

**EXISTENCE OF A POSITIVE INFIMUM EIGENVALUE  
FOR THE  $p(x)$ -LAPLACIAN NEUMANN PROBLEMS  
WITH WEIGHTED FUNCTIONS**

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ABSTRACT. We study the following nonlinear problem

$$-\operatorname{div}(w(x)|\nabla u|^{p(x)-2}\nabla u) + |u|^{p(x)-2}u = \lambda f(x, u) \quad \text{in } \Omega$$

which is subject to Neumann boundary condition. Under suitable conditions on  $w$  and  $f$ , we give the existence of a positive infimum eigenvalue for the  $p(x)$ -Laplacian Neumann problem.

**1. Introduction**

In the present paper, we are concerned with the existence of a positive infimum eigenvalue for the  $p(x)$ -Laplacian problem with the degeneracy subject to Neumann boundary condition

$$(B) \quad \begin{cases} -\operatorname{div}(w(x)|\nabla u|^{p(x)-2}\nabla u) + |u|^{p(x)-2}u = \lambda f(x, u) & \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases}$$

where  $\Omega$  is a bounded domain in  $\mathbb{R}^N$  with Lipschitz boundary  $\partial\Omega$ ,  $\frac{\partial u}{\partial n}$  denotes the outer normal derivative of  $u$  with respect to  $\partial\Omega$ , the variable exponent  $p : \overline{\Omega} \rightarrow (1, \infty)$  is a continuous function,  $g \in L^\infty(\Omega)$ ,  $w$  is a

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weighted function in  $\Omega$  and  $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  satisfies a Carathéodory condition.

The studies for the  $p(x)$ -Laplacian problems have been extensively considered by several authors in various ways; see for example [2–6,8] and references therein. Compared to the  $p$ -Laplacian equation, an analysis for the  $p(x)$ -Laplacian equation has to be carried out more carefully because it has complicated nonlinearities (it is nonhomogeneous) and includes a weighted function. Unlike the  $p$ -Laplacian case, under some conditions on  $p(x)$ , the first eigenvalue for the  $p(x)$ -Laplacian Neumann problems is not isolated (see [6]), that is, the infimum of all eigenvalues of the problem might be zero (see [5] for Dirichlet boundary condition). The goal of this paper is to give sufficient conditions on  $w$  and  $f$  to satisfy the positivity of the infimum of all eigenvalues for (B) still. To the best of our knowledge, there are no papers concerned with the positivity of the infimum of all eigenvalues for the  $p(x)$ -Laplacian Neumann problems with weighted functions.

To make a self-contained paper, we recall some definitions and basic properties of the weighted variable exponent Lebesgue spaces  $L^{p(\cdot)}(w, \Omega)$  and the weighted variable exponent Lebesgue-Sobolev spaces  $W^{1,p(\cdot)}(w, \Omega)$ .

Set

$$C_+(\overline{\Omega}) = \left\{ h \in C(\overline{\Omega}) : \min_{x \in \overline{\Omega}} h(x) > 1 \right\}.$$

For any  $h \in C_+(\overline{\Omega})$  we define

$$h_+ = \sup_{x \in \Omega} h(x) \quad \text{and} \quad h_- = \inf_{x \in \Omega} h(x).$$

Let  $w$  is a measurable positive and a.e. finite function in  $\Omega$ . For any  $p \in C_+(\overline{\Omega})$ , we introduce the weighted variable exponent Lebesgue space  $L^{p(x)}(w, \Omega)$  that consists of all measurable real-valued functions  $u$  satisfying

$$\int_{\Omega} w(x) |u(x)|^{p(x)} dx < \infty,$$

endowed with the Luxemburg norm

$$\|u\|_{L^{p(\cdot)}(w, \Omega)} = \inf \left\{ \lambda > 0 : \int_{\Omega} w(x) \left| \frac{u(x)}{\lambda} \right|^{p(\cdot)} dx \leq 1 \right\}.$$

The weighted variable exponent Sobolev space  $X = W^{1,p(\cdot)}(w, \Omega)$  is defined by

$$X = \{ u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(w, \Omega) \},$$

where the norm is

$$(1.1) \quad \|u\|_X = \|u\|_{L^{p(\cdot)}(\Omega)} + \|\nabla u\|_{L^{p(\cdot)}(w,\Omega)}.$$

This paper is organized as follows. We first introduce some basic results for the weighted variable exponent Lebesgue-Sobolev spaces which is given in [8]. Next we show the existence of a positive infimum eigenvalue for the  $p(x)$ -Laplacian problem with the degeneracy subject to Neumann boundary condition.

