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# STEADY NONLINEAR HYDROMAGNETIC FLOW OVER A STRETCHING SHEET WITH VARIABLE THICKNESS AND VARIABLE SURFACE TEMPERATURE

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ABSTRACT. This work is focused on the boundary layer and heat transfer characteristics of hydromagnetic flow over a stretching sheet with variable thickness. Steady, two dimensional, nonlinear, laminar flow of an incompressible, viscous and electrically conducting fluid over a stretching sheet with variable thickness and power law velocity in the presence of variable magnetic field and variable temperature is considered. Governing equations of the problem are converted into ordinary differential equations utilizing similarity transformations. The resulting non-linear differential equations are solved numerically by utilizing Nachtsheim-Swigert shooting iterative scheme for satisfaction of asymptotic boundary conditions along with fourth order Runge-Kutta integration method. Numerical computations are carried out for various values of the physical parameters and the effects over the velocity and temperature are analyzed. Numerical values of dimensionless skin friction coefficient and non-dimensional rate of heat transfer are also obtained.

## 1. INTRODUCTION

During the last few decades, the dynamics of boundary layer flow over a stretching sheet has received great attention owing to its abundance of practical applications in chemical and manufacturing processes, such as polymer extrusion, hot rolling, spinning of filaments, metal extrusion, crystal growing, glass fiber production, paper production, continuous casting of metals, copper wires drawing and glass blowing [1–3]. The pioneering work of Sakiadis [4] originated the problem on continuous moving solid surface. Following his path, some closed form of exponential solution of two-dimensional flow past a stretching plane was established by Crane [5]. Later Banks [6] obtained the similarity solutions of the boundary layer equations for a stretching wall. Since then, research area of stretching sheet has been flooded with many research articles with multiple dimensions enriched by the innovative researchers.

During the metallurgical processes, the rate of cooling can be controlled by drawing such strips into an electrically conducting fluid subjected to a magnetic field in order to get the final

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products of desired characteristics; as such a process greatly depends on the rate of cooling. In view of this, the study has been extended to hydromagnetic flow over a stretching sheet along with heat transfer characteristics and was investigated by many researchers [7–9]. Sparrow and Cess [7] reported the effect of magnetic field on the natural convection heat transfer. Chakrabarti and Gupta [8] analyzed the hydromagnetic flow and heat transfer over a stretching sheet. The hydromagnetic convective flow over the continuous moving surface was studied by Vajravelu [9].

New dimension in the field of stretching sheet has arrived that it can be stretched nonlinearly. Similarity solutions of the boundary layer equations for a nonlinearly stretching sheet was analysed by Talay Akyildiz et al. [10]. Heat transfer over a nonlinearly stretching sheet with non-uniform heat source and variable wall temperature was studied by Nandeppanavara et al. [11]. Significance of magnetic field over stretching sheet with power law velocity was enlightened by many authors. Chaim [12] examined the hydromagnetic flow over a surface with a power-law velocity. Behrouz et al. [13] obtained the solution to the MHD flow over a non-linear stretching sheet.

Practically, the stretching sheet need not be flat. Sheet with variable thickness can be encountered more often in real world applications. Plates with variable thickness are often used in machine design, architecture, nuclear reactor technology, naval structures and acoustical components. Variable thickness is one of the significant properties in the analysis of vibration of orthotropic plates [14]. Historically, the concepts of variable thickness sheets originate through linearly deforming substance such as needles and nozzles. Idea about the variable thickness sheet was initiated by Lee [15]. Following that, the characteristics of continuously moving thin needle in a parallel free stream was investigated by Ishak et al. [16].

Recently, Tiegang Fang et al. [17] studied the behavior of boundary layer flow over a stretching sheet with variable thickness. The numerical solution for boundary layer flow due to a nonlinearly stretching sheet with variable thickness and slip velocity has been obtained by Khader and Megahed [18]. The concept of variable surface temperature becomes significant in the studies over a deforming object like stretching sheet with variable thickness slendering away from the slot. Grubka and Bobba [19] worked on the heat transfer characteristics of a continuous stretching surface with variable temperature. The steady nonlinear hydromagnetic flow and heat transfer over a stretching surface of variable temperature was analyzed by Anjali Devi and Thiyagarajan [20].

