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Evaluating the ANSS and ATS Values of the Multivariate EWMA Control Charts with Markov Chain Method

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Abstract

Average number of samples to signal (ANSS) and average time to signal (ATS) are the most widely used criterion for comparing the efficiencies of the quality control charts. In this study the method of evaluating ANSS and ATS values of the multivariate exponentially weighted moving average (EWMA) control charts with Markov chain approach was presented when the production process is in control state or out of control state. Through numerical results, it is found that when the number of transient state r is less than 50, the calculated ANSS and ATS values are unstable; and $ATS(r)$ tends to be stabilized when r is greater than 100; in addition, when the properties of multivariate EWMA control chart is evaluated using Markov chain method, the number of transient state r requires bigger values when the smoothing constatnt λ becomes smaller.

Key words: Asymptotic ATS, Control Chart, Markov Chain Method

1. Introduction

The main purpose of statistical process control (SPC) is to improve the quality and productivity. One of the efficient methods for checking shifts or variations in the production process is control chart. Average run length (ARL), ANSS and ATS are the most widely used criterion for comparing the efficiency of the quality control charts.

The ability of a control chart detecting process changes is determined by the length of time required for the chart to signal. Thus, a good control chart detects any changes quickly in the process while producing few false alarms. In traditional control chart, the run length (RL) is defined as the random number of samples required for the chart to signal and the ARL is the expected value of the RL. Therefore, the expected time to signal is simply the product of the ARL and the length of the fixed sampling interval in fixed sampling interval (FSI) chart, so the ARL can be thought of as the expected time to signal. If there are no changes in the process then the ARL should be large so that the frequency of false alarm is low, and if there is a change in the process then ARL should be small so that the change is quickly detected.

Integral equation method can be used to approximate the ATS of the control chart when the control statistic is a continuous random variable. Page[1] developed the integral equation method for the FSI CUSUM chart. The integral equation method to evaluate the approximation of the RL distribution has been used by Crowder[2] for the EWMA control chart.

Brook and Evans[3] originally developed a Markov chain method to get the RL distribution for a discrete one-sided CUSUM chart and to approximate the RL distribution for a continuous one-sided CUSUM chart by discretizing the continuous state space so that the CUSUM is restricted to a finite set of values. Cui and Reynolds[4] considered variable sampling interval (VSI) Shewhart $\overline{X}$-chart with runs rules using Markov chain method. As compared to the integral equation method, the Markov chain method offers more flexibility for computing some quantities which are not easy to obtain with the integral equation method.

VSI procedures were first investigated by Arnold[5]. Recent years, application of VSI chart has become quite frequent. The basic idea of VSI chart is that the time interval should be short if there is some indication of any process change and should be long if there is no

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indication of any process change. Reynolds[6] showed that the optimal VSI uses only two sampling intervals d_1, d_2 $(d_1 < d_2)$ spaced as apart as possible.

In VSI control chart, the time required to signal is not the product of the number of samples and a fixed sampling interval. Thus, for the performances of a VSI chart, it is necessary to keep track of both the time to signal and the number of samples to signal. Following the definitions of Reynolds *et al.*[7], the number of samples to signal (NSS) is the number of samples taken from the start of the process to the time when the chart signals, and the ANSS is the expected value of the NSS. They also defined the time to signal (TS) as the time from the start of the process to the time when the chart signals, and the ATS as the expected value of the TS. Therefore, there is no difference between the definitions of ANSS and ARL.

In this paper it is evaluated the ANSS and ATS of the multivariate EWMA control charts for monitoring mean vector of the several correlated quality variables with Markov chain method when the production process is in control state or out of control state.

2. Multivariate Control Statistic

The quality of an output is often characterized by joint levels of several correlated variables. In the case, a multivariate quality control procedure for simultaneously monitoring several correlated variables is useful. The multivariate approach to quality control was first introduced by Hotelling[8]. A multivariate EWMA control chart for monitoring mean vector of a multivariate normal process using accmulate-combine approach was presented by Lowry *et al.*[9]. Through simulation, they showed that the performance of the multivariate EWMA procedure performs better than the multivariate CUSUM procedures Pignatiello and Runger[10].

