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THE BASES OF PRIMITIVE NON-POWERFUL COMPLETE SIGNED GRAPHS

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Abstract. The base of a signed digraph S is the minimum number k such that for any vertices u, v of S, there is a pair of walks of length k from u to v with different signs. Let K be a signed complete graph of order n, which is a signed digraph obtained by assigning $+1$ or -1 to each arc of the n-th order complete graph K_n considered as a digraph. In this paper we show that for $n \geq 3$ the base of a primitive non-powerful signed complete graph K of order n is 2, 3 or 4.

1. Introduction

A *sign pattern matrix* M is a square matrix with entries in $\{1, 0, -1\}$. In multiplying two sign pattern matrices, we use the operating rules of entries that continues to hold the signs of the usual addition and multiplication, that is
$1{+}1 = 1$; $(-1){+}(-1) = -1$; $1{+}0 = 0{+}1 = 1$; $(-1){+}0 = 0{+}(-1) = -1$; $0{\cdot}a = a{\cdot}0 = 0$; $1{\cdot}1 = (-1){\cdot}(-1) = 1$; $1{\cdot}(-1) = (-1){\cdot}1 = -1$ for any $a \in \{1, 0, -1\}$.

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In this case we contact the ambiguous situations $1+(-1)$ and $(-1)+1$, which we will use the notation "$\sharp$" as in [2]. Define the addition and multiplication which involving the symbol "$\sharp$" as follows: For any $a \in \Gamma = \{1,\ 0, -1,\ \sharp\}$,

$$(-1)+1 = 1+(-1) = \sharp; \quad a+\sharp = \sharp + a = \sharp$$

$$0 \cdot \sharp = \sharp \cdot 0 = 0; \quad a \cdot \sharp = \sharp \cdot a = \sharp \text{ (when } \ a \neq 0).$$

Matrices with entries in Γ are called *generalized sign pattern matrices.* The addition and multiplication of the entries of generalized sign pattern matrices are defined in the usual way such that they coincide with the operations in sign pattern matrices.

Definition 1. A square generalized sign pattern matrix M is *powerful* if each power of M contains no $\sharp$ entry. A square generalized sign pattern matrix M is called *non-powerful* if it is not powerful.

Definition 2. Let M be a square generalized sign pattern matrix of order n. The smallest number l such that $M^l = M^{l+p}$ for some p is called the *(generalized) base* of M and denoted by $l(M)$. The least positive integer p such that $M^l = M^{l+p}$ for $l = l(M)$ is called to be the *(generalized) period* of M and denote it by $p(M)$.

We introduce some graph theoretic concepts of generalized sign pattern matrices.

A *signed digraph* $S = (V, A, f)$ is a digraph with vertex set V, arc set A and a sign function f defined on A with its value 1 , -1. For $v, w \in V$ we say $f(vw)$ the *sign* of an arc vw, and we denote it by $\text{sgn}(vw)$. The *sign* of a (directed) walk W in S, denoted by $\text{sgn}(W)$ or $f(W)$, is the product of signs of all arcs in W. For example if $W = v_1v_2v_3v_4$, then $\text{sign}(W) = f(W) = f(v_1v_2)f(v_2v_3)f(v_3v_4)$. If two walks W_1 and W_2 have the same initial points, the same terminal points, the same lengths and different signs, then we say that W_1 and W_2 are a *pair of SSSD walks.*

A (signed) digraph S is *primitive* if there is a positive integer k such that for all vertices v, w of S there is a walk of length k from v to w. A signed digraph S is *powerful* if S contains no pair of SSSD walks. Also S is *non-powerful* if it is not powerful. Hence every non-powerful primitive signed digraph contains a pair of SSSD walks. Let $M = M(S) = [a_{ij}]$ be the adjacency matrix of a signed digraph S, that is, the arc (i, j) has sign $\text{sgn}(i, j) = \alpha$ if and only if $a_{ij} = \alpha$ with $\alpha = 1$, or -1. Hence the adjacency (signed) matrix M of a signed digraph S is a sign pattern matrix which satisfies that the $(i, j)-$entry of M^k is 0 if and only if S contains no walk of length k from i to j. Also $(i, j)-$entry of M^k is 1 (or -1) if and only if all walks of length k from i to j in S are of sign 1 (or, -1). The $(i, j)-$entry of M^k is $\sharp$ if and only if S contains a pair of SSSD walks of length k from i to j. We see from the above relations between matrices and digraphs that each power of a signed digraph S contains no pair of SSSD walks if and only if the adjacency matrix M is powerful. Henceforth we may also say that a signed digraph S is powerful or non-powerful if its adjacency sign pattern matrix M is powerful or non-powerful respectively.