So far no attempt has been tried towards hydromagnetic flow and heat transfer characteristics with variable surface temperature over a stretching sheet with variable thickness. In this work, a special form of magnetic field is considered along with the variable surface temperature to analyze various aspects of the flow and heat transfer effects.

## 2. Formulation of the Problem

Steady, two dimensional, nonlinear, laminar hydromagnetic flow of an incompressible, viscous and electrically conducting fluid over a stretching sheet with variable thickness and power law velocity in the presence of variable magnetic field and variable temperature is considered.

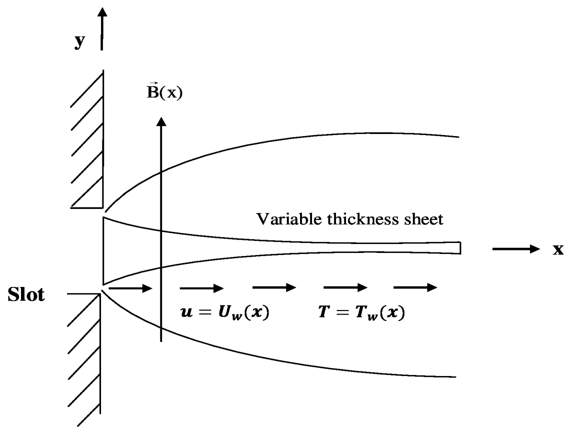


**Fig. 1 Schematic of the problem**

The $x$-axis is chosen in the direction of the sheet motion and the $y$-axis is perpendicular to it. The following assumptions are made

- The wall is impermeable with $v_w = 0$.
- The sheet is stretching with the velocity $U_w(x) = U_0(x+b)^m$ where $U_0$ is constant, $b$ is the physical parameter related to stretching sheet and $m$ is the velocity power index.
- The sheet is not flat and is described with a given profile which is specified as $y = A(x+b)^{\frac{1-m}{2}}$, where the coefficient $A$ is chosen as small for the sheet to be sufficiently thin, to avoid pressure gradient along the sheet $\left(\frac{\partial p}{\partial x} = 0\right)$.
- The problem is valid for $m \neq 1$ since $m = 1$ refers to the flat sheet case.
- The magnetic Reynolds number is assumed as so small so that the induced magnetic field is negligible. As the induced magnetic field is assumed to be negligible and since $B(x)$ is independent of time, $curl\vec{E} = 0$. In the absence of surface charge density, $div\vec{E} = 0$ . Hence the external electric field is assumed as negligible.
- The viscous and Joule dissipation are considered to be negligible.

Under the above assumptions, the steady boundary layer equations are given by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2.1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} - \frac{\sigma B(x)^2 u}{\rho}, \tag{2.2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{k}{\rho C_p}\frac{\partial^2 T}{\partial y^2} \tag{2.3}$$

with the boundary conditions

$$\begin{cases} u\left(x, A(x+b)^{\frac{1-m}{2}}\right) = U_w(x) = U_0(x+b)^m, v\left(x, A(x+b)^{\frac{1-m}{2}}\right) = 0, \\ T\left(x, A(x+b)^{\frac{1-m}{2}}\right) = T_w\left(x\right), \\ u\left(x,\infty\right) = 0, T\left(x,\infty\right) = T_\infty, \end{cases} \tag{2.4}$$

where $u, v$ are the velocity components in the $x$ and $y$ directions respectively, $\nu$ is the kinematic viscosity, $\sigma$ is the electrical conductivity of the fluid, $\rho$ is the density of the fluid, $k$ is the thermal conductivity of the fluid, $m$ is the velocity power index, $C_p$ is the specific heat at constant pressure, $T$ is the fluid temperature, $T_w(x)$ is the wall temperature, $T_\infty$ is the temperature far away from the sheet.

