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ON THE CARDINALITY OF SEMISTAR OPERATIONS OF FINITE CHARACTER ON INTEGRAL DOMAINS

Gyu Whan Chang

Abstract. Let D be an integral domain with $Spec(D)$ finite, K the quotient field of D, $[D, K]$ the set of rings between D and K, and $SFc(D)$ the set of semistar operations of finite character on D. It is well known that $|Spec(D)| \leq |SFc(D)|$. In this paper, we prove that $|Spec(D)| = |SFc(D)|$ if and only if D is a valuation domain, if and only if $|Spec(D)| = |[D, K]|$. We also study integral domains D such that $|Spec(D)| + 1 = |SFc(D)|$.

1. Introduction

Let D be an integral domain, K the quotient field of D, $\bar{D}$ the integral closure of D, $[D, K]$ the set of rings between D and K, and $Spec(D)$ the set of prime ideals of D. Let $\bar{F}(D)$ be the set of nonzero D-submodules of K, $F(D)$ the subset of $\bar{F}(D)$ consisting of all nonzero fractional ideals of D, and $f(D)$ the set of nonzero finitely generated fractional ideals of D; so $f(D) \subseteq F(D) \subseteq \bar{F}(D)$. A mapping $* : \bar{F}(D) \to \bar{F}(D)$, $A \mapsto A^*$, is called a *semistar operation on D* if the following three conditions are satisfied for all $0 \neq a \in K$ and $E, F \in \bar{F}(D)$:

1. $(aE)^* = aE^*$,
2. $E \subseteq E^*$,

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3. $E \subseteq F$ implies $E^* \subseteq F^*$, and $(E^*)^* = E^*$.

Let $*$ be a semistar operation on D. If $D^* = D$, then the map $*|_{F(D)} : F(D) \to F(D)$, given by $E^{*|_{F(D)}} = E^*$, is a *star operation on* D. Conversely, if $*_1$ is a star operation on D, then the map $*_1^l : \bar{F}(D) \to \bar{F}(D)$, defined by $E^{*_1^l} = E^{*_1}$ for $E \in F(D)$ and $E^{*_1^l} = K$ for $E \in \bar{F}(D) \setminus F(D)$, is a semistar operation on D. For each $E \in \bar{F}(D)$, let $E^{*_f} = \bigcup\{F^* | F \subseteq E$ and $F \in f(D)\}$. Then $*_f$ is also a semistar operation on D. It is clear that $(*_f)_f = *_f$ and $F^* = F^{*_f}$ for $F \in f(D)$. If $* = *_f$, then $*$ is called a *semistar operation of finite character.* So $*_f$ is of finite character. The v-, t-, and d-operations are the most well-known examples of semistar operations. The v-operation is defined by $E^v = (D : (D : E))$ and the t-operation is defined by $t = v_f$. The d-operation is just the identity function on $\bar{F}(D)$, that is, $E^d = E$ for all $E \in \bar{F}(D)$. The notion of semistar operations was introdced by Okabe and Matsuda [10] and have been studied by many researchers (cf. [1, 2, 6, 7, 8, 9, 11]).

Let $S(D)$ be the set of semistar operations on D and $SFc(D)$ the set of semistar operations of finite character on D; so $SFc(D) \subseteq S(D)$. Let $\dim(D)$ be the (Krull) dimension of D and let $|A|$ denote the cardinality of a set A. Assume that $|SFc(D)| < \infty$. In [9, Theorem 7], the authors proved that $\dim(D) + 1 = |SFc(D)|$ if and only if D is a valuation domain; hence D is not a valuation domain if and only if $\dim(D) + 2 \leq |SFc(D)|$. In [8, Theorem 4.3], Mimouni showed that if D is not quasi-local, then $\dim(D) + 3 \leq |SFc(D)|$ and the equality holds if and only if D is a Prüfer domain with exactly two maximal ideals M and N such that every prime ideal of D is contained in $M \cap N$. He also proved that $|SFc(D)| = 2+ \dim(D)$ if and only if $\bar{D}$ is a valuation domain, $D \subsetneq \bar{D}$, there is no proper overring between D and $\bar{D}$, each overring of D is comparable to $\bar{D}$, and each nonzero finitely generated ideal I of D is divisorial, i,e., $I^v = I$ [8, Theorem 4.4].