Suppose that the production process of interest has p quality characteristics whose distribution is multivariate normal with mean vector $\underline{\mu}$ and variance-covariance matrix Σ, and $(\underline{\mu}_0, \Sigma_0)$ is the known target process values for $(\underline{\mu}, \Sigma)$. For simplicity, we assume that $\underline{\mu}_0 = \underline{0}$, all diagonal and off-diagonal elements of Σ_0 are 1 and 0.3, respectively.

At each sampling time i $(i = 1, 2, 3, \cdots)$, we can obtain a sequence of random vector $\underline{X}_i{}' = (\underline{X}_{i1}{}', \underline{X}_{i2}{}', \cdots, \underline{X}_{in}{}')$ of the samples where $\underline{X}_{ij}{}' = (X_{ij1}, X_{ij2}, \cdots, X_{ijp})$. Thus $\underline{X}_i$ is an $np \times 1$ column vector. Then the jkth element X_{ijk} of $\underline{X}_i$ is the jth observation for kth quality characteristic at each sampling occasion i $(j = 1, 2, \cdots, n\,;\, k = 1, 2, \cdots, p)$. We assume that the sequential observation vectors between and within samples are independent and identically distributed.

Because a control chart can be viewed as repeated tests of significance, we can obtain multivariate control statistic for monitoring $\underline{\mu}$ by using the likelihood ratio test (LRT) statistic for testing $H_0 : \underline{\mu} = \underline{\mu}_0$ vs $H_1 : \underline{\mu} \neq \underline{\mu}_0$ where Σ_0 is known. Thus, the control statistic

$$Z_i^2 = n(\overline{\underline{X}}_i - \underline{\mu})' \Sigma_0^{-1} (\overline{\underline{X}}_i - \underline{\mu}) \tag{2.1}$$

can be used as the control statistic for monitoring mean vector of p correlated quality variables.

Since the control statistic Z_i^2 has a chi-square distribution with p degrees of freedom, the percentage point of Z_i^2 can be obtained from a chi-square distribution. When the process has shifted to $\underline{\mu}$ from the target $\underline{\mu}_0$, Z_i^2 has a non-central chi-square distribution with p degrees of freedom and noncentrality parameter $\tau^2 = n(\underline{\mu} - \underline{\mu}_0)' \Sigma_0^{-1} (\underline{\mu} - \underline{\mu}_0)$.

Multivariate FSI Shewhart chart for monitoring $\underline{\mu}$ signals whenever $Z_i^2 > \chi_\alpha(p)$, where $\chi_\alpha(p)$ is the upper 100αth percentile of a chi-square distribution with p degrees of freedom. Thus, LRT statistic Z_i^2 can be used as the control statistic for monitoring $\underline{\mu}$ of p correlated quality variables.

For the VSI Shewhart chart based on LRT statistic, suppose that the two sampling interval

$$\begin{aligned} &d_1 \text{ is used when } Z_i^2 \in (g_S, h_S] \\ &d_2 \text{ is used when } Z_i^2 \in (0, g_S], \end{aligned} \tag{2.2}$$

where $d_1 < d_2$. The process parameters g_S, h_S can be obtained from chi-square distribution to guarantee a desired ANSS and ATS. When the smoothing constant λ

is 1, the multivariate EWMA chart in (3.1) changes to multivariate Shewhart chart.

3. ANSS and ATS of Multivariate EWMA Chart

The ANSS and ATS of a multivariate EWMA control chart can be approximated using a procedure similar to that described by Brook and Evans[3]. They described discretizing the control statistic and then evaluated the exact properties of the discretized statistic, but in this study the properties of the continuous state Markov chain are evaluated by discretizing the infinite-state transition probability matrix.

In designing and evaluating the performances of the multivariate FSI and VSI EWMA control charts in (3.1) and (3.4), it is convenient to calculate ANSS and ATS, either the process in control or out of control state, when the distribution of control statistic Z_i^2 is known. Since the control statistic is continuous, the continuous state space of the chart statistic $Y_{Z^2,i}$ is partitioned into a finite number of discrete intervals and the probability distribution of the chart statistic is discretized to apply Markov chain method.