## 3. Similarity Transformations

The special form of magnetic field and wall temperature are considered as

$$B\left(x\right) = B_0\left(x+b\right)^{\frac{m-1}{2}} \tag{3.1}$$

and

$$T_w\left(x\right) = T_\infty + T_0(x+b)^r, \tag{3.2}$$

where $r$ is the temperature index parameter. The above forms are chosen to obtain the similarity solutions. Following Khader and Megahed [18], the stream function and similarity transformation are introduced to solve the equations (2.1)-(2.3) subject to (2.4).

$$\psi\left(x,y\right) = f\left(\eta\right)\sqrt{\frac{2}{m+1}\,\nu\, U_0\left(x+b\right)^{m+1}}, \tag{3.3}$$

$$\eta = y\sqrt{\frac{m+1}{2}\frac{U_0\left(x+b\right)^{m-1}}{\nu}}, (m \neq 1)\,. \tag{3.4}$$

Considering the similarity transformation $\theta$ as:

$$\theta = \frac{T - T_\infty}{T_w(x) - T_\infty}. \tag{3.5}$$

Equations (3.3)-(3.5) are proposed based on the standard practice for similarity transformation of partial differential equations. The stream function $\psi$ is defined as

$$u = \frac{\partial\psi}{\partial y} \qquad \text{and} \qquad v = -\frac{\partial\psi}{\partial x}. \tag{3.6}$$

Using the equations (3.3), (3.4) and (3.6), the velocity components are expressed as follows

$$u = U_0\,\left(x+b\right)^m\, f'\left(\eta\right), \tag{3.7}$$

$$v = -\sqrt{\frac{m+1}{2}\,\nu\, U_0\,\left(x+b\right)^{m-1}}\,\left[f'\left(\eta\right)\,\eta\left(\frac{m-1}{m+1}\right) + f\left(\eta\right)\right], (m \neq 1)\,. \tag{3.8}$$

Equation of continuity (2.1) is automatically satisfied. Using the similarity transformations (3.1)-(3.5), the nonlinear partial differential equations (2.2) and (2.3) with boundary conditions (2.4) are reduced to the following nonlinear ordinary differential equations:

$$f''' = \left[ \left( \frac{2m}{m+1} \right) (f')^2 - f\, f'' + M^2 f' \right], \tag{3.9}$$

$$\theta'' = Pr \left[ \left( \frac{2r}{m+1} \right) f'\, \theta - f\, \theta' \right], \tag{3.10}$$

with the boundary conditions, where $(m \neq 1)$

$$f(\alpha) = \alpha \left( \frac{1-m}{m+1} \right), \; f'(\alpha) = 1, \; \theta(\alpha) = 1,$$

$$f'(\infty) = 0, \; \theta(\infty) = 0, \tag{3.11}$$

where $\alpha = A\sqrt{\frac{m+1}{2}\frac{U_0}{\nu}}$ is the wall thickness parameter and $\eta = \alpha = A\sqrt{\frac{m+1}{2}\frac{U_0}{\nu}}$ indicates the plate surface. Equations (3.9) and (3.10) with the boundary conditions (3.11) are the nonlinear differential equations with a domain $[\alpha, \infty)$. In order to facilitate the computation and to transform the domain into $[0, \infty)$, we define $F(\xi) = F(\eta - \alpha) = f(\eta)$ and $\Theta(\xi) = \Theta(\eta - \alpha) = \theta(\eta)$. The similarity equations become

$$F''' = \left[ \left( \frac{2m}{m+1} \right) (F')^2 - F\, F'' + M^2 F' \right], \tag{3.12}$$

$$\Theta'' = Pr \left[ \left( \frac{2r}{m+1} \right) F'\, \Theta - F\, \Theta' \right], \tag{3.13}$$

with the boundary conditions, where $(m \neq 1)$

$$F(0) = \alpha \left( \frac{1-m}{m+1} \right), \; F'(0) = 1, \; \Theta(0) = 1,$$

$$F'(\infty) = 0, \; \Theta(\infty) = 0, \tag{3.14}$$

where the prime indicates differentiation with respect to $\xi$, $M^2 = \dfrac{2\sigma {B_0}^2}{\rho U_0 (m+1)}$ is the magnetic interaction parameter, $Pr = \dfrac{\mu C_p}{k}$ is the Prandtl number. Based on the variable transformation, the solution domain will be fixed from 0 to $\infty$.

