. G. Lieberman in [15] derived directly the local Morrey estimates for some second-order nondivergence elliptic and parabolic operators. For parabolic nondivergence operators of Hörmander type, S. Tang and P. Niu in [22] checked the local Sobolev-Morrey estimates. In this paper, we investigate the global weak Morrey estimates for the operator L. Moreover, the Hölder estimates of the operator L under some certain conditions are given.

To state our main results, we introduce some notations and function spaces.

Definition 1.1 (Morrey space)**.** We say that a measurable function $f \in L^p_{loc}(\mathbb{R}^{N+1})$ belongs to the Morrey space $L^{p,\lambda}(\mathbb{R}^{N+1})$ with $p \in (1, +\infty)$ and $\lambda \in [0, Q+2]$, if the norm

$$\|f\|_{L^{p,\lambda}(\mathbb{R}^{N+1})} = \left(\sup_{r>0} \frac{1}{r^\lambda}\int_{B_r} |f(x)|^p dx\right)^{1/p}$$

is finite, where Q and B_r are described in (2.2) and (2.5), separately.

Letting $Y_i = \partial_{x_i}$ $(i = 1, 2, \ldots, p_0)$, and noting $Y_0 = \langle x, B\nabla\rangle - \partial_t$, we use the simplified notations:

$$\|Du\|_{L^{p,\lambda}(\mathbb{R}^{N+1})} = \sum_{j=1}^{p_0} \|Y_j u\|_{L^{p,\lambda}(\mathbb{R}^{N+1})};$$

$$\|D^2u\|_{L^{p,\lambda}(\mathbb{R}^{N+1})} = \sum_{i,j=1}^{p_0} \|Y_iY_ju\|_{L^{p,\lambda}(\mathbb{R}^{N+1})} + \|Y_0u\|_{L^{p,\lambda}(\mathbb{R}^{N+1})}; \tag{1.8}$$

and

$$\|D^ku\|_{L^{p,\lambda}(\mathbb{R}^{N+1})} = \sum \|Y_{j_1}\cdots Y_{j_l}u\|_{L^{p,\lambda}(\mathbb{R}^{N+1})} \text{ for } k > 2, \tag{1.9}$$

where the sum is taken over all monomials $Y_{j_1}\cdots Y_{j_l}$ which are homogeneous of degree k. (Remark that Y_0 has degree two while the remaining fields have degree one.)

Definition 1.2. Let $p \in (1, +\infty)$, $\lambda \in [0, Q+2]$ and k be a nonnegative integer. We define Sobolev-Morrey spaces $S^{k,p,\lambda}(\mathbb{R}^{N+1})$ which consists of all $L^{p,\lambda}(\mathbb{R}^{N+1})$ functions with k-th derivatives with respect to vector fields Y_i's $(i = 0, 1, \dots, p_0)$. The Sobolev-Morrey norm is defined by

$$\|u\|_{S^{k,p,\lambda}(\mathbb{R}^{N+1})} = \sum_{h=0}^{k} \|D^hu\|_{L^{p,\lambda}(\mathbb{R}^{N+1})}.$$

Definition 1.3. Let $f \in L^p(\mathbb{R}^{N+1})$, $z_0 \in \mathbb{R}^{N+1}$, $\rho > 0$, $\tau \geqslant 0$, and set

$$A_{\tau,\rho}(f) = \{z \in B_\rho(z_0) \,|\, |f(z)| > \tau\},$$

where $B_\rho(z_0)$ is given in (2.5). The function

$$\lambda_f(\tau,\rho) = |A_{\tau,\rho}(f)|$$

is called a distribution function of f.

Definition 1.4 (Weak Morrey space)**.** For $p \in (1,\infty)$, $\lambda \in [0, Q+2]$, a measurable function f is said to belong to a weak Morrey space (denoted by $L^{p,\lambda}_w(\mathbb{R}^{N+1})$), if

$$\|f\|_{L^{p,\lambda}_w(\mathbb{R}^{N+1})} = \sup_{\rho>0} \inf_{\tau>0} \{A \,|\, \lambda_f(\tau,\rho) \leqslant \rho^\lambda \tau^{-p} A^p\}$$

is finite.

The main results in this paper are as follows.

Theorem 1.1. *For every $p \in (1,\infty)$, $\lambda \in [0, Q+2)$, there exists a positive constant C, depending on p, p_0, the matrix B and the number ν in* (1.7) *such that*

$$\|D^{k+2}u\|_{L^{p,\lambda}_w(\mathbb{R}^{N+1})} \leqslant C\|Lu\|_{S^{k,p,\lambda}(\mathbb{R}^{N+1})}$$

for every $u \in C_0^\infty(\mathbb{R}^{N+1})$ and nonnegative integer k.

Theorem 1.2. *If $2p + \lambda > Q + 2$, $p + \lambda < Q + 2$ and $\theta = \frac{2p+\lambda-(Q+2)}{p}$, then there exists a positive constant C, depending only on p, λ and the operator L, such that for every $u \in C_0^\infty(\mathbb{R}^{N+1})$,*

$$\frac{|u(z) - u(w)|}{\|z^{-1} \circ w\|^\theta} \leqslant C\|Lu\|_{L^{p,\lambda}(\mathbb{R}^{N+1})},$$

for every $z, w \in \mathbb{R}^{N+1}$, $z \neq w$, *where* $\circ$ *is the group law given in Section 2;*

If $p+\lambda > Q+2$ *and* $\delta = \frac{p+\lambda-(Q+2)}{p}$, *then there exists a positive constant* C, *depending only on* p, λ *and the operator* L, *such that for every* $u \in C_0^\infty(\mathbb{R}^{N+1})$,

$$\frac{|\partial_{x_j} u(z) - \partial_{x_j} u(w)|}{\|z^{-1} \circ w\|^\delta} \leqslant C\|Lu\|_{L^{p,\lambda}(\mathbb{R}^{N+1})}$$

for every $z, w \in \mathbb{R}^{N+1}$, $z \neq w$ *and* $j = 1, 2, \ldots, p_0$.