3.1. Fixed Sampling Interval Case

A multivariate FSI EWMA chart based on LRT statistic in (2.1) is given by

$$Y_{Z^2,i} = (1-\lambda)\, Y_{Z^2,i-1} + \lambda\, Z_i^2 \tag{3.1}$$

where $Y_{Z^2,0} = \omega\,(\omega \geq 0)$ and $\lambda\,(0 < \lambda \leq 1)$ is a smoothing constant. This chart signals whenever $Y_{Z^2,i} \geq h$.

Let the interval $(0,\infty)$ of chart statistic Y_j which depends on $\underline{X}_1, \underline{X}_2, \cdots, \underline{X}_j$ be divided into in control region $C = (0,h]$ and out of control region $C' = (h,\infty)$. Suppose that the in control region C is partitioned into r states $E_1, E_2, \cdots, E_r$ where each interval corresponds to a state of Markov chain, and absorbing state $C' = \{\underline{x} \mid Y_j > h\}$ is a signal region.

Since chart statistic Y_j is continuous, let a discretized version $\widetilde{Y}_j$ of $Y_j \in E_i$ be the midpoint of E_i. Assume that the in control region is divided into r states then $\omega = h/r$. The probability of moving from any state E_i to any other state E_j can be denoted as $p_{ij}(k) = P(Y_{k+1} \in E_j \mid Y_k \in E_i)$ for $i, j = 1, 2, \cdots, r+1$ and $k = 0, 1, 2, \cdots$.

Then the transition probability $p_{ij}(k)$ can be evaluated as follows. For $i = 1, 2, \cdots, r$,

$$\begin{aligned} p_{ij}(k) &= P[(1-\lambda)(i-\tfrac{1}{2})w + \lambda Z_k^2 \in [(j-1)w, jw)] \\ &= F\left[\frac{(j-i+0.5)\,w}{\lambda} + (i-0.5)w\right] \\ &\qquad - F\left[\frac{(j-i-0.5)w}{\lambda} + (i-0.5)w\right], \end{aligned}$$

where $j = 1, 2, \cdots, r$

$$\begin{aligned} p_{i,r+1}(k) &= P[\text{ go to } E_{r+1} \text{ state} \mid \text{in } E_i \text{ state}] \\ &= P[(1-\lambda)(i-\tfrac{1}{2})w + \lambda Z_k^2 \geq h] \\ &= 1 - F\left[\frac{h-(1-\lambda)(i-0.5)\,w}{\lambda}\right]. \end{aligned}$$

Here, $p_{ij}(k)$ will be written briefly as p_{ij} in the transition matrix and we denote $F(\cdot)$ as the distribution function of control statistic. Then the transition probability matrix P has the following form

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1r} & p_{1,r+1} \\ p_{21} & p_{22} & \cdots & p_{2r} & p_{2,r+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ p_{r1} & p_{r2} & \cdots & p_{rr} & p_{r,r+1} \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}_{(r+1)\times(r+1)}$$

and the above transition probability matrix can be partitioned as

$$P = \begin{bmatrix} Q & (I-Q)\cdot\underline{1} \\ \underline{1}' & 1 \end{bmatrix} \tag{3.2}$$

where Q is the $r \times r$ transition matrix corresponding to the transient state, I is the identity matrix, $\underline{0}$ is an $r \times 1$ vector of $0'$s, and $\underline{1}$ is the $r \times 1$ vector of 1's.

From the transition matrix P, we can obtain the fundamental matrix M as

$$M = (I-Q)^{-1} = [m_{ij}] \tag{3.3}$$

where m_{ij} is the expected number of visits to the transient state E_j before absorption, given that the Markov chain starts in transient state E_i. Hence when the process starts in state E_i, the ANSS and variance of N_i are given as

$$E(N_i) = \sum_{j=1}^{r} m_{ij}$$

and

$$V(N_i) = 2\sum_{k=1}^{r}\sum_{j=1}^{r} m_{ik}m_{kj} - \sum_{j=1}^{r} m_{ij} - (\sum_{j=1}^{r} m_{ij})^2.$$

3.2 Variable Sampling Interval Case

For the two sampling intervals VSI EWMA chart based on LRT statistic in (2.1), suppose the sampling interval ;

$$d_1 \text{ is used when } Y_{Z^2,i} \in (g,h] \quad (3.4)$$
$$d_2 \text{ is used when } Y_{Z^2,i} \in (0,g]$$

Suppose that the above VSI EWMA chart signals when $Y_{Z^2,i} \in C'$, the sampling interval d_2 is used when $Y_{Z^2,i} \in (g,h]$, and d_1 is used when $Y_{Z^2,i} \in (0,g]$. Assume that the interval $(0,g]$ is divided into m states and $(g,h]$ is divided into $(r-m)$ states, then ω and v are g/m and $(h-g)/(r-m)$, respectively.