From now on we assume that $S = (V, A, f)$ is a primitive non-powerful signed digraph of order n. For each pair of vertices u, v of S, we define the *local base* $l_S(u, v)$ from u to v to be the smallest integer l such that for each $k \geq l$, there is a pair of SSSD walks of length k from u to v in S. The *base* $l(S)$ of S is defined to be $\max\{l_S(u, v)|u, v \in V(S)\}$. It follows directly from the definitions that $l(S) = l(M)$ where M is the adjacency matrix of S.

The upper bounds for the bases of primitive nonpowerful sign pattern matrices are found by You et al. [5]. They also characterized extremal cases completely. Gao et al.[1], Shao and Gao[4] and Li and Liu [3] studied the base and the local base of a primitive non-powerful signed symmetric digraphs with loops.

Let us assume that K is a complete non-powerful signed digraph of order n which is the n-th order complete graph (considered as a digraph) by assigning signs to each arc such that it becomes a non-powerful signed digraph. In this paper we prove that the base of K is less than or equals to 4. As a consequence if all the entries of a non-powerful sign pattern matrix A are nonzero except diagonals, then the all entries of A^4 are $\sharp$. We also provide the examples when the base of K is 2, 3 and 4 respectively.

2. Main theorems

Let $K = (V, A, f)$ be a complete non-powerful signed digraph of order n. That is, K is the n-th order digraph which has unique arc for each ordered pair of vertices of K and signs are assigned to each arc such that K becomes a non-powerful signed digraph. Let $v_1,\ v_2, \cdots, v_r$

be vertices of K. If C is a directed walk from v_1 to v_r which goes through $v_2, v_3, \cdots v_{r-1}$, then we denote C by $v_1v_2\cdots v_{r-1}v_r$ and the sign $f(v_1v_2)f(v_2,v_3)\cdots f(v_{r-1}v_r)$ of C by $f(C) = f(v_1v_2\cdots v_{r-1}v_r) = \mathrm{sgn}(C) = \mathrm{sgn}(v_1v_2\cdots v_{r-1}v_r)$. Throughout this paper we use the notation $u \xrightarrow{k} v$ if there is a walk of length k from a vertex u to another vertex v. The sum $W_1 + W_2$ of two walks $W_1 = v_1v_2\cdots v_n$ and $W_2 = w_1w_2\cdots w_m$ such that $v_n = w_1$ and the inverse $-W_1$ of W_1 are defiened by $W_1 + W_2 = v_1v_2\cdots v_nw_2w_3\cdots w_m$ and $-W_1 = v_nv_{n-1}\cdots v_1$.

THEOREM 1. *The base $l(K)$ of the complete non-powerful signed digraph K of order $n \geq 4$ is less than or equals to* 4.

Proof. It suffices to show that there is a pair of SSSD walks of common length 4 from u to v. Let u, v be vertices of K. Since $n \geq 4$, we can choose a vertex w of K different from u and v. Let σ be the sign of the walk uwu. If there is a vertex x of K such that $x \neq u$ and the sign of the walk uxu is $-\sigma$, then $uwuwv$ and $uxuwv$ are a pair of SSSD walks of length 4 from u to v.

If the sign of the walk uxu is σ for any vertex x of K and there are distinct vertices y, z of K such that $z \neq u$ and the sign of the walk yzy is $-\sigma$, then both y and z are different from u. If $y \neq v$, then $uwuyv$ and $uyzyv$ are a pair of SSSD walks with common length 4 from u to v. If $y = v$, then since $z \neq v$, $uwuzv$ and $uzyzv$ are desired pair of SSSD walks with common length 4 from u to v.

Assume that the sign of the walk yzy is σ for all distinct vertices y, z.