The paper is organized as follows: In Section 2, we introduce some preliminary and known results which will be used later. Section 3 is first devoted to obtaining global second order Morrey estimates and the higher order Morrey estimates by using global L^p estimate. And then, we check the relation between the global Morrey space and the global weak Morrey space. Using these conclusions, Theorem 1.1 is proved. The proof of Theorem 1.2 is given in Section 4.

2. Preliminary

It is proved in [14] that the operator L is left-invariant with respect to the Lie group $\mathcal{K}$ whose underlying manifold is $\mathbb{R}^{N+1}$, endowed with the composition law

$$(x,t) \circ (\xi, \tau) = (\xi + E(\tau)x, t+\tau),$$

where $E(\tau) = \exp(-\tau B^T)$ and B^T denotes the transpose of B. Note that

$$(\xi,\tau)^{-1} = (-E(-\tau)\xi, -\tau).$$

There exists a group of dilations on $\mathbb{R}^{N+1}$, which we denote by $(D(\lambda))_{\lambda>0}$. More precisely, $D(\lambda)$ is defined by

$$D(\lambda) = \operatorname{diag}(\lambda^{\alpha_1}, \lambda^{\alpha_2}, \ldots, \lambda^{\alpha_N}, \lambda^2), \tag{2.1}$$

where

$$\begin{aligned}&\alpha_1 = \cdots = \alpha_{p_0} = 1,\ \alpha_{p_0+1} = \cdots = \alpha_{p_0+p_1} = 3, \ldots,\\ &\alpha_{p_0+\cdots+p_{r-1}+1} = \cdots = \alpha_N = 2r+1.\end{aligned}$$

Therefore, we can write

$$D(\lambda) = \operatorname{diag}(\lambda I_{p_0}, \lambda^3 I_{p_1}, \ldots, \lambda^{2r+1} I_{p_r}, \lambda^2),$$

where I_{p_j}, $D(\lambda)$ denote the $p_j \times p_j$ identity matrix and the matrix of dilations on $\mathbb{R}^{N+1}$, respectively. Note that

$$\det(D(\lambda)) = \lambda^{Q+2},$$

where

$$Q+2 = p_0 + 3p_1 + \cdots + (2r+1)p_r + 2 \tag{2.2}$$

is called the homogeneous dimension of $\mathbb{R}^{N+1}$ with respect to $(D(\lambda))_{\lambda>0}$.

Definition 2.1. We say that a differential operator Y on $\mathbb{R}^{N+1}$ is homogeneous of degree $\beta > 0$, if

$$Y(\varphi(D(\lambda)z)) = \lambda^\beta (Y\varphi)(D(\lambda)z),\ z \in \mathbb{R}^{N+1},\ \lambda > 0$$

for every test function φ. Also, we say that a function f is homogeneous of degree α if

$$f((D(\lambda)z)) = \lambda^\alpha f(z),\ \lambda > 0,\ z \in \mathbb{R}^{N+1}.$$

Clearly, if Y is a homogeneous differential operator of degree β and f is a homogeneous function of degree α, then Yf is homogeneous of degree $\alpha - \beta$. By Definition 2.1, it is easy to show that the operator L is homogeneous of degree two with respect to the dilations $D(\lambda)$, i.e.,

$$L(u(D(\lambda)z)) = \lambda^2 (Lu)(D(\lambda)z),\ z \in \mathbb{R}^{N+1},\ \lambda > 0$$

for every $u \in C_0^\infty(\mathbb{R}^{N+1})$.

We introduce a norm and a quasidistance in $\mathbb{R}^{N+1}$, related to the groups of translations and dilations defined above.

Definition 2.2. Let $z = (x_1, x_2, \dots, x_N, t) \in \mathbb{R}^{N+1}$, if $z = 0$ we set $\|z\| = 0$, while if $z \in \mathbb{R}^{N+1} \backslash \{0\}$ we define $\|z\| = \varrho$, where ϱ is the unique positive solution to the equation

$$\frac{x_1^2}{\varrho^{2\alpha_1}} + \frac{x_2^2}{\varrho^{2\alpha_2}} + \cdots + \frac{x_N^2}{\varrho^{2\alpha_N}} + \frac{t^2}{\varrho^4} = 1,$$

where $\alpha_1, \alpha_2, \dots, \alpha_N$ are the positive integers in (2.1).

Bramanti and Cerutti in [4] showed that the norm $\| \cdot \|$ satisfies

$$\|z^{-1}\| \leqslant c_1 \|z\|,\ z \in \mathbb{R}^{N+1} \tag{2.3}$$

and

$$\|z \circ \zeta\| \leqslant c_2(\|z\| + \|\zeta\|),\ z, \zeta \in \mathbb{R}^{N+1}, \tag{2.4}$$

where the positive constants c_1 and c_2 depend only on the matrix B.