This paper is motivated by Mimoumi's results [8, Theorems 4.3 and 4.4] and the following observation: *For each $T \in [D, K]$, the map $*_T : \bar{F}(D) \to \bar{F}(D)$ defined by $E \mapsto E^{*_T} := ET$ is a semistar operation of finite character on D* [10]*. In particular, if P is a prime ideal of D, then $*_P := *_{D_P} \in SFc(D)$.*

Note that $\dim(D) + 1 \leq |Spec(D)|$ and $\{D_P \mid P \in Spec(D)\} \subseteq [D, K]$; so we have $\dim(D) + 1 \leq |Spec(D)| \leq |[D, K]| \leq |SFc(D)|$ (see Lemma 1(1)). In this paper, we prove that $|Spec(D)| = |SFc(D)|$ if and only if $|Spec(D)| = |[D, K]|$, if and only if D is a valuation domain and that if $Spec(D)$ is linearly ordered, then $|Spec(D)|+1 = |SFc(D)|$ if and only if $|[D,K]| = |Spec(D)| + 1$ and $t = d$ on D, if and only if $D \subsetneq \bar{D}$, $[D,K] = \{D_P \mid P \in Spec(D)\} \cup \{\bar{D}\}$, and $t = d$ on D. We also prove that if $Spec(D)$ is not linearly ordered, then $|Spec(D)| + 1 = |SFc(D)|$ if and only if D is a Prüfer domain with two maximal ideals P_1 and P_2 such that each non-maximal prime ideal of D is contained in $P_1 \cap P_2$, if and only if $[D,K] = \{D_P \mid P \in Spec(D)\} \cup \{D\}$, if and only if $|[D,K]| = |Spec(D)| + 1$.

2. Main Results

Throughout this paper, D is an integral domain with $|Spec(D)| < \infty$, K is the quotient field of D, $\bar{D}$ is the integral closure of D, and $[D,K]$ is the set of rings between D and K. Let $*$ be a semistar operation on D, and let R be an overring of D, i.e., $R \in [D,K]$. Then $R^*R^* \subseteq (R^*R^*)^* = (RR)^* = R^*$, and thus R^* is an overring of D [10, Proposition 5]. In particular, D^* is an overring of D. Also, it is easy to see that the map $*_T : \bar{F}(D) \to \bar{F}(D)$ defined by $E \mapsto E^{*_T} := ET$ is a semistar operation of finite character on D.

Lemma 1. 1. *$dim(D) + 1 \leq |Spec(D)| \leq |[D,K]| \leq |SFc(D)| \leq |S(D)|$.*
2. *$dim(D) + 1 = |Spec(D)|$ if and only if $Spec(D)$ is linearly ordered.*

Proof. (1) Let P be a prime ideal of D, and let $E^{*_P} = ED_P$ for all $E \in \bar{F}(D)$. Then $*_P$ is a semistar operation of finite character on D (in particular, if $P = (0)$, then $E^{*_P} = K$ for all $E \in \bar{F}(D)$). It is clear that if P and Q are prime ideals of D, then $P = Q \Leftrightarrow D_P = D_Q \Leftrightarrow *_P = *_Q$. Thus the second and third inequalities hold. The first and fourth inequalities are clear.