### 2. Preliminaries

In this section, we first state some elementary properties for the (weighted) variable exponent Lebesgue-Sobolev spaces which play a crucial role in obtaining our main result. The basic properties of the variable exponent Lebesgue-Sobolev spaces, that is, when  $w(x) \equiv 1$  can be found from [3].

LEMMA 2.1. ([3]) *The space  $L^{p(\cdot)}(\Omega)$  is a separable, uniformly convex Banach space, and its conjugate space is  $L^{p'(\cdot)}(\Omega)$  where  $1/p(x) + 1/p'(x) = 1$ . For any  $u \in L^{p(\cdot)}(\Omega)$  and  $v \in L^{p'(\cdot)}(\Omega)$ , we have*

$$\left| \int_{\Omega} uv \, dx \right| \leq \left( \frac{1}{p_-} + \frac{1}{(p')_-} \right) \|u\|_{L^{p(\cdot)}(\Omega)} \|v\|_{L^{p'(\cdot)}(\Omega)} \leq 2 \|u\|_{L^{p(\cdot)}(\Omega)} \|v\|_{L^{p'(\cdot)}(\Omega)}.$$

LEMMA 2.2. ([8]) *Denote*

$$\rho(u) = \int_{\Omega} w(x)|u|^{p(x)} \, dx, \quad \text{for all } u \in L^{p(\cdot)}(w, \Omega).$$

Then

- (1)  $\rho(u) > 1$  ( $= 1$ ;  $< 1$ ) if and only if  $\|u\|_{L^{p(\cdot)}(w,\Omega)} > 1$  ( $= 1$ ;  $< 1$ ), respectively;
- (2) If  $\|u\|_{L^{p(\cdot)}(w,\Omega)} > 1$ , then  $\|u\|_{L^{p(\cdot)}(w,\Omega)}^{p_-} \leq \rho(u) \leq \|u\|_{L^{p(\cdot)}(w,\Omega)}^{p_+}$ ;
- (3) If  $\|u\|_{L^{p(\cdot)}(w,\Omega)} < 1$ , then  $\|u\|_{L^{p(\cdot)}(w,\Omega)}^{p_+} \leq \rho(u) \leq \|u\|_{L^{p(\cdot)}(w,\Omega)}^{p_-}$ .

LEMMA 2.3. ([11]) *Let  $q \in L^{\infty}(\Omega)$  be such that  $1 \leq p(x)q(x) \leq \infty$  for almost all  $x \in \Omega$ . If  $u \in L^{q(\cdot)}(\Omega)$  with  $u \neq 0$ , then*

- (1) If  $\|u\|_{L^{p(\cdot)q(\cdot)}(w,\Omega)} > 1$ , then  $\|u\|_{L^{p(\cdot)q(\cdot)}(w,\Omega)}^{q_-} \leq \| |u|^{q(x)} \|_{L^{p(\cdot)}(w,\Omega)} \leq \|u\|_{L^{p(\cdot)q(\cdot)}(w,\Omega)}^{q_+}$ ;

(2) If  $\|u\|_{L^{p(\cdot)q(\cdot)}(w,\Omega)} < 1$ , then  
 $\|u\|_{L^{p(\cdot)q(\cdot)}(w,\Omega)}^{q_+} \leq \| |u|^{q(x)} \|_{L^{p(\cdot)}(w,\Omega)} \leq \|u\|_{L^{p(\cdot)q(\cdot)}(w,\Omega)}^{q_-}$ .

We assume that  $w$  is a measurable positive and a.e. finite function in  $\Omega$  satisfying that

- (w1)  $w \in L^1_{\text{loc}}(\Omega)$  and  $w^{-1/(p(x)-1)} \in L^1_{\text{loc}}(\Omega)$ ;
- (w2)  $w^{-s(x)} \in L^1(\Omega)$  with  $s(x) \in \left(\frac{N}{p(x)}, \infty\right) \cap \left[\frac{1}{p(x)-1}, \infty\right)$ .