If $\sigma = -1$, then

$$\begin{aligned}\mathrm{sgn}(uvwuv)\mathrm{sgn}(uwvuv) &= f(uv)f(vw)f(wu)f(uv)f(uw)f(wv)f(vu)f(uv) \\ &= (f(uv)f(vw))(f(vw)f(wv))(f(uw)f(wu))(f(uv))^2 \\ &= \mathrm{sgn}(uvu)\mathrm{sgn}(vwv)\mathrm{sgn}(uwu) = \sigma^3 = -1.\end{aligned}$$

Hernce $uvwuv$ and $uwvuv$ are a pair of SSSD walks with common length 4 from u to v.

If $\sigma = 1$, then since K is non-powerful, there is an even cycle of sign -1, or there are two odd cycles with different signs. Assume that there is an even cycle $x_1x_2\cdots x_kx_1$ with sign -1. If $x_i \neq u$ for all $i = 1, 2, \cdots, k$, then since

$$\begin{aligned}\operatorname{sgn}(ux_1x_2u)&\operatorname{sgn}(ux_2x_3u)\cdots\operatorname{sgn}(ux_{k-1}x_ku)\operatorname{sgn}(ux_kx_1u)\\&=(f(ux_1)f(x_1x_2)f(x_2u))(f(ux_2)f(x_2x_3)f(x_3u))\\&\cdots(f(ux_{k-1})f(x_{k-1}x_k)f(x_ku))(f(ux_k)f(x_kx_1)f(x_1u))\\&=f(x_1x_2)f(x_2x_3)\cdots f(x_{k-1}x_k)f(x_kx_1)\\&=\operatorname{sgn}(x_1x_2\cdots x_kx_1)=-1,\end{aligned}$$

among the walks ux_1x_2u, ux_2x_3u, $\cdots$,$ux_{k-1}x_ku$, ux_kx_1u, there are two walks C_1, C_2 with different signs. Thus C_1+uv and C_2+uv are a pair of SSSD walks SSSD walks of common length 4 from u to v.

Let $x_i = u$ for some i. Similarly among the walks

$$ux_1x_2u, ux_2x_3u, \cdots ux_{i-2}x_{i-1}u,\ ux_{i+1}x_{i+2}u,\ ux_{i+2}x_{i+3}u,\ \cdots, ux_{k-1}x_ku,$$

we can find a pair, say C_1' and C_2', of SSSD walks. As a consequence, we nave a pair $C_1'+uv$ and $C_2'+uv$ of SSSD walks of common length 4 from u to v.

Let us assume that there are two odd cycles $y_1y_2\cdots y_ly_1$ and $z_1z_2\cdots z_mz_1$ with signs 1 and -1 respectively. We want to show that there is a walk $C_3 = uy_ty_{t+1}u$ (or $C_3 = uy_ly_1u$) of sign $+1$. If $u \neq y_i$ for all $i = 1, 2, \cdots, l$, then since

$$\operatorname{sgn}(uy_1y_2u)\operatorname{sgn}(uy_2y_3u)\cdots\operatorname{sgn}(uy_{l-1}y_lu)\operatorname{sgn}(uy_ly_1u)=\operatorname{sgn}(y_1y_2\cdots y_ly_1)=1,$$

among the walks uy_1y_2u, uy_2y_3u, $\cdots$,$uy_{l-1}y_lu$, uy_ly_1u, there is a walk C_3 with sign $+1$. If $u = y_i$ for some i, then since

$$\begin{aligned}&\operatorname{sgn}(uy_1y_2u)\operatorname{sgn}(uy_2y_3u)\cdots\operatorname{sgn}(uy_{i-2}y_{i-1}u)\\&\operatorname{sgn}(uy_{i+1}y_{i+2}u)\cdots\operatorname{sgn}(uy_{l-1}y_lu)\operatorname{sgn}(uy_ly_1u)\\&=\operatorname{sgn}(y_1y_2\cdots y_ly_1)=1,\end{aligned}$$

we have a walk from u to v of length 3 with sign 1.

Similarly among the walks uz_1z_2u, uz_2z_3u,$\cdots$,$uz_{m-1}z_mu$, uz_mz_1u, there is a walk C_4 of sign -1. Thus C_3+uv and C_4+uv are a pair of SSSD walks with common length 4 from u to v. As a consequence, we have $l(K) \leq 4$. □

We will show the upper bound 4 in Theorem 1 is extremal by constructing a complete nonpowerful signed digraph of base at least 4.