(2) This follows directly from the definition of the (Krull) dimension. □

Proposition 2. *The following statements are equivalent.*

1. *$dim(D) + 1 = |SFc(D)|$.*
2. *$|Spec(D)| = |SFc(D)|$; so $SFc(D) = \{*_P \mid P \in Spec(D)\}$.*
3. *D is a valuation domain.*
4. *$|Spec(D)| = |[D,K]|$; so $[D,K] = \{D_P \mid P \in Spec(D)\}$.*

Proof. (1) $\Leftrightarrow$ (3) [9, Theorem 7].

(1) $\Rightarrow$ (2) $\Rightarrow$ (4) This follows directly from Lemma 1(1) and the fact that for $P, Q \in Spec(D)$, $D_P = D_Q \Leftrightarrow P = Q \Leftrightarrow *_P = *_Q$.

(4) $\Rightarrow$ (3) First, note that D is a Prüfer domain [3, page 334] since each overring of D is a quotient ring of D. Also, since $D \in [D, K]$, we have $D = D_P$ for some $P \in Spec(D)$, and hence D is quasi-local. Thus, D is a valuation domain. □

Corollary 3. *$|Spec(D)| = |S(D)|$ if and only if D is a strongly discrete valuation domain.*

Proof. Note that $|Spec(D)| = |S(D)|$ implies $|Spec(D)| = |SFc(D)| = |S(D)|$. Hence D is a valuation domain by Proposition 2, and hence D is strongly discrete [9, Theorem 10]. Conversely, assume that D is a strongly discrete valuation domain. Then $|SFc(D)| = |S(D)|$ [9, Theorem 10] and $|Spec(D)| = |SFc(D)|$ by Proposition 2. Thus $|Spec(D)| = |S(D)|$. □

By Proposition 2, if D is an integral domain that is not a valuation domain, then $|Spec(D)|+1 \leq |SFc(D)|$. We next study integral domains D with $|Spec(D)| + 1 = |SFc(D)|$ when $Spec(D)$ is linearly ordered (Theorem 4) and $Spec(D)$ is not linearly ordered (Theorem 6). .

Theorem 4. *If $Spec(D)$ is linearly ordered, then the following are equivalent.*

1. $|Spec(D)| + 1 = |SFc(D)|$.
2. $D \subsetneq \bar{D}$ *and* $SFc(D) = \{*_P \mid P \in Spec(D)\} \cup \{*_{\bar{D}}\}$.
3. $D \subsetneq \bar{D}$, $[D, K] = \{D_P \mid P \in Spec(D)\} \cup \{\bar{D}\}$ *and* $t = d$ *on* D.
4. $|[D, K]| = |Spec(D)| + 1$ *and* $t = d$ *on* D.

In this case, $\bar{D}$ and D_P are valuation domains such that $\bar{D} \subsetneq D_P$ for all non-maximal prime ideals P of D.

Proof. (1) $\Rightarrow$ (2) By [8, Theorem 4.4] and Lemma 1(2), $D \subsetneq \bar{D}$, and hence $*_{\bar{D}} \neq *_P$ for all $P \in Spec(D)$. Hence $|\{*_P \mid P \in Spec(D)\} \cup \{*_{\bar{D}}\}| = |Spec(D)| + 1 = |SFc(D)|$. Thus $SFc(D) = \{*_P \mid P \in Spec(D)\} \cup \{*_{\bar{D}}\}$.

(2) $\Rightarrow$ (1) Clear.

(2) $\Rightarrow$ (3) Let T be an overring of D. Then $*_T \in SFc(D)$, and hence either $*_T = *_{\bar{D}}$ or $*_T = *_P$ for some $P \in Spec(D)$. If $*_T = *_P$, then $T = T^{*_T} = T^{*_P} = TD_P \supseteq D_P = (D_P)^{*_P} = (D_P)^{*_T} = (D_P)T \supseteq T$, and thus $T = D_P$. Similarly, if $*_T = *_{\bar{D}}$, then $T = \bar{D}$. Thus $[D, K] = \{D_P \mid$

$P \in Spec(D)\} \cup \{\bar{D}\}$. Also, since $t \in SFc(D)$ and $D^t = D$, we have $t = d$ on D.