Remark 2.1. There is a natural homogeneous norm in $\mathbb{R}^{N+1}$, induced by dilation $D(\lambda)$:

$$\|(x,t)\| = \sum_{j=1}^{N} |x_j|^{1/\alpha_j} + |t|^{1/2}.$$

Clearly, we have

$$\|D(\lambda)z\| = \lambda \|z\|,\ \lambda > 0,\ z \in \mathbb{R}^{N+1}.$$

Definition 2.3. For every $z, w \in \mathbb{R}^{N+1}$, define a quasidistance by

$$d(z, w) = \|w^{-1} \circ z\|.$$

The ball with respect to d is denoted by

$$B(z,r) = B_r(z) = \{w \in \mathbb{R}^{N+1} : d(z,w) < r\}. \tag{2.5}$$

Since $B(0,r) = D(r)B(0,1)$ and $\det(D(\lambda)) = \lambda^{Q+2}$, we also have

$$|B_r(0)| = r^{Q+2}|B_1(0)|,$$

where $|B_1(0)| = \omega_{N+1}$ is the Lebesgue measure of the Euclidean unit ball of $\mathbb{R}^{N+1}$. This implies that the Lebesgue measure dz is a doubling measure with respect to d, since

$$|B(z,2r)| = 2^{Q+2}|B(z,r)|, \; z \in \mathbb{R}^{N+1}, \; r > 0.$$

Therefore, the space $(\mathbb{R}^{N+1}, dz, d)$ is a space of homogenous type. Recall that if f and g are functions on $\mathbb{R}^{N+1}$, their convolution $f * g$ is defined by

$$f * g(x) = \int_{\mathbb{R}^{N+1}} f(x \circ y^{-1}) g(y) dy = \int_{\mathbb{R}^{N+1}} g(y^{-1} \circ x) f(y) dy.$$

Lemma 2.1 ([14])**.** *The operator L possesses a fundamental solution*

$$\Gamma(z) = \begin{cases} 0, & t \leqslant 0, \\ \frac{(4\pi)^{-N/2}}{\sqrt{\det C(t)}} \exp(-\frac{1}{4}\langle C^{-1}(t)x, x\rangle), & t > 0, \end{cases} \tag{2.6}$$

where $z = (x,t)$ and $C(t)$ is as in (1.2)*. Moreover, $\Gamma \in C^\infty(\mathbb{R}^{N+1} \setminus \{0\})$.*

The authors in [8] and [21] proved a representation formula:

$$u(z) = -(Lu * \Gamma)(z) = -\int_{\mathbb{R}^{N+1}} \Gamma(\zeta^{-1} \circ z) Lu(\zeta) d\zeta. \tag{2.7}$$

And the following formula was given by Bramanti in [5]:

$$\partial^2_{x_i x_j} u(z) = -PV(Lu * \partial^2_{x_i x_j}\Gamma)(z) + c_{ij} Lu(z) \tag{2.8}$$

for every $u \in C_0^\infty(\mathbb{R}^{N+1})$ and some constants c_{ij}, $i,j = 1,2,\ldots,p_0$. The principal value in (2.8) is understood as

$$PV(Lu * \partial^2_{x_i x_j}\Gamma)(z) = \lim_{\varepsilon \to 0} \int_{\|\zeta^{-1} \circ z\| > \varepsilon} (\partial^2_{x_i x_j}\Gamma)(\zeta^{-1} \circ z) Lu(\zeta) d\zeta.$$

Set

$$\Gamma_i(z) = \partial_{x_i}\Gamma(z), \; \Gamma_{ij}(z) = \partial_{x_i}\partial_{x_j}\Gamma(z), \; i,j = 1,2,\ldots,p_0.$$

We also observe that $\Gamma(z)$ is homogeneous of degree $-Q$ with respect to the group $(D(\lambda))_{\lambda>0}$ and $\Gamma_i(z)$ $(i,j = 1,2,\ldots,p_0)$ are homogeneous of degree $-(Q+1)$. Recall that $\Gamma_{ij}(\cdot)$ has the following properties.

Lemma 2.2 ([4])**.** *For $i,j = 1,2,\ldots,p_0$, one has*

(a) $\Gamma_{ij}(\cdot) \in C^\infty(\mathbb{R}^{N+1} \backslash \{0\})$;
(b) $\Gamma_{ij}(\cdot)$ *is homogeneous of degree $-Q-2$;*
(c) *for every $R > r > 0$,*

$$\int_{r<\|z\|<R} \Gamma_{ij}(z) dz = \int_{\|z\|=1} \Gamma_{ij}(z) d\sigma(z) = 0.$$

Lemma 2.3 ([8, 9])**.** *Let K_h be a kernel in $C^\infty(\mathbb{R}^{N+1}\backslash\{0\})$ and homogeneous of degree $h-Q-2$, for some integer h with $0<h<Q+2$. Denote*

$$T_h f = f * K_h$$

and let P^h be a homogeneous left invariant differential operator of degree h. Then

(d)

$$P^h T_h f = P.V.(f * P^h K_h) + \alpha f,$$

where α is a constant depending on P^h and K_h;

(e) *the singular integral operator*

$$f \mapsto P.V.(f * P^h K_h)$$

is continuous on $L^p(\mathbb{R}^{N+1})$ for $1<p<\infty$.

3. Proof of Theorem 1.1

In this section, we first establish global second order Morrey estimates and the higher order Morrey estimates by applying Lemmas 2.2 and 2.3 and then give the relation between the global Morrey space and the global weak Morrey space. Based on these, Theorem 1.1 is proved.