The important physical quantities of interest, the local skin-friction coefficient $C_f$ and local Nusselt number $Nu_x$ are defined as

$$C_f = \frac{\mu \left(\frac{\partial u}{\partial y}\right)_{y=A(x+b)^{\frac{1-m}{2}}}}{\frac{1}{2}\rho {U_w}^2} = 2\sqrt{\frac{m+1}{2}}\,(Re_x)^{-\frac{1}{2}}\,F''(0), \tag{3.15}$$

$$Nu_x = \frac{(x+b)\left(\frac{\partial T}{\partial y}\right)_{y=A(x+b)^{\frac{1-m}{2}}}}{(T_w(x)-T_\infty)} = -\sqrt{\frac{m+1}{2}}\,(Re_x)^{\frac{1}{2}}\,\Theta'(0), \tag{3.16}$$

where $Re_x = \dfrac{U_w X}{\nu}$ is the local Reynolds number and $X = (x+b)$.

## 4. Numerical solution

The set of non-linear differential equations (3.12) and (3.13) constitute the nonlinear boundary value problem prescribed at the boundaries. The governing system of partial differential equations are first reduced to a system of ordinary differential equations. The crux of the problem is that we have to make an initial guess for the values of $F''(0)$ and $\Theta'(0)$. The system of transformed equations together with the boundary conditions are solved numerically using Nachtsheim-Swigert shooting iteration technique [21] for the satisfaction of the asymptotic boundary conditions along with fourth order Runge-Kutta integration method. In this particular shooting method, the governing system of partial differential equations are at first to initiate the shooting process. The success of the procedure depends on the appropriate choice of the guess. The different initial guesses were made taking into account of the convergence. The process is repeated until the results are corrected upto desired accuracy of $10^{-5}$ level. The numerical solutions are obtained for several values of the physical parameters over the flow field and dimensionless temperature distribution. Numerical values of dimensionless skin friction coefficient and non dimensional rate of heat transfer are also obtained.

## 5. Results and Discussion

The main objective of this work is to establish the influence of magnetic field, wall thickness parameter and temperature index parameter over the stretching sheet with variable thickness. The reliability of the numerical procedure of this work has been tested through the comparison analysis.

**Table: 1.** *Comparison of numerical values of* $-F''(0)$ *when* $M^2 = 0$, $r = 0$ *and* $\lambda = 0$

| $\alpha$ | $m$ | *Khader and Megahed [18]* | *Present work* |
|---|---|---|---|
| 0.25 | -0.5 | 0.0832 | 0.083333 |
| 0.25 | -1/3 | 0.5000 | 0.500001 |
| 0.5 | -0.5 | 1.1666 | 1.166668 |

In the absence of magnetic interaction parameter $\left(M^2 = 0\right)$, temperature index parameter $(r = 0)$ and slip parameter $(\lambda = 0)$, the numerical values of $-F''(0)$ are found to be in excellent agreement with that of Khader and Megahed [18] which are displayed in Table 1.

The solution of this problem is valid for $m \neq 1$. The analysis have been carried out for various values of $M^2 \left(0 \leq M^2 \leq 9\right), m\,(-0.9 \leq m \leq -0.25), \alpha\,(0.25 \leq \alpha \leq 1.25), (Pr = 0.71)$ for air and $(Pr = 7.02)$ for water at $20°C\ (293\,K)$ and $r\,(0 \leq r \leq 4)$. In order to get the clear insight of the problem, the computed results are displayed graphically through Fig. 2 - Fig. 12.