The reasons that we assume (w1) and (w2) can be found in [8].

For  $p, s \in C_+(\overline{\Omega})$ , let us denote

$$p_s(x) := \frac{p(x)s(x)}{1 + s(x)} < p(x),$$

where  $s(x)$  is given in (w2) and

$$(2.1) \quad p_s^*(x) := \begin{cases} \frac{p(x)s(x)N}{(s(x)+1)N - p(x)s(x)} & \text{if } N > p_s(x), \\ +\infty & \text{if } N \leq p_s(x), \end{cases}$$

for almost all  $x \in \Omega$ .

We shall frequently make use of the following (compact) imbedding theorem for the weighted variable exponent Lebesgue-Sobolev space in the next sections.

LEMMA 2.4. ([8]) *Let  $\Omega \subset \mathbb{R}^N$  be an open, bounded set with Lipschitz boundary and  $p \in C_+(\overline{\Omega})$  with  $1 < p_- \leq p_+ < \infty$ . If assumptions (w1) and (w2) hold and  $r \in L^\infty(\Omega)$  with  $r_- > 1$  satisfies  $1 < r(x) \leq p_s^*$  for all  $x \in \Omega$ , then we have*

$$W^{1,p(x)}(w, \Omega) \hookrightarrow L^{r(x)}(\Omega)$$

and the imbedding is compact if  $\inf_{x \in \Omega} (p_s^*(x) - r(x)) > 0$ .

### 3. Main Result

In this section, we shall give the sufficient conditions on  $w$  and  $f$  to obtain the positivity of the infimum eigenvalue for the problem (B). Let us consider the following quantity

$$(3.1) \quad \lambda^* = \inf_{u \in X \setminus \{0\}} \frac{\int_{\Omega} \frac{w(x)}{p(x)} |\nabla u|^{p(x)} dx + \int_{\Omega} \frac{1}{p(x)} |u|^{p(x)} dx}{\int_{\Omega} F(x, u) dx}.$$

For the case of  $F(x, u) = m(x)|u|^{q(x)}$  and  $w \equiv 1$ , where  $m(x)$  satisfies a suitable condition, Benouhiba [1] proved  $\lambda^* > 0$  when  $\Omega = \mathbb{R}^N$ . Under Neumann boundary conditions, we shall generalize the condition on  $f$  and give the condition on  $w$  to satisfy  $\lambda^* > 0$  still.

DEFINITION 3.1. We say that  $u \in X$  is a weak solution of the problem (B) if

$$\int_{\Omega} w(x) |\nabla u(x)|^{p(x)-2} \nabla u(x) \cdot \nabla \varphi(x) \, dx + \int_{\Omega} |u(x)|^{p(x)-2} u(x) \varphi(x) \, dx = \lambda \int_{\Omega} f(x, u) \varphi(x) \, dx$$

for all  $\varphi \in X$ .

For almost all  $x \in \Omega$ , we assume that

- (H1)  $p, q \in C_+(\overline{\Omega})$ ,  $p(x) < N$ , and  $1 < p_- \leq p_+ < q_- \leq q_+ < p_s^*(x)$ .
- (H2)  $m(x) \in L^{\frac{r(\cdot)}{r(\cdot)-q(\cdot)}}(\Omega)$  for some  $r \in C_+(\overline{\Omega})$  with  $q(x) < r(x) < p_s^*(x)$  and  $\text{meas}\{x \in \Omega : m(x) > 0\} > 0$ .
- (F1)  $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  satisfies the Carathéodory condition in the sense that  $f(\cdot, t)$  is measurable for all  $t \in \mathbb{R}$  and  $f(x, \cdot)$  is continuous for almost all  $x \in \Omega$ .
- (F2)  $f$  satisfies the following growth condition: For all  $(x, t) \in \Omega \times \mathbb{R}$ ,  $f(x, t)t \geq 0$  and

$$|f(x, t)| \leq m(x) |t|^{q(x)-1}.$$

where  $q$  and  $m$  are given in (H1) and (H2), respectively.