Lemma 3.1. *Let $a\in\mathbb{R}^1$ and $\gamma\in C(\mathbb{R}^{N+1}\backslash\{0\})$ be a homogeneous function with degree a with respect to the group $(D(\lambda))_{\lambda>0}$. Then*

$$|\gamma(z)| \leqslant c\|z\|^a,$$

where $c=\sup_{\Sigma_{N+1}}|\gamma(z)|$, Σ_{N+1} denotes the unit sphere of $\mathbb{R}^{N+1}$.

Proof. Since γ is a homogeneous function with degree a, one has

$$\gamma(z) = \|z\|^a\gamma(D(\|z\|^{-1})z),\ z\in\mathbb{R}^{N+1}\backslash\{0\}.$$

It is clear that $\|D(\|z\|^{-1})z\|=1$, hence $|\gamma(D(\|z\|^{-1})z)|\leqslant c,\ z\in\mathbb{R}^{N+1}\backslash\{0\}$. Therefore, $|\gamma(z)|\leqslant c\|z\|^a$. The proof is completed. □

Let us define a singular integral operator

$$T_{ij}g(z) = \lim_{\varepsilon\to 0}\int_{\|\zeta^{-1}\circ z\|>\varepsilon}(\partial_{x_ix_j}\Gamma)(\zeta^{-1}\circ z)g(\zeta)d\zeta,\ i,j=1,\dots,p_0 \tag{3.1}$$

for every measurable function g.

Lemma 3.2. *Let $p\in(1,\infty),\ \lambda\in[0,Q+2)$. For every $g\in L^{p,\lambda}(\mathbb{R}^{N+1})$, there exists a positive constant c depending on p, λ and the operator L, such that*

$$\|T_{ij}g\|_{L^{p,\lambda}(\mathbb{R}^{N+1})} \leqslant c\|g\|_{L^{p,\lambda}(\mathbb{R}^{N+1})}.$$

Proof. Fix y and $r>0$. For any measurable set $E\subset\mathbb{R}^{N+1}$, its characteristic function is denoted by χ_E. Then we set

$$g_0 = g\chi_{B_r(y)},\quad g_k = g\chi_{B_{2^kr}(y)\backslash B_{2^{k-1}r}(y)},\ k=1,2,\dots,$$

and $\delta = 1/(2c_1^2c_2)$, where c_1 and c_2 are the constants in (2.3) and (2.4). It immediately follows from Lemma 2.3 that

$$\|T_{ij}g\|_{L^p(\mathbb{R}^{N+1})} \leqslant c\|g\|_{L^p(\mathbb{R}^{N+1})},$$

where c is a positive constant. Hence,

$$\|T_{ij}g_0\|_{L^p(B_{\delta r}(y))} \leqslant \|T_{ij}g_0\|_{L^p(\mathbb{R}^{N+1})} \leqslant c\|g\|_{L^p(B_r(y))} \leqslant cr^{\frac{\lambda}{p}}\|g\|_{L^{p,\lambda}(\mathbb{R}^{N+1})}, \tag{3.2}$$

where the last inequality is obtained from the definition of $L^{p,\lambda}(\mathbb{R}^{N+1})$. We now consider $g_k(z)$, where $k \in \mathbb{N}^+$ and $z \in B_{\delta r}(y)$. By Lemma 3.1, there exists $c > 0$ such that

$$|\Gamma_{ij}(\zeta^{-1} \circ z)| \leqslant c\|\zeta^{-1} \circ z\|^{-(Q+2)}. \tag{3.3}$$

Moreover, if $\zeta \in B_{2^k r}(y)\backslash B_{2^{k-1}r}(y)$, then by (2.3) and (2.4),

$$\begin{aligned} \frac{2^{k-1}r}{c_1} &\leqslant \|\zeta^{-1} \circ y\| \\ &\leqslant c_2(\|\zeta^{-1} \circ z\| + \| z^{-1} \circ y\|) \\ &\leqslant c_2(\|\zeta^{-1} \circ z\| + c_1\delta r), \end{aligned}$$

and

$$\|\zeta^{-1} \circ z\| \geqslant \frac{1}{4c_1c_2}2^k r.$$

Thus, from (3.1) and (3.3), it follows

$$\begin{aligned} |T_{ij}g_k(z)| &\leqslant \int_{2^{k-1}r \leqslant \|y^{-1}\circ\zeta\| \leqslant 2^k r} \frac{c}{\|\zeta^{-1} \circ z\|^{Q+2}}|g(\zeta)|d\zeta \\ &\leqslant \frac{c(4c_1c_2)^{Q+2}}{(2^k r)^{Q+2}} \int_{B_{2^k r}(y)} |g(\zeta)|d\zeta \\ &\leqslant c'\|g\|_{L^{p,\lambda}(\mathbb{R}^{N+1})}(2^k r)^{\frac{\lambda-(Q+2)}{p}}; \end{aligned}$$

and by integration over $B_{\delta r}(y)$,

$$\|T_{ij}g_k\|_{L^p(B_{\delta r}(y))} \leqslant c'' r^{\frac{\lambda}{p}}\|g\|_{L^{p,\lambda}(\mathbb{R}^{N+1})}(2^k)^{\frac{\lambda-(Q+2)}{p}}, \tag{3.4}$$

where c'' is a positive constant depending only on p and the constants c_1 and c_2. By (3.2) and (3.4),

$$\frac{1}{(\delta r)^{\frac{\lambda}{p}}}\|T_{ij}g\|_{L^p(B_{\delta r}(y))} \leqslant \delta^{-\frac{\lambda}{p}}\|g\|_{L^{p,\lambda}(\mathbb{R}^{N+1})}\left(c + c''\sum_{k=1}^{\infty}(2^{\frac{\lambda-(Q+2)}{p}})^k\right),$$