THEOREM 2. *Let* $V = \{v_1, v_2, \cdots, v_n\}$, $A = \{(v_i, v_j) | 1 \leq i, j \leq n, i \neq j\}$ *and* $f : A \longrightarrow \{-1, 1\}$ *such that*

$$f(v_i, v_j) = \begin{cases} -1, & \text{if } j = 3 \text{ and } i \neq 1, \text{ or } (i,j) = (3,2); \\ 1, & \text{otherwise.} \end{cases}$$

The signed digraph $G = (V, A, f)$ *is primitive non-powerful and* $l(G) \geq 4$.

Proof. Let W be a walk of length 3 from v_1 to v_2. Then $W = v_1 v_i v_j v_2$ for some i, j. If $i = 2$, then for all $j \neq 2$ since $f(v_2 v_j v_2) = f(v_2 v_j) f(v_j v_2) = 1$, we have $\text{sgn}(v_1 v_2 v_j v_2) = 1$. If $i = 3$, then $j \neq 3$. Hence $f(v_1 v_3 v_j v_2) = f(v_1 v_3) f(v_3 v_j) f(v_j v_2) = 1$. If $i \geq 4$ and $j = 3$, then $\text{sgn}(v_1 v_i v_3 v_2) = f(v_1 v_i) f(v_i v_3) f(v_3 v_2) = 1(-1)(-1) = 1$. If $i \geq 4$ and $j \neq 3$, then $\text{sgn}(v_1 v_i v_j v_2) = f(v_1 v_i) f(v_i v_j) f(v_j v_2) = 1$. Hence the sign of a walk of length 3 from v_1 and v_2 is always 1. We have $l(v_1, v_2) \geq 4$, and hence $l(G) \geq 4$. By Theorem 1, we conclude that $l(G) = 4$. □

We can easily see that the base of a primitive non-powerful digraph is at least 2. In the following examples we provide two complete signed graphs of order $n \geq 4$ with base 2 and 3 respectively. As a result, the possible base of a complete signed graph of order $n \geq 4$ is 2, 3 and 4.

EXAMPLE 1. Let $n \geq 4$, $V = \{v_1, v_2, \cdots, v_n\}$, $A = \{(v_i, v_j) | 1 \leq i, j \leq n, i \neq j\}$ and $f : A \longrightarrow \{-1, 1\}$ such that

$$f(v_i v_j) = \begin{cases} -1, & \text{if } j = 3 \text{ and } i \neq 1, (i,j) = (3,2), \text{ or } (i,j) = (1,2), \\ 1, & \text{otherwise} \end{cases}.$$

We find a pair of SSSD walks of length 2 from v_i to v_j as follows for each i and j.

$$\begin{array}{ll}
v_1 v_2 v_1 \text{ and } v_1 v_3 v_1 & \text{if } i = 1 \text{ and } j = 2, \\
v_1 v_3 v_2 \text{ and } v_1 v_4 v_2 & \text{if } i = 1 \text{ and } j = 2, \\
v_1 v_2 v_3 \text{ and } v_1 v_4 v_3 & \text{if } i = 1 \text{ and } j = 3, \\
v_1 v_2 v_j \text{ and } v_1 v_3 v_j & \text{if } i = 1 \text{ and } j \geq 4, \\
v_2 v_3 v_1 \text{ and } v_2 v_4 v_1 & \text{if } i = 2 \text{ and } j = 1, \\
v_2 v_1 v_2 \text{ and } v_2 v_3 v_2 & \text{if } i = 1 \text{ and } j = 2, \\
v_2 v_1 v_3 \text{ and } v_2 v_4 v_3 & \text{if } i = 2 \text{ and } j = 3, \\
v_2 v_1 v_j \text{ and } v_2 v_3 v_j & \text{if } i = 2 \text{ and } j \geq 4,
\end{array}$$