Denoting  $F(x, t) = \int_0^t f(x, s) \, ds$ , it follows from (F2) that

(F2')  $F$  satisfies the following growth condition:

$$0 \leq F(x, t) \leq \frac{m(x)}{q(x)} |t|^{q(x)}, \text{ for all } (x, t) \in \Omega \times \mathbb{R}.$$

Define the functionals  $\Phi, \Psi, I_\lambda : X \rightarrow \mathbb{R}$  by

$$\begin{aligned} \Phi(u) &= \int_{\Omega} \frac{w(x)}{p(x)} |\nabla u|^{p(x)} \, dx + \int_{\Omega} \frac{1}{p(x)} |u|^{p(x)} \, dx, \\ \Psi(u) &= \int_{\Omega} F(x, u) \, dx, \quad \text{and} \quad I_\lambda(u) = \Phi(u) - \lambda \Psi(u). \end{aligned}$$

Then  $\Phi \in C^1(X, \mathbb{R})$  ([8]) and it is easy to check that  $\Psi \in C^1(X, \mathbb{R})$  and its Gateaux derivatives are

$$\begin{aligned} \langle \Phi'(u), \varphi \rangle &= \int_{\Omega} w(x) |\nabla u(x)|^{p(x)-2} \nabla u(x) \cdot \nabla \varphi(x) dx \\ &\quad + \int_{\Omega} |u(x)|^{p(x)-2} u(x) \varphi(x) dx \end{aligned}$$

and

$$\langle \Psi'(u), \varphi \rangle = \int_{\Omega} f(x, u) \varphi(x) dx$$

for any  $u, \varphi \in X$ . Denote

$$\gamma(x) = \frac{r(x)}{r(x) - q(x)} \quad \text{for almost all } x \in \Omega,$$

where  $r$  and  $q$  are given in (H2).

The following existence result of a measurable function is important to estimate  $\Psi(u)$ ; see [10].

LEMMA 3.2. *Assume that (w2) and (H1) hold. Then there exist  $\delta$  with  $0 < \delta < 1$  and a measurable function  $\ell(x)$  that*

$$(3.2) \quad \max \left\{ \frac{p(x)\gamma(x)}{p(x) + \delta\gamma(x)}, \frac{p_s^*(x)}{p_s^*(x) + \delta - q(x)} \right\} \leq \ell(x) \\ \leq \min \left\{ \frac{p_s^*(x)\gamma(x)}{p_s^*(x) + \delta\gamma(x)}, \frac{p(x)}{p(x) + \delta - q(x)} \right\}$$

holds for almost all  $x \in \Omega$  and

$$(3.3) \quad \delta \left( \frac{\ell_+}{\ell_-} + 1 \right) < q_-.$$

Moreover, we have  $\ell \in L^\infty(\Omega)$  and  $1 < \ell(x) < \gamma(x)$ .

The following Lemma plays a key role in obtaining the main result in this section. The proof of this lemma proceeds the same way as in that of Lemma 4.3 in [10]. However, we will consider it because our problem has a Neumann boundary condition.

LEMMA 3.3. *Assume that (w1), (w2), (H1), (H2), (F1) and (F2) hold and satisfy*

$$(H3) \quad q_+ - \frac{1}{2}p_- < \delta,$$

where  $\delta$  is given in (3.3), then the functionals  $\Phi$  and  $\Psi$  satisfy the following relations:

$$(3.4) \quad \lim_{\|u\|_X \rightarrow 0} \frac{\Phi(u)}{\Psi(u)} = \infty,$$

and

$$(3.5) \quad \lim_{\|u\|_X \rightarrow \infty} \frac{\Phi(u)}{\Psi(u)} = \infty.$$

*Proof.* Applying Lemmas 2.1, 2.3 and 2.4, we get

$$(3.6) \quad \begin{aligned} |\Psi(u)| &= \left| \int_{\Omega} F(x, u) \, dx \right| \\ &\leq \int_{\Omega} \left| \frac{m(x)}{q(x)} |u|^{q(x)} \right| \, dx \\ &\leq \frac{2}{q_-} \|m\|_{L^{\gamma(\cdot)}(\Omega)} \| |u|^{q(x)} \|_{L^{\frac{r(\cdot)}{q(\cdot)}(\Omega)}} \\ &\leq \frac{2}{q_-} \|m\|_{L^{\gamma(\cdot)}(\Omega)} \left( \|u\|_{L^{r(\cdot)}(\Omega)}^{q_+} + \|u\|_{L^{r(\cdot)}(\Omega)}^{q_-} \right) \\ &\leq \frac{2c}{q_-} \|m\|_{L^{\gamma(\cdot)}(\Omega)} \left( \|u\|_X^{q_+} + \|u\|_X^{q_-} \right) \end{aligned}$$