Fig. 2 portrays the velocity distribution for different values of $M^2$. As the magnetic interaction parameter $(M^2)$ increases, the velocity distribution gets decreased which happens eventually due to the effect of Lorentz force. Further the boundary layer thickness is decreased due to the influence of magnetic field. This happens due to Lorentz force arising from the interaction of magnetic and electric fields during the motion of an electrically conducting fluid. The generated Lorentz force opposes the fluid motion in boundary layer region and thereby reducing the momentum boundary layer thickness.

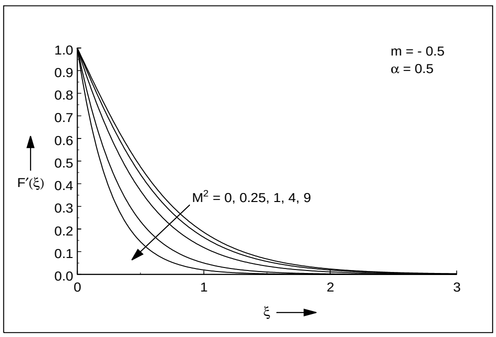


**Fig. 2 Velocity distribution for various values of $M^2$**

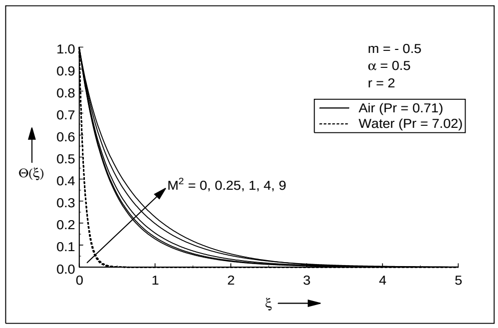


**Fig. 3 Temperature distribution for different values of $M^2$**

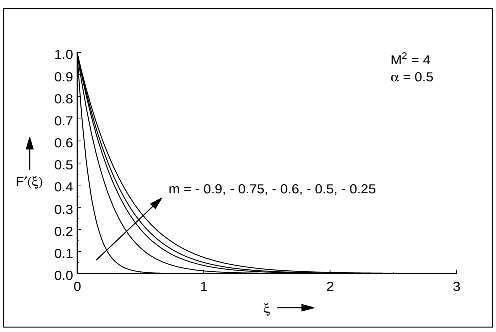


**Fig. 4 Velocity distribution for various values of m**

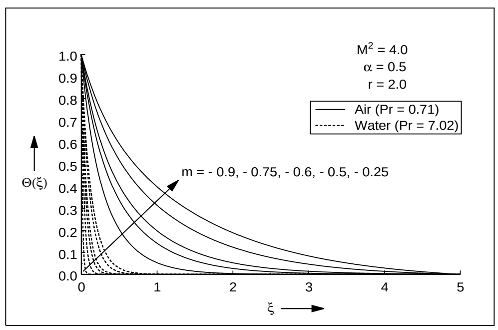


**Fig. 5 Temperature distribution for different values of m**

The impact of magnetic field over dimensionless temperature distribution for both air $(Pr = 0.71)$ and water $(Pr = 7.02)$ were highlighted in Fig. 3. It is noted that for both cases of air $(Pr = 0.71)$ and water $(Pr = 7.02)$, the increase in the value of magnetic interaction parameter $(M^2)$ leads to increase in the temperature distribution. It also reveals that, the influence of magnetic field is less significant in the case of water ($Pr = 7.02$) than air ($Pr = 0.71$). Also the thermal boundary layer thickness is enhanced due to increasing strength of the applied magnetic field.

Velocity distribution for different values of velocity power index is presented graphically in Fig. 4. It shows the obvious result that due to the increase in the value of power index, the velocity distribution gets increased. Eventually the boundary layer gets thicker as the velocity power index is increased.

Fig. 5 depicts the effect of velocity power index over the temperature distribution for both air ($Pr = 0.71$) and water ($Pr = 7.02$). The velocity power index (m) has the tendency to enhance temperature distribution of air ($Pr = 0.71$) and water ($Pr = 7.02$) as it is increased. It is also viewed that increase in velocity power index leads to thickening of the thermal boundary layer of air ($Pr = 0.71$) than water ($Pr = 7.02$).