To design and evaluate the performances of the VSI EWMA chart, let a finite number of interval lengths be $d_1, d_2, \cdots, d_\eta$ where $d_1 < d_2 < \cdots < d_\eta$. In addition let the continuous in control region C be partitioned into η regions $C_1, C_2, \cdots, C_\eta$ where C_i is the region in which the interval d_i is used when $Y_j \in C_i$.

Let $\underline{b} = (b_1, b_2, \cdots, b_r)$, $\underline{N} = (N_1, N_2, \cdots, N_r)$ and $\underline{T} = (T_1, T_2, \cdots, T_r)$ are vectors of sampling interval, NSS and TS, respectively. The ANSS vector is

$$E(\underline{N}) = M\underline{1} \quad (3.5)$$

and

$$V(\underline{N}) = (2M-I)\cdot E(\underline{N}) - [E(\underline{N})]^{(2)} \quad (3.6)$$

where $[E(\underline{N})]^{(2)}$ is a vector whose i th component is the square of the i th component of $E(\underline{N})$.

Following Reynolds[11], the ATS vector is

$$E(\underline{T}) = M\underline{b} \quad (3.7)$$

and

$$V(\underline{T}) = MB(2M-I)\underline{b} - [M\underline{b}]^{(2)} \quad (3.8)$$

where B is a diagonal matrix with elements of corresponding sampling interval and $[M\underline{b}]^{(2)}$ is a vector whose i th component is the square of the i th component of $M\underline{b}$. Hence when the process starts in state i, the ATS and variance of T_i are given as

$$E(T_i) = \sum_{j=1}^{r} m_{ij}b_j \quad (3.9)$$

and

$$V(T_i) = 2\sum_{k=1}^{r}\sum_{j=1}^{r} m_{ik}m_{kj}b_k b_j - \sum_{j=1}^{r} m_{ij}b_j^2 - (\sum_{j=1}^{r} m_{ij}b_j)^2.$$

Then the transition probability $p_{ij}(k)$ can be evaluated as follows. For $i = 1, 2, \cdots, m$,

$$\begin{aligned} p_{ij}(k) &= P[(1-\lambda)(i-\tfrac{1}{2})w + \lambda Z_k^2 \in E_j] \\ &= F\left[\frac{(j-i+0.5)w}{\lambda} + (i-0.5)w\right] \\ &\quad - F\left[\frac{(j-i-0.5)w}{\lambda} + (i-0.5)w\right], \end{aligned}$$

where $j = 1, 2, \cdots, m$

$$\begin{aligned} p_{ij}(k) &= F\left[PA + \frac{\{g+(j-m)v-(i-0.5)w\}}{\lambda}\right] \\ &\quad - F\left[PA + \frac{\{g+(j-m-1)v-(i-0.5)w\}}{\lambda}\right], \end{aligned}$$

where $j = m+1, \cdots, r$ and $PA = (i-\frac{1}{2})w$

Table 1. ANSS and ATS of multivariate EWMA charts with Markov chain method ($p=2$, $\lambda=0.1$)