which plainly proves the conclusion, since the above series is convergent. □

Theorem 3.1. *For every $p \in (1,\infty)$, $\lambda \in [0, Q+2)$, there exists a constant $C > 0$, depending on p, p_0, the matrix B and the number ν in* (1.7) *such that for every $u \in C_0^\infty(\mathbb{R}^{N+1})$,*

$$\|\partial^2_{x_ix_j}u\|_{L^{p,\lambda}(\mathbb{R}^{N+1})} \leqslant C\|Lu\|_{L^{p,\lambda}(\mathbb{R}^{N+1})},\ i,j = 1,2,\ldots,p_0;$$
$$\|Y_0u\|_{L^{p,\lambda}(\mathbb{R}^{N+1})} \leqslant C\|Lu\|_{L^{p,\lambda}(\mathbb{R}^{N+1})},$$

where $Y_0 = \langle x, B\nabla\rangle - \partial_t$.

Proof. It follows from (2.8) that

$$\partial^2_{x_ix_j}u(z) = -PV(Lu * \partial^2_{x_ix_j}\Gamma)(z) + c_{ij}Lu(z).$$

By Lemma 3.2, there exists a constant C such that

$$\|\partial^2_{x_ix_j}u\|_{L^{p,\lambda}(\mathbb{R}^{N+1})} \leqslant C\|Lu\|_{L^{p,\lambda}(\mathbb{R}^{N+1})},\ i,j = 1,2,\ldots,p_0.$$

The estimate of $\|Y_0u\|_{L^{p,\lambda}(\mathbb{R}^{N+1})}$ yields from

$$Y_0u = Lu - \sum_{i,j=1}^{p_0} a_{ij}\partial^2_{x_ix_j}u. \tag{3.5}$$

This ends the proof. ☐

Lemma 3.3. *Let*

$$P^k = Y_{j_1}\cdots Y_{j_l},\ Y_{j_i} = \partial x_{j_i},\ i = 1,\ldots,l \tag{3.6}$$

for $k = 1,2,\ldots$ *and* $0 \leqslant j_i \leqslant p_0,\ i = 1,2,\ldots,l,$ *where* $Y_{j_1}\cdots Y_{j_l}$ *is homogeneous of degree* k*. Then for any test function* τ*, we have*

$$(P^k\Gamma) * \tau = \Gamma * P^k\tau.$$

Proof. It is easy to check that for any test function τ,

$$(Y_i\Gamma) * \tau = \Gamma * Y_i\tau,\ i = 1,2,\ldots,p_0. \tag{3.7}$$

By (3.5),

$$Y_0 = L - \sum_{i,j=1}^{p_0} a_{ij}\partial^2_{x_ix_j}.$$

Consequently, we have

$$\begin{aligned}(Y_0\Gamma) * \tau = Y_0(\Gamma * \tau) &= L(\Gamma * \tau) - \sum_{i,j=1}^{p_0} a_{ij}\partial^2_{x_ix_j}(\Gamma * \tau)\\ &= (\Gamma * L\tau) - (\Gamma * \sum_{i,j=1}^{p_0} a_{ij}\partial^2_{x_ix_j}\tau)\\ &= \Gamma * (L - \sum_{i,j=1}^{p_0} a_{ij}\partial^2_{x_ix_j})\tau = \Gamma * Y_0\tau.\end{aligned} \tag{3.8}$$

It follows from (3.7) and (3.8) that

$$(P^k\Gamma) * \tau = \Gamma * P^k\tau.$$

This finishes the proof. ☐

Theorem 3.2. *For $p \in (1, \infty)$, $\lambda \in [0, Q+2)$, there exists a constant $C > 0$, depending on p, p_0, the matrix B and the number ν in* (1.7) *such that for every $u \in C_0^\infty(\mathbb{R}^{N+1})$ and nonnegative integer k,*

$$\|D^{k+2}u\|_{L^{p,\lambda}(\mathbb{R}^{N+1})} \leqslant C\|Lu\|_{S^{k,p,\lambda}(\mathbb{R}^{N+1})}. \tag{3.9}$$

Proof. Due to (1.8), (3.6) and Theorem 3.1, we have

$$\|P^2u\|_{L^{p,\lambda}(\mathbb{R}^{N+1})} \leqslant C\|Lu\|_{L^{p,\lambda}(\mathbb{R}^{N+1})}. \tag{3.10}$$

In order to obtain the estimate of $\|D^3u\|_{L^{p,\lambda}(\mathbb{R}^{N+1})}$, it is enough to consider $\|P^3u\|_{L^{p,\lambda}(\mathbb{R}^{N+1})}$ from (1.9) and (3.6). Note that the P^3 can be split as P^2Y_i and Y_iY_0, $i = 1, \ldots, p_0$. In the first case, by Lemma 3.3,

$$P^2Y_iu = P^2Y_i(\Gamma * Lu) = P^2(\Gamma * Y_iLu).$$

Thus, on the basis of Lemma 3.2, there exists a constant C such that

$$\|P^2Y_iu\|_{L^{p,\lambda}(\mathbb{R}^{N+1})} \leqslant \|P^2(\Gamma * Y_iLu)\|_{L^{p,\lambda}(\mathbb{R}^{N+1})} \leqslant C\|Y_iLu\|_{L^{p,\lambda}(\mathbb{R}^{N+1})}. \tag{3.11}$$