Fig. 6 and Fig. 7 elucidate the variation in velocity distribution and temperature distribution respectively due to the variation in wall thickness parameter ($\alpha$). For the variable thickness sheet slendering away from the slot, the wall thickness decrease as it stretches away. Decelerating wall thickness parameter enriched the boundary layer thickness due to acelerating velocity which is illustrated in Fig. 6.

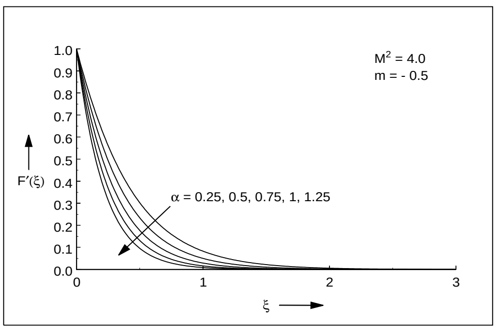


**Fig. 6 Velocity distribution for different values of** $\alpha$

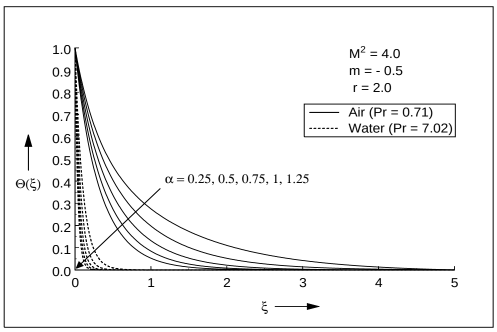


**Fig. 7 Temperature distribution for different values of** $\alpha$

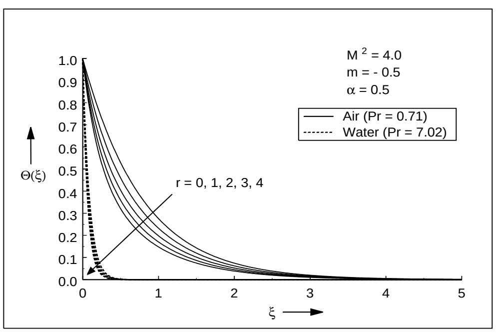


**Fig. 8 Temperature distribution for different values of r**

It is evident through Fig. 7 that increase in wall thickness parameter is to reduce the temperature distribution. The thermal boundary layer becomes thinner for higher values of wall thickness parameter.

Variation in dimensionless temperature distribution due to temperature index parameter (r) for air and water is visualized through Fig. 8. It is inferred from the figure that for both the fluids, increase in the temperature index parameter reduces the dimensionless temperature. The effect of temperature index parameter over the thermal boundary layer thickness becomes significantly less, for water ($Pr = 7.02$) than air ($Pr = 0.71$).

Fig. 9 and Fig. 10 represent the influence of various physical parameters like Magnetic interaction parameter ($M^2$), Velocity power index (m) and wall thickness parameter ($\alpha$) over the dimensionless skin friction coefficient. Skin friction coefficient against wall thickness parameter ($\alpha$) for different values of magnetic interaction parameter ($M^2$) is presented in Fig. 9. It is evident that the skin friction coefficient increases in magnitude for both increased values of magnetic interaction parameter and wall thickness parameter.