for some positive constant  $c$ . Let  $u$  in  $X$  with  $\|u\|_X \leq 1$ . Then it follows from the above inequality and Lemma 2.2 (3) that

$$(3.7) \quad \begin{aligned} \left| \frac{\Phi(u)}{\Psi(u)} \right| &\geq \frac{\int_{\Omega} \frac{w(x)}{p(x)} |\nabla u|^{p(x)} \, dx + \int_{\Omega} \frac{1}{p(x)} |u|^{p(x)} \, dx}{\frac{4c}{q_-} \|m\|_{L^{\gamma(\cdot)}(\Omega)} \|u\|_X^{q_-}} \\ &\geq \frac{\frac{1}{p_+} \|u\|_X^{p_+}}{\frac{4c}{q_-} \|m\|_{L^{\gamma(\cdot)}(\Omega)} \|u\|_X^{q_-}}. \end{aligned}$$

Since  $q_- > p_+$ , we conclude that

$$\frac{\Phi(u)}{\Psi(u)} \rightarrow \infty \quad \text{as} \quad \|u\|_X \rightarrow 0.$$

Next we will show that the relation (3.5) holds. From (H3) and (3.3), we get that

$$(3.8) \quad p_+ > p_- > 2(q_+ - \delta) > 2(q_- - \delta) > \delta \frac{\ell_+}{\ell_-}.$$

Let  $u \in X$  with  $\|u\|_X > 1$ . Then it follows from (F2') and Lemma 2.1 that

$$\begin{aligned} |\Psi(u)| &\leq \frac{1}{q_-} \int_{\Omega} m(x) |u|^{\delta} |u|^{q(x)-\delta} dx \\ &\leq \frac{2}{q_-} \|m |u|^{\delta}\|_{L^{\ell(\cdot)}(\Omega)} \| |u|^{q(x)-\delta}\|_{L^{\ell'(\cdot)}(\Omega)}. \end{aligned}$$

Therefore, without loss of generality we may suppose that  $\|m |u|^{\delta}\|_{L^{\ell(\cdot)}(\Omega)} > 1$ . From the above inequality, by using Lemma 2.2, Lemma 2.1 and Lemma 2.3 in order, we have

$$\begin{aligned} |\Psi(u)| &\leq \frac{2}{q_-} \left( \int_{\Omega} m^{\ell(x)} |u|^{\delta \ell(x)} \right)^{\frac{1}{\ell_-}} \left( \|u\|_{L^{(q(\cdot)-\delta)\ell'(\cdot)}(\Omega)}^{q_+-\delta} + \|u\|_{L^{(q(\cdot)-\delta)\ell'(\cdot)}(\Omega)}^{q_--\delta} \right) \\ &\leq \frac{4}{q_-} \|m^{\ell(x)}\|_{L^{\frac{\gamma(\cdot)}{\ell(\cdot)}(\Omega)}^{\frac{1}{\ell_-}}} \| |u|^{\delta \ell(x)}\|_{L^{(\frac{\gamma(\cdot)}{\ell(\cdot)})'(\Omega)}^{\frac{1}{\ell_-}}} \left( \|u\|_{L^{(q(\cdot)-\delta)\ell'(\cdot)}(\Omega)}^{q_+-\delta} + \|u\|_{L^{(q(\cdot)-\delta)\ell'(\cdot)}(\Omega)}^{q_--\delta} \right) \\ &\leq \frac{4}{q_-} \|m\|_{L^{\gamma(\cdot)}(\Omega)}^{\alpha} \left( \|u\|_{L^{\delta \ell(\cdot)(\frac{\gamma(\cdot)}{\ell(\cdot)})'(\Omega)}^{\frac{\delta \ell_{\pm}}{\ell_-}} + \|u\|_{L^{\delta \ell(\cdot)(\frac{\gamma(\cdot)}{\ell(\cdot)})'(\Omega)}^{\delta} \right) \\ &\quad \times \left( \|u\|_{L^{(q(\cdot)-\delta)\ell'(\cdot)}(\Omega)}^{q_+-\delta} + \|u\|_{L^{(q(\cdot)-\delta)\ell'(\cdot)}(\Omega)}^{q_--\delta} \right), \end{aligned}$$