$$\begin{array}{ll} v_3v_2v_1 \text{ and } v_3v_4v_1 & \text{if } i=3 \text{ and } j=1, \\ v_3v_1v_2 \text{ and } v_3v_4v_2 & \text{if } i=1 \text{ and } j=2, \\ v_3v_1v_3 \text{ and } v_3v_4v_3 & \text{if } i=3 \text{ and } j=2, \\ v_3v_1v_j \text{ and } v_3v_2v_j & \text{if } i=3 \text{ and } j\geq 4, \\ v_iv_2v_1 \text{ and } v_iv_3v_1 & \text{if } i=1 \text{ and } j=1, \\ v_iv_1v_2 \text{ and } v_iv_3v_2 & \text{if } i\geq 4 \text{ and } j=2, \\ v_iv_1v_3 \text{ and } v_iv_2v_3 & \text{if } i\geq 4 \text{ and } j=3, \\ v_iv_1v_j \text{ and } v_iv_3v_j & \text{if } i\geq 4 \text{ and } j\geq 4. \end{array}$$

As a consequence, the signed digraph $G = (V, A, f)$ is primitive non-powerful and $l(G) = 2$.

EXAMPLE 2. Let $n \geq 4$, $V = \{v_1, v_2, \cdots, v_n\}$, $A = \{(v_i, v_j)|1 \leq i, j \leq n, i \neq j\}$ and $f : A \longrightarrow \{-1, 1\}$ such that

$$f(v_iv_j) = \begin{cases} -1, & \text{if } (i,j) = (1,2), \\ 1, & \text{otherwise} \end{cases} .$$

We can see for each walk of length 2 from v_1 to v_2 is of sign $+1$. Thus $l(G) \geq 3$. By the same method used in above example, there are a pair of SSSD walks of length 3 from v_i to v_j as follows for each i and j It follows that the signed digraph $G = (V, A, f)$ is primitive non-powerful and $l(G) = 3$.

A consequence of the above theorems and examples is that the base of a sign pattern matrix such that every diagonal entry is zero and every non diagonal entries is of sign 1 or -1 is 2, 3 and 4. Also we can consider the sign pattern matrix without zero entries. The corresponding digraph is a complete graph with loops on each vertices. In this case we have the following theorem.

THEOREM 3. *If $n \geq 3$ and K is a non-powerful signed digraph over n-th order complete graph with loops on each vertices, then $l(K) \leq 3$.*

Proof. Suppose that $l(K) \geq 4$. There are $v, w \in V$ and $\sigma \in \{+1, -1\}$ such that the sign of every walk from v to w of length 3 is always σ. Let τ be the sign of the loop incident on v. For all $x \in V$, since $\text{sgn}(vvxw) = \text{sgn}(vv)\text{sgn}(vxw) = \tau\text{sgn}(vxw) = \sigma$, we have $\text{sgn}(xxw) = \sigma$. Since $\text{sgn}(vxxw) = f(vx)f(xx)f(xw) = f(xx)\text{sgn}(vxw) = f(xx)\sigma\tau = \sigma$, we have $f(xx) = \tau$.

Let $C = x_1x_2\cdots x_kx_1$ be a cycle of length k in K. We have

$$\begin{aligned}\sigma^k &= \mathrm{sgn}(vx_1x_2w)\mathrm{sgn}(vx_2x_3w)\cdots\mathrm{sgn}(vx_kx_1w)\\ &= (f(vx_1)f(x_1x_2)f(x_2w))(f(vx_2)f(x_2x_3)f(x_3w))\\ &\qquad\cdots(f(vx_k)f(x_kx_1)f(x_1w))\\ &= (f(vx_1)f(x_1w))(f(vx_2)f(x_2w))\cdots(f(vx_k)f(x_kw))f(x_1x_2)f(x_2x_3)\\ &\qquad\cdots f(x_kx_1)\\ &= (\sigma\tau)^k\mathrm{sgn}(x_1x_2\cdots x_kx_1)\\ &= \sigma^k\tau^kf(C).\end{aligned}$$

Thus the signs of all even and odd cycles are 1 and τ respectively. Therefore K is powerful. This is a contradiction. Hence $l(K) \le 3$. □