(3) ⇒ (2) Let $V \in [D, K]$ be a valuation domain such that $Spec(D) = \{Q \cap D \mid Q \in Spec(V)\}$ (cf. [3, Corollary 19.7]). Then $V \neq D_P$ for all $P \in Spec(D)$, and thus $V = \bar{D}$ by (3). Similarly, we have that D_P is a valuation domain and $\bar{D} \subsetneq D_P$ for each non-maximal prime ideal P of D. Let $*$ be a semistar operation of finite character on D. If $D^* = D$, then $*|_{F(D)}$ is a star operation of finite character, and hence $t = *|_{F(D)} = d$ as star operations. Note that $*$, t and d are of finite character; so $* = d$ as semistar operations. Next, assume that $D^* \neq D$. Then D^* is a proper overring of D, and thus $D^* = D_P$ for some non-maximal $P \in Spec(D)$ or $D^* = \bar{D}$ by (3). For any $A \in f(D)$, since D_P is a valuation domain, there exists an $a \in A$ such that $AD_P = aD_P$. Thus $A^* = (AD)^* = (AD^*)^* = (AD_P)^* = (aD_P)^* = a(D_P)^* = aD_P = AD_P = A^{*_P}$. Also, since $*$ is of finite character, we have $* = *_P$. Similarly, if $D^* = \bar{D}$, then $* = *_{\bar{D}}$. Thus the proof is completed.

(3) ⇒ (4) Clear.

(4) ⇒ (3) Note that $\bar{D} \neq D_P$ for all non-maximal ideals P of D; so it suffices to show that $D \subsetneq \bar{D}$ by (4).

Assume that $D = \bar{D}$. Then D is not a valuation domain by (4) and Proposition 2, and hence there is a valuation domain V such that $D \subseteq V$ and $Spec(D) = \{Q \cap D \mid Q \in Spec(D)\}$ [3, Corollary 19.7]. Clearly, $V \neq D_P$ for all $P \in Spec(D)$, and so $[D, K] = \{D_P \mid P \in SpecD\} \cup \{V\}$. Since D is not a valuation domain, there are $a, b \in D$ such that $(a, b)D$ is not invertible and $\frac{b}{a} \in V \setminus D$. Let $f = aX - b \in D[X]$, where $D[X]$ is the polynomial ring over D, and let $\varphi : D[X] \to D[\frac{b}{a}]$, defined by $\varphi(g(X)) = g(\frac{b}{a})$, be the canonical ring homomorphism. Then φ is onto and the kernel of φ is $Q_f := fK[X] \cap D[X]$. Hence $D[X]/Q_f = D[\frac{b}{a}]$. Note that if $Q_f \nsubseteq P[X]$ for all $P \in Spec(D)$, then there is a polynomial $g \in K[X]$ such that $D = (A_{fg})_v = (A_f A_g)_v$, where A_h is the fractional ideal of D generated by the coefficients of a polynomial h, ([4, Theorem 1.4] and [3, Corollary 34.8]) because $D = \bar{D}$. Also, since each prime ideal P of D is a t-ideal, i.e., $P^t = P$ [5, Theorem 3.19], $A_f A_g = D$, and hence $A_f = (a, b)D$ is invertible, a contradiction. Thus if P is the maximal ideal of D, then $Q_f \subseteq P[X]$, and so $(D/P)[X] = (D[X]/Q_f)/(P[X]/Q_f)$. Thus $D[\frac{b}{a}] = D[X]/Q_f$ is not quasi-local since $(D/P)[X]$ has infinitely many maximal ideals. Thus $D \subsetneq D[\frac{b}{a}] \subsetneq V$, whence $|[D, K]| \geq |Spec(D)| + 2$, a contradiction. Therefore, $D \subsetneq \bar{D}$. □

We need a lemma for the proof of Theorem 6.