In the second case, note that $Y_0\Gamma(z) = -\sum_{i,j=1}^{p_0} a_{ij}\partial^2_{x_ix_j}\Gamma(z)$, $z \in \mathbb{R}^{N+1} \setminus \{0\}$. By Lemmas 3.2 and 3.3, there exists a constant C such that

$$\begin{aligned}
\|Y_iY_0u\|_{L^{p,\lambda}(\mathbb{R}^{N+1})} &= \|Y_iY_0(\Gamma * Lu)\|_{L^{p,\lambda}(\mathbb{R}^{N+1})} \\
&\leqslant \|(Y_iY_0\Gamma * Lu)\|_{L^{p,\lambda}(\mathbb{R}^{N+1})} \\
&= \|(Y_0\Gamma * Y_iLu)\|_{L^{p,\lambda}(\mathbb{R}^{N+1})} \\
&\leqslant C\|Y_iLu\|_{L^{p,\lambda}(\mathbb{R}^{N+1})} \\
&\leqslant C\|DLu\|_{L^{p,\lambda}(\mathbb{R}^{N+1})}.
\end{aligned} \tag{3.12}$$

By (3.11) and (3.12), we have

$$\|P^3u\|_{L^{p,\lambda}(\mathbb{R}^{N+1})} \leqslant C\|DLu\|_{L^{p,\lambda}(\mathbb{R}^{N+1})}. \tag{3.13}$$

For $k \geqslant 2$, we will consider the estimates of $D^{k+2}u$. In fact, by (1.9) and (3.6), we only to show the estimates of $P^{k+2}u$. Let us split the $P^{k+2}u$ as P^2P^k (with $P^2 = Y_0$) and P^3P^{k-1} (with $P^3 = Y_iY_0$ for some $i = 1, 2, \ldots, p_0$). In the first case, using Lemmas 3.2 and 3.3, leads to that there exists a constant C such that

$$\begin{aligned}
\|P^2P^ku\|_{L^{p,\lambda}(\mathbb{R}^{N+1})} &= \|P^2(P^k\Gamma * Lu)\|_{L^{p,\lambda}(\mathbb{R}^{N+1})} \\
&= \|P^2(\Gamma * P^kLu)\|_{L^{p,\lambda}(\mathbb{R}^{N+1})} \\
&\leqslant C\|P^kLu\|_{L^{p,\lambda}(\mathbb{R}^{N+1})} \\
&\leqslant C\|D^kLu\|_{L^{p,\lambda}(\mathbb{R}^{N+1})}.
\end{aligned} \tag{3.14}$$

In the second case, by Lemmas 3.2, 3.3 and Theorem 3.1, there exists a constant C such that

$$\begin{aligned}
\|Y_iY_0P^{k-1}u\|_{L^{p,\lambda}(\mathbb{R}^{N+1})} &= \|Y_iY_0(P^{k-1}\Gamma * Lu)\|_{L^{p,\lambda}(\mathbb{R}^{N+1})} \\
&\leqslant \|(Y_iY_0\Gamma * P^{k-1}Lu)\|_{L^{p,\lambda}(\mathbb{R}^{N+1})} \\
&= \|(Y_0\Gamma * Y_iP^{k-1}Lu)\|_{L^{p,\lambda}(\mathbb{R}^{N+1})} \\
&\leqslant C\|P^kLu\|_{L^{p,\lambda}(\mathbb{R}^{N+1})} \\
&\leqslant C\|D^kLu\|_{L^{p,\lambda}(\mathbb{R}^{N+1})}.
\end{aligned} \tag{3.15}$$

Hence, by (3.14) and (3.15),

$$\|P^{k+2}u\|_{L^{p,\lambda}(\mathbb{R}^{N+1})} \leqslant C\|D^kLu\|_{L^{p,\lambda}(\mathbb{R}^{N+1})}. \tag{3.16}$$

Therefore, (3.9) is followed from (3.10), (3.13) and (3.16). The proof is completed. □

Theorem 3.3. *For every $p \in (1,\infty)$, there exists a constant $C > 0$, such that for every $f \in L^{p,\lambda}(\mathbb{R}^{N+1})$,*

$$\|f\|_{L_w^{p,\lambda}(\mathbb{R}^{N+1})} \leqslant C\|f\|_{L^{p,\lambda}(\mathbb{R}^{N+1})}.$$

Proof. For every $f \in L^{p,\lambda}(\mathbb{R}^{N+1})$,

$$\begin{aligned}
\rho^{-\lambda}\tau^p|A_{\tau,\rho}(f)| &\leqslant \rho^{-\lambda}\int_{A_{\tau,\rho}}|f(z)|^p dz \\
&\leqslant \rho^{-\lambda}\int_{\mathbb{R}^{N+1}}|f(z)|^p dz \\
&\leqslant C\|f\|^p_{L^{p,\lambda}(\mathbb{R}^{N+1})},
\end{aligned}$$

where C is a constant. Then,

$$\|f\|_{L_w^{p,\lambda}(\mathbb{R}^{N+1})} \leqslant C\|f\|_{L^{p,\lambda}(\mathbb{R}^{N+1})}.$$

It ends the proof. □

Proof of Theorem 1.1. For every nonnegative integer k, by Theorem 3.2, we have

$$\|D^{k+2}u\|_{L^{p,\lambda}(\mathbb{R}^{N+1})} \leqslant C\|Lu\|_{S^{k,p,\lambda}(\mathbb{R}^{N+1})}.$$

Hence, from Theorem 3.3, there exists a constant C such that

$$\|D^{k+2}u\|_{L_w^{p,\lambda}(\mathbb{R}^{N+1})} \leqslant C\|Lu\|_{S^{k,p,\lambda}(\mathbb{R}^{N+1})}.$$

The proof is completed. □

4. Hölder continuity

In this section, by demonstrating Hölder estimates of two integral operators, we prove Theorem 1.2.