	EWMA ($r=10$)		EWMA ($r=40$)		EWMA ($r=100$)		EWMA ($r=110$)		EWMA ($r=160$)		Shewhart chart ($\lambda=1.0$)	
	ANSS	ATS	ANSS	ATS	ANSS	ATS	ANSS	ATS	ANSS	ATS	ANSS	ATS
In control	366.09	385.99	494.91	503.41	500.00	499.98	500.16	500.08	500.59	499.68	500.04	500.04
$\tau=0.5$	150.39	126.73	172.24	133.21	173.27	131.61	173.31	131.62	173.40	131.47	265.77	246.31
$\tau=1.0$	38.16	26.89	38.74	26.08	38.84	26.04	38.84	26.04	38.86	26.05	84.95	63.15
$\tau=1.5$	15.56	13.13	15.75	13.41	15.80	13.48	15.81	13.49	15.82	13.51	28.33	15.21
$\tau=2.0$	8.74	8.42	8.90	8.72	8.94	8.79	8.94	8.79	8.95	8.81	11.00	4.36
$\tau=2.5$	5.74	5.99	5.87	6.24	5.90	6.29	5.90	6.29	5.91	6.31	5.09	1.88
$\tau=3.0$	4.13	4.56	4.24	4.76	4.26	4.80	4.26	4.81	4.26	4.81	2.81	1.27
$\tau=3.5$	3.17	3.65	3.25	3.82	3.27	3.85	3.27	3.85	3.27	3.86	1.83	1.10
$\tau=4.0$	2.54	3.03	2.61	3.16	2.62	3.19	2.62	3.19	2.63	3.20	1.37	1.04
$\tau=4.5$	2.12	2.58	2.17	2.69	2.18	2.71	2.18	2.71	2.18	2.71	1.16	1.02
$\tau=5.0$	1.81	2.26	1.86	2.33	1.87	2.35	1.87	2.35	1.87	2.35	1.06	1.01
$\tau=5.5$	1.57	2.06	1.61	2.10	1.62	2.11	1.62	2.11	1.62	2.11	1.02	1.00
$\tau=6.0$	1.36	1.96	1.40	1.98	1.40	1.98	1.40	1.98	1.41	1.98	1.01	1.00

Table 2. ANSS and ATS of multivariate EWMA charts with Markov chain method ($p=2$, $\lambda=0.3$)

	EWMA ($r=10$)		EWMA ($r=40$)		EWMA ($r=100$)		EWMA ($r=110$)		EWMA ($r=160$)		Shewhart chart ($\lambda=1.0$)	
	ANSS	ATS	ANSS	ATS	ANSS	ATS	ANSS	ATS	ANSS	ATS	ANSS	ATS
In control	489.73	460.71	499.30	498.75	499.96	499.97	500.00	499.90	500.05	499.74	500.04	500.04
$\tau=0.5$	212.40	166.13	215.33	175.33	215.56	175.48	215.57	175.43	215.60	175.33	265.77	246.31
$\tau=1.0$	48.23	24.09	48.62	23.92	48.67	23.87	48.67	23.86	48.68	23.85	84.95	63.15
$\tau=1.5$	14.50	6.51	14.61	6.43	14.64	6.43	14.64	6.43	14.64	6.44	28.33	15.21
$\tau=2.0$	6.56	3.83	6.64	3.87	6.66	3.88	6.66	3.88	6.66	3.88	11.00	4.36
$\tau=2.5$	3.88	2.88	3.94	2.92	3.95	2.93	3.95	2.93	3.96	2.93	5.09	1.88
$\tau=3.0$	2.67	2.38	2.72	2.41	2.73	2.42	2.73	2.42	2.73	2.42	2.81	1.27
$\tau=3.5$	2.02	2.12	2.06	2.13	2.06	2.14	2.06	2.14	2.06	2.14	1.83	1.10
$\tau=4.0$	1.62	1.98	1.65	1.99	1.65	1.99	1.65	1.99	1.66	1.99	1.37	1.04
$\tau=4.5$	1.35	1.93	1.38	1.93	1.38	1.93	1.38	1.93	1.38	1.93	1.16	1.02
$\tau=5.0$	1.18	1.91	1.20	1.91	1.20	1.91	1.20	1.91	1.20	1.91	1.06	1.01
$\tau=5.5$	1.08	1.90	1.09	1.90	1.09	1.90	1.09	1.90	1.09	1.90	1.02	1.00
$\tau=6.0$	1.03	1.90	1.03	1.90	1.03	1.90	1.03	1.90	1.03	1.90	1.01	1.00

Table 3. ANSS and ATS of multivariate EWMA charts with Markov chain method ($p=5$, $\lambda=0.1$)