LEMMA 5. *Let P_1, P_2 be incomparable prime ideals of D, and let $*$ be the semistar operation on D defined by $E^* = ED_{P_1} \cap ED_{P_2}$ for all $E \in \bar{F}(D)$. Then $* \neq *_P$ for all $P \in Spec(D)$. In particular, $|Spec(D)| + 1 \leq |SFc(D)|$.*

Proof. Let P be a prime ideal of D.

Case 1. $P \subsetneq P_1$. Then $P_1^* = P_1D_{P_1} \cap P_1D_{P_2} = P_1D_{P_1} \cap D_{P_2} \neq D_P = P_1D_P = P_1^{*_P}$. So $* \neq *_P$.

Case 2. $P = P_1$. Then $P_2^* = P_2D_{P_1} \cap P_2D_{P_2} = D_{P_1} \cap P_2D_{P_2} \neq D_P = P_2D_P = P_2^{*_P}$. So $* \neq *_P$.

Case 3. $P_1 \subsetneq P$. Then $P \nsubseteq P_2$, and hence $P^{*_P} = PD_P \neq D_{P_1} \cap D_{P_2} = PD_{P_1} \cap PD_{P_2} = P^*$. So $* \neq *_P$.

Case 4. P is not comparable to P_1. If P is comparable to P_2, then $* \neq *_P$ by Cases 1,2, and 3. If P is not comparable to P_2, then $P^{*_P} = PD_P \neq D_{P_1} \cap D_{P_2} = PD_{P_1} \cap PD_{P_2} = P^*$, and thus $* \neq *_P$.

For the "in particular" part, note that $*$ is of finite character [8, Theorem 2.4], and hence $\{*_P \mid P \in Spec(D)\} \cup \{*\} \subseteq SFc(D)$ and $|\{*_P \mid P \in Spec(D)\} \cup \{*\}| = |Spec(D)| + 1$. Thus $|Spec(D)| + 1 \leq |SFc(D)|$. □

In [8, Theorem 4.3], Mimoumi proved the equivalence of (2) and (3) of Theorem 6 under the assumption that D is not quasi-local.

THEOREM 6. *If $Spec(D)$ is not linearly ordered, then the following are equivalent.*

1. $|Spec(D)| + 1 = |SFc(D)|$.
2. $|SFc(D)| = 3 + dim(D)$.
3. *D is a Prüfer domain with two maximal ideals P_1 and P_2 such that each non-maximal prime ideal of D is contained in $P_1 \cap P_2$.*
4. $SFc(D) = \{*_D\} \cup \{*_P \mid P \in Spec(D)\}$.
5. $[D, K] = \{D_P \mid P \in Spec(D)\} \cup \{D\}$.
6. $|[D, K]| = |Spec(D)| + 1$.

Proof. (1) $\Rightarrow$ (2) and (3) Let P_1, P_2 be incomparable prime ideals of D, and let $*$ be the semistar operation on D defined by $E^* = ED_{P_1} \cap ED_{P_2}$ for all $E \in \bar{F}(D)$. Then $*$ is a semistar operation of finite character [8, Theorem 2.4] and $* \neq *_P$ for all $P \in Spec(D)$ by Lemma 5. Hence $SFc(D) = \{*_P \mid P \in Spec(D)\} \cup \{*\}$ by (1). Note that if there is a prime ideal P of D such that P is not comparable to P_1 or P_2, then the semistar operation defined by $E^{*_i} = ED_P \cap ED_{P_i}$ is different form $*$ and $*_P$; so $|Spec(D)| + 2 \leq |SFc(D)|$, a contradiction. Thus P_1, P_2 are comparable to each prime ideal in $Spec(D) \setminus \{P_1, P_2\}$. The same argument also shows that $Spec(D) \setminus \{P_1, P_2\}$ is linearly ordered.