Lemma 4.1 ([17]). *Let $b \in \mathbb{R}^1$ and $K \in C^1(\mathbb{R}^{N+1}\backslash\{0\})$ be a homogeneous function with degree b with respect to the group $(D(\lambda))_{\lambda>0}$ and there exist two constants $c > 0$ and $M > 1$ such that if $\|z\| \geqslant M\|z^{-1} \circ \zeta\|$. Then*

$$|K(\zeta) - K(z)| \leqslant c\|z^{-1} \circ \zeta\| \cdot \|z\|^{b-1}.$$

Lemma 4.2. *Let $p \in (1, \infty)$ and $\lambda \in [0, Q+2)$. Fixed $w \in \mathbb{R}^{N+1}$, $\alpha \in [0, Q+2)$, $\beta \in (0, Q+2)$ and $\sigma > 0$, for every $g \in L^{p,\lambda}(\mathbb{R}^{N+1})$, we set*

$$T'_\alpha g(z) = \int_{\|\zeta^{-1}\circ z\| \geqslant \sigma\|w^{-1}\circ z\|} \frac{g(\zeta)}{\|\zeta^{-1} \circ z\|^{Q+2-\alpha}} d\zeta$$

and

$$T''_\beta g(z) = \int_{\|\zeta^{-1}\circ z\| < \sigma\|w^{-1}\circ z\|} \frac{g(\zeta)}{\|\zeta^{-1} \circ z\|^{Q+2-\beta}} d\zeta.$$

Then, if $\lambda + p\alpha < Q + 2$, then there exists $c = c(p, \lambda, \alpha, \sigma) > 0$ such that

$$|T'_\alpha g(z)| \leqslant c\|g\|_{L^{p,\lambda}(\mathbb{R}^{N+1})}\|w^{-1} \circ z\|^{\frac{p\alpha+\lambda-(Q+2)}{p}}. \tag{4.1}$$

Moreover, if $\lambda + p\beta > Q + 2$, then there exists $c = c(p, \lambda, \beta, \sigma) > 0$ such that

$$|T''_\beta g(z)| \leqslant c\|g\|_{L^{p,\lambda}(\mathbb{R}^{N+1})}\|w^{-1} \circ z\|^{\frac{p\beta+\lambda-(Q+2)}{p}}. \tag{4.2}$$

Proof. Observing that

$$\begin{aligned}
|T'_\alpha g(z)| &\leqslant \sum_{k=1}^{\infty} \int_{2^{k-1}\sigma\|w^{-1}\circ z\| \leqslant \|\zeta^{-1}\circ z\| < 2^k\sigma\|w^{-1}\circ z\|} \frac{g(\zeta)}{\|\zeta^{-1} \circ z\|^{Q+2-\alpha}} d\zeta \\
&\leqslant \sum_{k=1}^{\infty} \left(\frac{2}{2^k\sigma\|w^{-1} \circ z\|}\right)^{Q+2-\alpha} \int_{B_{2^k c_1\sigma\|w^{-1}\circ z\|}(z)} |g(\zeta)| d\zeta \\
&\leqslant c\|g\|_{L^{p,\lambda}(\mathbb{R}^{N+1})}\|w^{-1} \circ z\|^{\frac{p\alpha+\lambda-(Q+2)}{p}} \sum_{k=1}^{\infty} (2^{\frac{p\alpha+\lambda-(Q+2)}{p}})^k,
\end{aligned}$$

we know that (4.1) is true, since the above series is convergent.

Similarly, by integrating on the set

$$\{\zeta \in \mathbb{R}^{N+1} : 2^{-k}\sigma\|w^{-1} \circ z\| \leqslant \|\zeta^{-1} \circ z\| < 2^{1-k}\sigma\|w^{-1} \circ z\|\},$$

it yields

$$\begin{aligned}
|T''_\beta g(z)| &\leqslant \sum_{k=1}^{\infty} \int_{2^{-k}\sigma\|w^{-1}\circ z\| \leqslant \|\zeta^{-1}\circ z\| < 2^{1-k}\sigma\|w^{-1}\circ z\|} \frac{g(\zeta)}{\|\zeta^{-1} \circ z\|^{Q+2-\beta}} d\zeta \\
&\leqslant \sum_{k=1}^{\infty} \left(\frac{2}{2^{1-k}\sigma\|w^{-1} \circ z\|}\right)^{Q+2-\beta} \int_{B_{2^{1-k} c_1\sigma\|w^{-1}\circ z\|}(z)} |g(\zeta)| d\zeta
\end{aligned}$$

$$\leqslant c\|g\|_{L^{p,\lambda}(\mathbb{R}^{N+1})}\|w^{-1}\circ z\|^{\frac{p\beta+\lambda-(Q+2)}{p}}\sum_{k=1}^{\infty}(2^{\frac{p\beta+\lambda-(Q+2)}{p}})^{-k}.$$