	EWMA ($r=10$)		EWMA ($r=40$)		EWMA ($r=100$)		EWMA ($r=110$)		EWMA ($r=160$)		Shewhart chart ($\lambda=1.0$)	
	ANSS	ATS	ANSS	ATS	ANSS	ATS	ANSS	ATS	ANSS	ATS	ANSS	ATS
In control	255.32	603.06	488.70	512.85	499.96	500.04	500.34	499.54	501.31	498.04	499.99	500.00
$\tau=0.5$	153.37	273.55	241.78	210.78	245.50	204.98	245.62	204.76	245.95	204.15	344.91	327.13
$\tau=1.0$	57.50	61.75	67.25	47.70	67.57	47.05	67.58	47.03	67.62	46.98	147.32	119.26
$\tau=1.5$	26.08	25.94	27.06	23.57	27.12	23.62	27.13	23.63	27.14	23.65	55.24	34.64
$\tau=2.0$	15.09	16.21	15.32	15.95	15.37	16.05	15.37	16.06	15.38	16.08	21.53	9.93
$\tau=2.5$	10.03	11.43	10.20	11.69	10.24	11.79	10.24	11.79	10.25	11.81	9.36	3.42
$\tau=3.0$	7.24	8.63	7.39	8.98	7.42	9.07	7.43	9.07	7.43	9.09	4.67	1.72
$\tau=3.5$	5.53	6.83	5.66	7.16	5.69	7.23	5.69	7.23	5.69	7.24	2.70	1.24
$\tau=4.0$	4.39	5.60	4.51	5.87	4.53	5.93	4.53	5.94	4.53	5.95	1.81	1.10
$\tau=4.5$	3.60	4.71	3.70	4.94	3.72	4.99	3.72	4.99	3.72	5.00	1.37	1.04
$\tau=5.0$	3.03	4.06	3.11	4.25	3.13	4.29	3.13	4.29	3.14	4.30	1.16	1.02
$\tau=5.5$	2.60	3.58	2.67	3.75	2.69	3.78	2.69	3.78	2.69	3.79	1.06	1.01
$\tau=6.0$	2.27	3.17	2.33	3.34	2.34	3.38	2.34	3.38	2.35	3.39	1.02	1.00

Table 4. ANSS and ATS of multivariate EWMA charts with Markov chain method ($p=5,\ \lambda=0.3$)

	EWMA ($r=10$)		EWMA ($r=40$)		EWMA ($r=100$)		EWMA ($r=110$)		EWMA ($r=160$)		Shewhart chart ($\lambda=1.0$)	
	ANSS	ATS	ANSS	ATS	ANSS	ATS	ANSS	ATS	ANSS	ATS	ANSS	ATS
In control	476.24	484.43	498.65	500.69	499.98	500.00	500.04	499.95	500.15	499.64	499.99	500.00
$\tau=0.5$	279.05	250.83	289.72	255.42	290.36	254.76	290.39	254.71	290.45	254.52	344.91	327.13
$\tau=1.0$	85.62	54.32	87.42	53.42	87.55	53.17	87.55	53.15	87.57	53.10	147.32	119.26
$\tau=1.5$	26.48	12.89	26.72	12.60	26.75	12.57	26.76	12.57	26.76	12.57	55.24	34.64
$\tau=2.0$	11.16	6.28	11.25	6.34	11.27	6.36	11.27	6.36	11.28	6.37	21.53	9.93
$\tau=2.5$	6.23	4.43	6.30	4.54	6.32	4.56	6.32	4.56	6.32	4.57	9.36	3.42
$\tau=3.0$	4.13	3.47	4.20	3.58	4.21	3.60	4.21	3.60	4.21	3.61	4.67	1.72
$\tau=3.5$	3.04	2.86	3.09	2.95	3.10	2.97	3.10	2.97	3.10	2.97	2.70	1.24
$\tau=4.0$	2.38	2.45	2.43	2.52	2.44	2.53	2.44	2.53	2.44	2.53	1.81	1.10
$\tau=4.5$	1.96	2.18	2.00	2.22	2.00	2.23	2.00	2.23	2.01	2.23	1.37	1.04
$\tau=5.0$	1.65	2.02	1.69	2.04	1.70	2.05	1.70	2.05	1.70	2.05	1.16	1.02
$\tau=5.5$	1.42	1.94	1.45	1.95	1.46	1.95	1.46	1.96	1.46	1.96	1.06	1.01
$\tau=6.0$	1.24	1.91	1.26	1.92	1.27	1.92	1.27	1.92	1.27	1.92	1.02	1.00

Table 5. ANSS and ATS of multivariate EWMA charts with Markov chain method ($p=10,\ \lambda=0.1$)