Next, assume that D is quasi-local with maximal ideal M. Then $M \neq P_i$ for $i = 1, 2$. Note that $*_{\bar{D}} \in SFc(D)$ and $D^{*_{\bar{D}}} = \bar{D}$; so if P is a non-maximal prime ideal of D, then $*_{\bar{D}} \neq *_P$. Also, note that $M^* = D_{P_1} \cap D_{P_2} \neq M\bar{D} = M^{*_{\bar{D}}}$; so $*_{\bar{D}} \neq *$. Hence $*_{\bar{D}} = *_M$ and $D = \bar{D}$. Consider the chain of prime ideals of D containing P_1, and let V be a valuation overring of D such that $Spec(V)$ is contracted to the chain, i.e., $\{Q \cap D \mid Q \in Spec(V)\} = Spec(D) \setminus \{P_2\}$ [3, Corollary 19.7]. Note that $*_V = *$ or $*_V = *_P$ for some $P \in Spec(D)$; so by the proof of "(3) $\Rightarrow$ (2)" of Theorem 5, either $V = D_{P_1} \cap D_{P_2}$ or $V = D_P$, a contradiction. Hence D is not quasi-local, and thus P_1 and P_2 are maximal ideals of D and $\dim(D) + 2 = |Spec(D)|$; so $|SFc(D)| = \dim(D) + 3$. Moreover, by Lemma 1(1), $[D : K] = \{D_P \mid P \in Spec(D)\} \cup \{D\}$; so each overring of D is a quotient ring of D. Thus, D is a Prüfer domain [3, page 334].

(2) $\Rightarrow$ (1) Note that $\dim(D) + 2 \leq |Spec(D)|$ by Lemma 1(2); so $|SFc(D)| = \dim(D) + 3 \leq |Spec(D)| + 1 \leq |SFc(D)|$ by (2) and Lemma 5. Thus $|Spec(D)| + 1 = |SFc(D)|$.

(3) $\Rightarrow$ (5) Note that each finitely generated ideal of D is principal [3, Proposition 7.4]; hence each overring of D is a quotient ring of D [3, Theorem 27.5]. Thus $[D, K] = \{D_P \mid P \in Spec(D)\} \cup \{D\}$.

(5) $\Rightarrow$ (4) Note that each overring of D is a quotient ring of D by (5), and thus D is a Prüfer domain [3, page 334]. Also, D has at most two maximal ideals because $D_{M_1} \cap D_{M_2} \neq D_P$ for any maximal ideals M_i and non-maximal prime ideal P. Next, let $*_1$ be a semistar operation of finite character on D, and let $T = D^{*_1}$. Then T is an overring of D, and hence either $T = D$ or $T = D_P$ for some prime ideal P of D. If $T = D$, then for any $A \in f(D)$, $A^{*_1} = (AD)^{*_1} = (aD)^{*_1} = aD^{*_1} = aD = A = A^{*_D}$ (note that D is a Bzeout domain, and hence $AD = aD$ for some $a \in A$). Thus $*_1 = *_D$. Similarly, we have $*_1 = *_P$ if $T = D_P$ for some $P \in Spec(D)$. This completes the proof.

(4) $\Rightarrow$ (1) Clear.

(5) $\Rightarrow$ (6) Clear.

(6) $\Rightarrow$ (5) Let P_1 and P_2 be incomparable prime ideals of D, and let $R = D_{P_1} \cap D_{P_2}$. Then $R \neq D_P$ for all $P \in Spec(D)$, and so $[D, K] = \{D_P \mid P \in Spec(D)\} \cup \{R\}$ by (6). As in the proof of (1) $\Rightarrow$ (2) and

(3), we can show that D is not quasi-local with maximal ideals P_1 and P_2. Hence $R = D$, and thus $[D, K] = \{D_P \mid P \in Spec(D)\} \cup \{D\}$ □

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Gyu Whan Chang
Department of Mathematics Education
Incheon National University
Incheon 406-772, Korea
E-mail: `whan@incheon.ac.kr`