where  $\alpha = \begin{cases} \ell_+/\ell_- & \text{if } \|m\|_{L^{\gamma(\cdot)}(\Omega)} > 1 \\ 1 & \text{if } \|m\|_{L^{\gamma(\cdot)}(\Omega)} \leq 1. \end{cases}$

By Young's inequality, we get

$$\begin{aligned} |\Psi(u)| &\leq \frac{4}{q_-} \|m\|_{L^{\gamma(\cdot)}(\Omega)}^{\alpha} \left( \|u\|_{L^{\delta \ell(\cdot)(\frac{\gamma(\cdot)}{\ell(\cdot)})'(\Omega)}^{\frac{2\delta \ell_{\pm}}{\ell_-}} + \|u\|_{L^{\delta \ell(\cdot)(\frac{\gamma(\cdot)}{\ell(\cdot)})'(\Omega)}^{2\delta} \right. \\ &\quad \left. + \|u\|_{L^{(q(\cdot)-\delta)\ell'(\cdot)}(\Omega)}^{2(q_+-\delta)} + \|u\|_{L^{(q(\cdot)-\delta)\ell'(\cdot)}(\Omega)}^{2(q_--\delta)} \right). \end{aligned}$$

From (3.2),

$$1 < \delta \ell(x) \left( \frac{\gamma(x)}{\ell(x)} \right)' \leq p_s^*(x), \quad 1 < (q(x) - \delta)\ell'(x) \leq p_s^*(x)$$

hold for almost all  $x \in \Omega$ . Hence it follows from Lemma 2.4 that

$$(3.9) \quad |\Psi(u)| \leq \frac{4c}{q_-} \|m\|_{L^{\gamma(x)}(\Omega)}^{\alpha} \left( \|u\|_X^{\frac{2\delta \ell_{\pm}}{\ell_-}} + \|u\|_X^{2(q_+-\delta)} \right),$$



for some positive constant  $c$ . Therefore, we obtain that

$$\left| \frac{\Phi(u)}{\Psi(u)} \right| \geq \frac{\frac{1}{p_+} \|u\|_X^{p_-}}{\frac{4c}{q_-} \|m\|_{L^{\gamma(\cdot)}(\Omega)}^\alpha \left( \|u\|_X^{2\delta \frac{\ell_+}{\ell_-}} + \|u\|_X^{2(q_+ - \delta)} \right)}.$$

From (3.8) with the above inequality, we conclude that the relation (3.5) holds. □

LEMMA 3.4. *Assume that (w1), (w2) and (H1) hold. Then  $\Phi$  is weakly lower semi-continuous, i.e.,  $u_n \rightharpoonup u$  in  $X$  implies that  $\Phi(u) \leq \liminf_{n \rightarrow \infty} \Phi(u_n)$ .*

*Proof.* Let  $u$  be fixed in  $X$ . Since  $\Phi$  is convex, we know that  $\Phi(v) \geq \Phi(u) + \langle \Phi'(u), v - u \rangle$  for all  $v \in X$ . Hence we know that  $\Phi$  is weakly lower semi-continuous. □

LEMMA 3.5. *Assume that (w1), (w2), (H1)–(H3), (F1) and (F2) hold. For almost all  $x \in \Omega$  and all  $t \in \mathbb{R}$  the following estimate holds:*

$$(3.10) \quad F(x, t) \leq \frac{1}{q_-} \left( \frac{m(x)^{\gamma(x)}}{\gamma_-} + \frac{|t|^{r(x)}}{(\gamma_+)' } \right)$$

Moreover, the Nemytskij operator

$$u \mapsto F(x, u(x))$$

is continuous from  $L^{r(\cdot)}(\Omega)$  into  $L^1(\Omega)$ .