Noting that the above series is convergent, (4.2) is proved. □

Proof of Theorem 1.2. For $u\in C_0^\infty(\mathbb{R}^{N+1})$, by Lemmas 3.1 and 4.1, there exist $M,c>0$ such that

$$\begin{aligned}|u(z)-u(w)| &\leqslant \int_{\mathbb{R}^{N+1}}|\Gamma(\zeta^{-1}\circ z)-\Gamma(\zeta^{-1}\circ w)||Lu(\zeta)|d\zeta\\ &\leqslant \int_{\|\zeta^{-1}\circ z\|\geqslant M\|z^{-1}\circ w\|}\frac{c\|z^{-1}\circ w\|}{\|\zeta^{-1}\circ z\|^{Q+1}}|Lu(\zeta)|d\zeta\\ &\quad+\int_{\|\zeta^{-1}\circ z\|<M\|z^{-1}\circ w\|}\frac{c}{\|\zeta^{-1}\circ z\|^{Q}}|Lu(\zeta)|d\zeta\\ &\quad+\int_{\|\zeta^{-1}\circ z\|<M\|z^{-1}\circ w\|}\frac{c}{\|\zeta^{-1}\circ w\|^{Q}}|Lu(\zeta)|d\zeta\\ &\equiv I_1+I_2+I_3.\end{aligned}$$

By applying Lemma 4.2 and choosing $\alpha=2$ and $\sigma=M/c_1$, there exists a positive constant c such that

$$|I_1|\leqslant c\|Lu\|_{L^{p,\lambda}(\mathbb{R}^{N+1})}\|z^{-1}\circ w\|^{\frac{2p+\lambda-(Q+2)}{p}};\tag{4.3}$$

choosing $\beta=2$ and $\sigma=Mc_1$ in Lemma 4.2, there exists a positive constant c such that

$$|I_2|\leqslant c\|Lu\|_{L^{p,\lambda}(\mathbb{R}^{N+1})}\|z^{-1}\circ w\|^{\frac{2p+\lambda-(Q+2)}{p}};\tag{4.4}$$

choosing $\beta=2$ and $\sigma=c_2(1+M)$ in Lemma 4.2, there exists a positive constant c such that

$$|I_3|\leqslant c\|Lu\|_{L^{p,\lambda}(\mathbb{R}^{N+1})}\|z^{-1}\circ w\|^{\frac{2p+\lambda-(Q+2)}{p}}.\tag{4.5}$$

Hence, by (4.3), (4.4) and (4.5), it is easy to obtain

$$\frac{|u(z)-u(w)|}{\|z^{-1}\circ w\|^{\theta}}\leqslant C\|Lu\|_{L^{p,\lambda}(\mathbb{R}^{N+1})},$$

where C is a positive constant, $z,w\in\mathbb{R}^{N+1},\ z\neq w$.

By (2.7), we write

$$\partial_{x_j}u(z)=-\int_{\mathbb{R}^{N+1}}\Gamma_j(\zeta^{-1}\circ z)Lu(\zeta)d\zeta$$

for every $z\in\mathbb{R}^{N+1}$ and $j=1,2,\dots,p_0$. Analogously, by Lemmas 3.1 and 4.1, we get that there exist $M,c>0$ such that

$$\begin{aligned}|\partial_{x_j}u(z)-\partial_{x_j}u(w)| &\leqslant \int_{\mathbb{R}^{N+1}}|\Gamma_j(\zeta^{-1}\circ z)-\Gamma_j(\zeta^{-1}\circ w)||Lu(\zeta)|d\zeta\\ &\leqslant \int_{\|\zeta^{-1}\circ z\|\geqslant M\|z^{-1}\circ w\|}\frac{c\|z^{-1}\circ w\|}{\|\zeta^{-1}\circ z\|^{Q+2}}|Lu(\zeta)|d\zeta\end{aligned}$$

$$+\int_{\|\zeta^{-1}\circ z\|<M\|z^{-1}\circ w\|}\frac{c}{\|\zeta^{-1}\circ z\|^{Q+1}}|Lu(\zeta)|d\zeta$$
$$+\int_{\|\zeta^{-1}\circ z\|<M\|z^{-1}\circ w\|}\frac{c}{\|\zeta^{-1}\circ w\|^{Q+1}}|Lu(\zeta)|d\zeta$$
$$\equiv I_1'+I_2'+I_3'.$$

By applying Lemma 4.2 and choosing $\alpha = 1$ and $\sigma = M/c_1$, there exists a positive constant c such that

$$|I_1'|\leqslant c\|Lu\|_{L^{p,\lambda}(\mathbb{R}^{N+1})}\|z^{-1}\circ w\|^{\frac{p+\lambda-(Q+2)}{p}}; \tag{4.6}$$

choosing $\beta = 1$ and $\sigma = Mc_1$ in Lemma 4.2, there exists a positive constant c such that

$$|I_2'|\leqslant c\|Lu\|_{L^{p,\lambda}(\mathbb{R}^{N+1})}\|z^{-1}\circ w\|^{\frac{p+\lambda-(Q+2)}{p}}; \tag{4.7}$$

choosing $\beta = 1$ and $\sigma = c_2(1+M)$ in Lemma 4.2, there exists a positive constant c such that

$$|I_3'|\leqslant c\|Lu\|_{L^{p,\lambda}(\mathbb{R}^{N+1})}\|z^{-1}\circ w\|^{\frac{p+\lambda-(Q+2)}{p}}. \tag{4.8}$$

Hence, by (4.6), (4.7) and (4.8), we derive

$$\frac{|\partial_{x_j}u(z)-\partial_{x_j}u(w)|}{\|z^{-1}\circ w\|^{\delta}}\leqslant C\|Lu\|_{L^{p,\lambda}(\mathbb{R}^{N+1})},\ j=1,2,\dots,p_0,$$

where C is a positive constant, $z,w\in\mathbb{R}^{N+1}$, $z\neq w$. This ends the proof. □

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