	EWMA ($r=10$)		EWMA ($r=40$)		EWMA ($r=100$)		EWMA ($r=110$)		EWMA ($r=160$)		Shewhart chart ($\lambda=1.0$)	
	ANSS	ATS	ANSS	ATS	ANSS	ATS	ANSS	ATS	ANSS	ATS	ANSS	ATS
In control	182.19	-	479.28	539.13	500.01	500.00	500.73	498.68	502.54	495.36	499.99	499.99
$\tau=0.5$	135.98	693.36	289.94	282.85	299.28	262.73	299.60	262.06	300.41	260.35	391.79	376.91
$\tau=1.0$	71.28	169.78	100.36	76.74	101.59	73.29	101.64	73.18	101.75	72.90	208.91	178.67
$\tau=1.5$	37.77	54.55	41.00	34.99	41.12	34.66	41.13	34.66	41.15	34.65	91.72	64.39
$\tau=2.0$	22.90	29.55	23.05	24.08	23.10	24.13	23.10	24.13	23.12	24.16	38.39	20.68
$\tau=2.5$	15.48	19.51	15.49	18.25	15.53	18.36	15.54	18.36	15.55	18.39	16.72	6.83
$\tau=3.0$	11.28	14.24	11.37	14.37	11.42	14.49	11.42	14.49	11.43	14.52	7.96	2.76
$\tau=3.5$	8.66	11.15	8.80	11.63	8.84	11.73	8.84	11.74	8.85	11.76	4.25	1.58
$\tau=4.0$	6.90	9.15	7.05	9.62	7.09	9.71	7.09	9.72	7.09	9.73	2.58	1.22
$\tau=4.5$	5.67	7.72	5.81	8.10	5.84	8.19	5.84	8.19	5.84	8.21	1.77	1.09
$\tau=5.0$	4.76	6.61	4.88	6.94	4.91	7.01	4.91	7.02	4.91	7.03	1.37	1.04
$\tau=5.5$	4.07	5.74	4.18	6.04	4.20	6.10	4.20	6.10	4.21	6.11	1.16	1.02
$\tau=6.0$	3.54	5.04	3.63	5.32	3.65	5.37	3.65	5.38	3.66	5.39	1.07	1.01

Table 6. ANSS and ATS of multivariate EWMA charts with Markov chain method ($p=10,\ \lambda=0.3$)

	EWMA ($r=10$)		EWMA ($r=40$)		EWMA ($r=100$)		EWMA ($r=110$)		EWMA ($r=160$)		Shewhart chart ($\lambda=1.0$)	
	ANSS	ATS	ANSS	ATS	ANSS	ATS	ANSS	ATS	ANSS	ATS	ANSS	ATS
In control	458.53	521.26	497.60	502.61	499.97	500.03	500.06	499.87	500.26	499.43	499.99	499.99
$\tau=0.5$	316.13	327.81	339.28	312.58	340.69	310.70	340.73	310.59	340.86	310.29	391.79	376.91
$\tau=1.0$	126.00	99.21	131.74	92.42	132.10	91.72	132.12	91.68	132.15	91.58	208.91	178.67
$\tau=1.5$	43.56	24.83	44.40	23.19	44.47	23.06	44.48	23.05	44.48	23.04	91.72	64.39
$\tau=2.0$	17.94	10.04	18.07	9.86	18.09	9.87	18.09	9.87	18.10	9.88	38.39	20.68
$\tau=2.5$	9.54	6.59	9.60	6.69	9.62	6.72	9.62	6.72	9.62	6.73	16.72	6.83
$\tau=3.0$	6.12	5.11	6.18	5.25	6.20	5.28	6.20	5.28	6.20	5.29	7.96	2.76
$\tau=3.5$	4.40	4.19	4.47	4.33	4.48	4.36	4.48	4.36	4.48	4.37	4.25	1.58
$\tau=4.0$	3.40	3.54	3.46	3.68	3.47	3.71	3.47	3.71	3.48	3.71	2.58	1.22
$\tau=4.5$	2.76	3.05	2.81	3.18	2.82	3.20	2.82	3.20	2.82	3.21	1.77	1.09
$\tau=5.0$	2.31	2.65	2.36	2.76	2.37	2.78	2.37	2.79	2.37	2.79	1.37	1.04
$\tau=5.5$	2.00	2.33	2.04	2.42	2.05	2.44	2.05	2.44	2.05	2.44	1.16	1.02
$\tau=6.0$	1.76	2.11	1.80	2.17	1.81	2.18	1.81	2.18	1.81	2.19	1.07	1.01

$$p_{i,r+1}(k) = P\left[\ Z_k^2 > \frac{h-(1-\lambda)(i-0.5)w}{\lambda}\ \right]$$
$$= 1 - F\left[\ \frac{h-(1-\lambda)(i-0.5)w}{\lambda}\ \right].$$