*Proof.* Since  $q(x)(\gamma(x))' = r(x)$ , the estimate (3.10) is obtained from (F2') and Young's inequality. For the remaining part, let  $\Psi_0 : L^{r(\cdot)}(\Omega) \rightarrow L^1(\Omega)$  be an operator defined by

$$\Psi_0(u)(x) = F(x, u(x)).$$

Let  $u_n \rightarrow u$  in  $L^{r(\cdot)}(\Omega)$  as  $n \rightarrow \infty$ . Then there exist a subsequence  $(u_{n_k})$  and measurable function  $v$  in  $L^{r(\cdot)}(\Omega)$  such that  $u_{n_k}(x) \rightarrow u(x)$  as  $k \rightarrow \infty$  for almost all  $x \in \Omega$  and  $|u_{n_k}(x)| \leq v(x)$  for all  $k \in \mathbb{N}$  and for almost all  $x \in \Omega$ . We have by (3.10) that

$$\begin{aligned} \|\Psi_0(u_{n_k}) - \Psi_0(u)\|_{L^1(\Omega)} &= \int_{\Omega} |F(x, u_{n_k}(x)) - F(x, u(x))| \, dx \\ &\leq C \int_{\Omega} (|m(x)|^{\gamma(x)} + |u_{n_k}(x)|^{r(x)} + |u(x)|^{r(x)}) \, dx \end{aligned}$$

for a positive constant  $C$ . Since  $F$  satisfies a Carathéodory condition by (F1) and (3.10), we obtain that  $F(x, u_{n_k}(x)) - F(x, u(x)) \rightarrow 0$  as  $k \rightarrow \infty$  for almost all  $x \in \Omega$ . Therefore the Lebesgue convergence theorem implies that  $\Psi_0(u_{n_k}) \rightarrow \Psi_0(u)$  in  $L^1(\Omega)$  as  $k \rightarrow \infty$ . We conclude that  $\Psi_0$  is continuous.  $\square$

LEMMA 3.6. *Assume that (w1), (w2), (H1), (H2), (F1) and (F2) hold. Then  $\Psi$  is weakly-strongly continuous, i.e.,  $u_n \rightharpoonup u$  in  $X$  implies that  $\Psi(u_n) \rightarrow \Psi(u)$ .*

*Proof.* Let  $\{u_n\}$  be a sequence in  $X$  such that  $u_n \rightharpoonup u$  in  $X$ . Since  $1 < r(x) < p_s^*(x)$ , Lemma 2.4 implies  $u_n \rightarrow u$  in  $L^{r(\cdot)}(\Omega)$ . This together with Lemma 3.5 yields that  $\Psi(u_n) \rightarrow \Psi(u)$  as  $n \rightarrow \infty$ . The proof is completed.  $\square$

LEMMA 3.7. *Assume that (w1), (w2), (H1), (H2), (F1) and (F2) hold. Then  $I_\lambda$  is coercive for all  $\lambda > 0$ .*

*Proof.* Let  $u \in X$  with  $\|u\|_X > 1$ . Proceeding as in the proof of relation (3.9) in Lemma 3.3, we deduce

$$I_\lambda(u) \geq \frac{c}{p_+} \|u\|_X^{p_-} - \lambda \frac{4C}{q_-} \|m\|_{L^{\gamma(\cdot)}(\Omega)}^\alpha \left( \|u\|_X^{2\delta \frac{\ell_+}{\ell_-}} + \|u\|_X^{2(q_+ - \delta)} \right).$$

Since  $p_- > 2(q_+ - \delta) > 2\delta(\ell_+/\ell_-)$ , the above inequality implies that  $I_\lambda(u) \rightarrow \infty$  as  $\|u\|_X \rightarrow \infty$  for all  $\lambda > 0$ , that is,  $I_\lambda$  is coercive. The proof is completed.  $\square$

We are in position to state the main results about the existence of the positive infimum eigenvalue for the problem (B). We can modify the proof of Theorem 3.1 in [1] with the aid of Lemmas 3.3, 3.4, and 3.6 to get the next theorem. For convenience, we consider the proof of the following assertion.