Remark 1. Let $n = 3$, $V = \{v_1, v_2, v_3\}$ and $A = \{(v_i, v_j)|i \ne j\}$. Since $v_1v_3v_2$ is the only $v_1 \xrightarrow{2} v_2$ walk in K, we have $l(K) \ge 3$.
If $\mathrm{sgn}(v_1v_2v_1) = \mathrm{sgn}(v_2v_3v_2) = \mathrm{sgn}(v_3v_1v_3) = 1$, then every 2-cycle in G is of sign 1. Since

$$\begin{aligned}&\mathrm{sgn}(v_1v_2v_3v_1)\mathrm{sgn}(v_1v_3v_2v_1)\\ &= f(v_1v_2)f(v_2v_3)f(v_3v_1)f(v_1v_3)f(v_3v_2)f(v_2v_3)f(v_3v_1)\\ &= (f(v_1v_2)f(v_2v_1))(f(v_2v_3)f(v_3v_2))(f(v_3v_1)f(v_1v_3))\\ &= \mathrm{sgn}(v_1v_2v_1)\mathrm{sgn}(v_2v_3v_2)\mathrm{sgn}(v_3v_1v_3) = 1\end{aligned},$$

all 3-cycles in K are of the same sign. It follows that K is powerful.
If $\mathrm{sgn}(v_1v_2v_1) = \mathrm{sgn}(v_2v_3v_2) = \mathrm{sgn}(v_3v_1v_3) = -1$ for all $v_i, v_j \in V$, then there is a $v_i \xrightarrow{2} v_j$ walk W in K. Since

$$\mathrm{sgn}(v_1v_2v_3v_1)\mathrm{sgn}(v_1v_3v_2v_1) = f(v_1v_2v_1)f(v_2v_3v_2)f(v_3v_1v_3) = -1,$$

there are two $v_i \xrightarrow{3} v_i$ walks W_1 and W_2 in K with different signs. Thus we see that $W + W_1$ and $W + W_2$ are a pair of SSSD walks with length 5. We have $l(K) \le 5$. Let $W = w_0w_1w_2w_3w_4$ be a $v_1 \xrightarrow{4} v_1$ walk in K. Hence we have $w_0 = w_4 = v_1$. We may assume that $w_1 = v_2$. If $w_2 = v_1$, then $f(W) = f(v_1v_2v_1)f(v_1w_4v_1) = 1$. If $w_2 = v_3$, then since $w_3 = v_2$, we have $f(W) = 1$. Therefore there is no $v_1 \xrightarrow{4} v_1$ walk in K with sign -1. Thus $l(K) = 5$.
If the signs of $f(v_1v_2v_1), f(v_2v_3v_2)$ and $f(v_3v_1v_3)$ are not equal, then we may assume that $f(v_1v_2v_1) = f(v_2v_3v_2) = -f(v_3v_1v_3)$. Let $v_i, v_j \in V$. Hence there is a $v_i \xrightarrow{2} v_j$ walk $W = v_iv_kv_j$ in K. If $i \ne 2$, then there

are two $v_i \xrightarrow{2} v_i$ walks W_1 and W_2 in K with different signs. It is clear that $W_1 + W$ and $W_2 + W$ are a pair of SSSD walks with length 4. Similarly, we have a pair of SSSD walks with length 4 for the case $j \neq 2$. If $i = j = 2$, then $k \neq 2$. Whence there are a pair of $v_k \xrightarrow{2} v_k$ walks X_1 and X_2 in K with different signs. Thus we see that $(v_i v_k) + X_1 + (v_k v_j)$ and $(v_i v_k) + X_2 + (v_k v_j)$ are a pair of SSSD walks with length 4. Hence $l(K) \leq 4$.
Let $f_1, f_2 : A \to \{1, -1\}$,

$$f_1(v_i v_j) = \begin{cases} -1, & i = 1 \text{ and } j = 2 \\ 1, & \text{otherwise}, \end{cases}$$

and

$$f_2(v_i v_j) = \begin{cases} -1, & i = 1 \text{ and } j = 2, 3 \\ 1, & \text{otherwise}. \end{cases}$$

Then (V, A, f_1) and (V, A, f_2) are examples of signed digraph over complete graphs with loops with bases 3 and 4 respectively. Hence the possible bases of signed digraph over complete graphs with loops on 3 vertices are $3, 4$ and 5.
Note that if

$$A = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix},$$

then

$$A^4 = \begin{pmatrix} 1 & \# & \# \\ \# & 1 & \# \\ \# & \# & 1 \end{pmatrix}.$$

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