And for $i = m+1, m+2, \cdots, r$,

$$p_{ij}(k) = P[(1-\lambda)\left\{g+(i-m-\frac{1}{2})v\right\}+\lambda Z_k^2 \in E_j]$$
$$= F\left[PB + \frac{jw-g-(i-m-0.5)v}{\lambda}\right]$$
$$-F\left[PB + \frac{(j-1)w-g-(i-m-0.5)v}{\lambda}\right],$$

where $j = 1,\cdots,m$ and $PB = g+(i-m-\frac{1}{2})v$

$$p_{ij}(k) = F\left[g+(i-m-\frac{1}{2})v+\frac{(j-i+0.5)\,v}{\lambda}\right]$$
$$-F\left[g+(i-m-\frac{1}{2})v+\frac{(j-i-0.5)\,v}{\lambda}\right],$$

where $j = m+1, m+2, \cdots, r$

$$p_{i,r+1} = P\left[Z_k^2 \geq \frac{h-(1-\lambda)[g+(i-m-0.5)v]}{\lambda}\right]$$
$$= 1 - F\left[\frac{h-(1-\lambda)[g+(i-m-0.5)v]}{\lambda}\right].$$

Let $ATS(r)$ be an asymptotic ATS calculated using r states. Lucas and Crosier[12] showed that an approximation of the continuous state ATS with

a second degree polynomial in $1/r^2$ is a good approximation. This polynomial is of the form

$$ATS(r) = asymptotic\ \ ATS + \frac{A}{r^2} + \frac{B}{r^4} \qquad (3.10)$$

where A and B are the coefficients.

For large r, we also obtain the ATS by taking the asymptotic ATS. The $ATS(r)$ tends to stabilize as the number of transient state r increases.

For the two sampling interval VSI chart, the sampling interval d_1 is used when $Y_j \in C_1$ and the sampling interval d_2 is used when $Y_j \in C_2$. Then, when the Markov chain starts in state i, the ATS is

$$ATS_i = d_2 \sum_{j=1}^{m} m_{ij} + d_1 \sum_{j=m+1}^{r} m_{ij} \qquad (3.11)$$

4. Computational Results and Concluding Remarks

To evaluate ad compare the properties for matched FSI and VSI multivariate EWMA charts with Markov chain method, in this study it is employed that the sampling interval of unit time $d = 1$ in FSI charts and $d_1 = 0.1$ and $d_2 = 1.9$ for the two sampling intervals VSI charts. In the computation, the ANSS and ATS of the chart were fixed to 500.0 for the process in control and the sample size for each characteristic was 5 for $p = 2,\ 5,\ 10$.

The design parameters h_S and g_S values of the Shewhart chart were obtained from the percentage points of chi-square distributions to satisfy an in control ATS and ANSS. After the smoothing constant λ of the proposed EWMA chart in (3.1) had been determined, the parameters h and g were calculated by Markov chain method when the number of transient states r is equal to 100.

The numerical results of ANSS and ATS using Markov chain method for the matched FSI and VSI charts are given in Table 1 through Table 6 when the process is in control state or out of control state. It is found that smaller values of smoothing constant λ are more efficient for small shifts, and that VSI features are more efficient than FSI. And, when ANSS and ATS is evaluated by using Makov chain method, the larger number of transient state r is required for smaller value of λ than the larger values, for the ANSS and ATS to be stabilized.

In addition, the asymptotic ANSS and ATS values with different number of transient state r are given in Tables. When r is less than 50, the ANSS and ATS values calculated by Markov chain method is appeared unstable, and it was found through numerical results that the calculated ANSS and ATS are stabilized as r is increasing. In our computation, ATS(r) tends to be stabilize when the number r is greater than 100 for various p.

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