THEOREM 3.8. *Assume that (w1), (w2), (H1)-(H3), (F1) and (F2) hold. Then  $\lambda^*$  is a positive eigenvalue of the problem (B). Moreover the problem (B) has a nontrivial weak solution for any  $\lambda \geq \lambda^*$ .*

*Proof.* It is trivial by (3.1) that  $\lambda^* \geq 0$ . Suppose to the contrary that  $\lambda^* = 0$ . Let  $\{u_n\}$  be a sequence in  $X \setminus \{0\}$  such that

$$\lim_{n \rightarrow \infty} \frac{\Phi(u_n)}{\Psi(u_n)} = 0.$$

As in (3.7), we have

$$\left| \frac{\Phi(u_n)}{\Psi(u_n)} \right| \geq C \|u_n\|_X^{p_+ - q_-}$$

for some positive constant  $C$ . Since  $p_+ < q_-$ , we obtain that  $\|u_n\|_X \rightarrow \infty$  as  $n \rightarrow \infty$ . Hence it follows from Lemma 3.3 that

$$\lim_{n \rightarrow \infty} \frac{\Phi(u_n)}{\Psi(u_n)} = \infty,$$

which contradicts with the hypothesis. Hence we get  $\lambda^* > 0$ .

Next, we prove that  $\lambda^*$  is an eigenvalue for the problem (B). Let  $\{u_n\} \subseteq X \setminus \{0\}$  be a minimizing sequence for  $\lambda^*$ , namely,

$$(3.11) \quad \lambda^* = \lim_{n \rightarrow \infty} \frac{\Phi(u_n)}{\Psi(u_n)}.$$

From Lemma 3.3,  $\{u_n\}$  is bounded in  $X$  and so  $u_n \rightharpoonup u$  in  $X$  as  $n \rightarrow \infty$  for some nonzero element  $u \in X$ . Indeed, suppose that  $u \equiv 0$ . Since  $\Psi$  is weakly-strongly continuous, we know that  $\Psi(u_n) \rightarrow 0$  as  $n \rightarrow \infty$ . Using (3.11), we assert that

$$\lim_{n \rightarrow \infty} \Phi(u_n) = \lim_{n \rightarrow \infty} \frac{\Phi(u_n)}{\Psi(u_n)} \Psi(u_n) = 0.$$

Since either  $\Phi(u_n) \geq \|u_n\|_X^{p_+}$  or  $\Phi(u_n) \geq \|u_n\|_X^{p_-}$  by Lemma 2.2, we know that  $\|u_n\|_X \rightarrow 0$  as  $n \rightarrow \infty$ . Applying Lemma 3.3, we deduce that

$$\lim_{n \rightarrow \infty} \frac{\Phi(u_n)}{\Psi(u_n)} = \infty,$$

contradiction. Thus we have that  $u \not\equiv 0$ . Since  $u_n \rightharpoonup u$  in  $X$  as  $n \rightarrow \infty$ , we get by Lemmas 3.4 and 3.6 that

$$(3.12) \quad \Phi(u) \leq \liminf_{n \rightarrow \infty} \Phi(u_n) \quad \text{and} \quad \Psi(u_n) \rightarrow \Psi(u).$$

Hence the definition of  $\lambda^*$  and relation (3.12) imply that  $\Phi(u) = \lambda^* \Psi(u)$ . Therefore  $\lambda^*$  is a positive eigenvalue of the problem (B).

Finally, we show that the problem (B) has a nontrivial weak solution for any  $\lambda \geq \lambda^*$ . Notice that  $u$  is a weak solution of (B) if and only if  $u$  is a critical point of  $I_\lambda$ . Assume that  $\lambda > \lambda^*$  is fixed. Since the functional  $I_\lambda$  is weakly lower semi-continuous and coercive by Lemmas 3.4, 3.6, and 3.7, we deduce that there exists a global minimizer  $u_0$  of  $I_\lambda$  in  $X$ ; see [9].

Since  $\lambda > \lambda^*$ , we ensure by definition (3.1) that there is an element  $\omega$  in  $X \setminus \{0\}$  such that  $\Phi(\omega)/\Psi(\omega) < \lambda$ . Then  $I_\lambda(\omega) < 0$ . So we get that

$$I_\lambda(u_1) = \inf_{v \in X \setminus \{0\}} I_\lambda(v) < 0.$$

Consequently, we conclude that  $u_0 \neq 0$ . This completes the proof.  $\square$

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