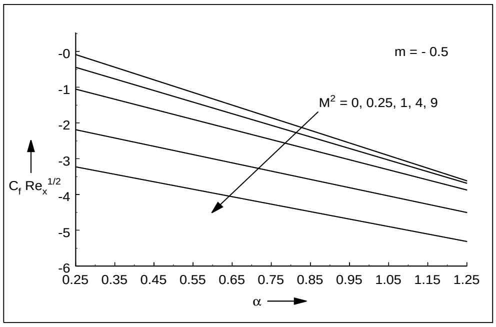


**Fig. 9 Non dimensional skin friction coefficient against α for different values of $M^2$**

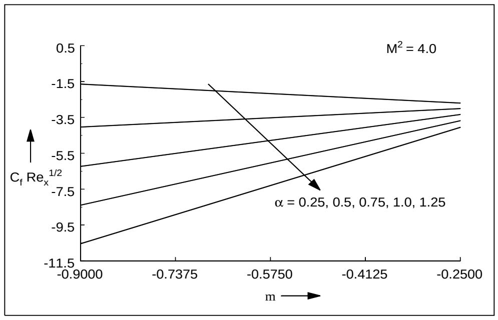


**Fig. 10 Non dimensional skin friction coefficient against m for different values of α**

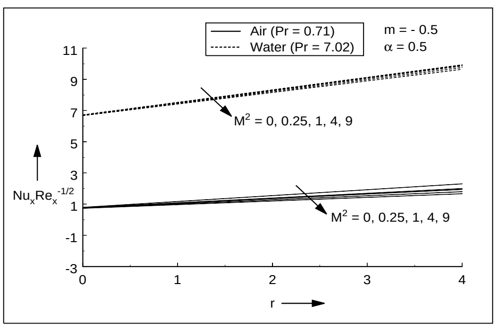


**Fig. 11 Non dimensional rate of heat transfer against r for different values of $M^2$**

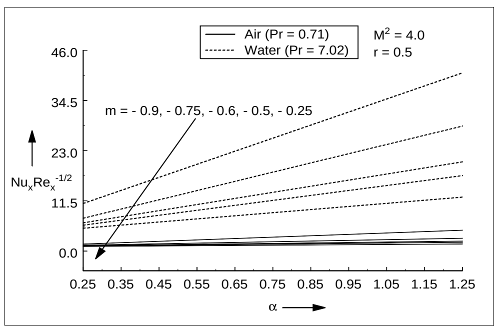


**Fig. 12 Non dimensional rate of heat transfer against α for different values of m**

The effect of velocity power index over the dimensionless skin friction coefficient can be viewed through Fig. 10. It shows that for increasing velocity power index parameter, the skin friction coefficient gets increased for ($\alpha > 0.25$), where as it gets decreased for ($\alpha \leq 0.25$).

The non dimensional rate of heat transfer for air ($Pr = 0.71$) and water ($Pr = 7.02$) against temperature index parameter (r) for various values of magnetic interaction parameter ($M^2$) is displayed through Fig. 11. For both the fluids, the results follow the similar trend. The magnetic interaction parameter suppress the non dimensional rate of heat transfer as it increases, whereas the temperature index parameter is to enhance the non dimensional heat transfer rate.

Fig. 12 reveals the state of dimensionless rate of heat transfer for air ($Pr = 0.71$) and water ($Pr = 7.02$) against the wall thickness parameter ($\alpha$) for different values of velocity power index (m). It depicts that, in both the fluids, for increasing the velocity power index parameter, the dimensionless rate of heat transfer gets reduced. The wall thickness parameter has significant effects over the dimensionless rate of heat transfer. It can be viewed that for both air ($Pr = 0.71$) and water ($Pr = 7.02$), as wall thickness parameter increases, the non dimensional heat transfer rate gets increased. For larger values of wall thickness parameter, water ($Pr = 7.02$) has more heat transfer rate than that of air ($Pr = 0.71$). From both Fig. 11 and Fig. 12, it can be noted that compared to air ($Pr = 0.71$), water ($Pr = 7.02$) has more non dimensional heat transfer rate.

During many industrial manufacturing process like glass blowing, metal extrusion, glass fiber production and polymer extrusion, in which the rate of cooling of the product is vital. It may affect the quality of the final product. Physically cooling is proportional to the rate of heat transfer from the hot surface. In such kind of problems, the main objective is to control the rate of heat transfer which depends on the manufacturing process and physical nature of the problem considered. So from the above results, it is meaningful to suggest some optimal conditions for the physical parameters involved in such kind of problems. Increase in the magnetic field strength suppresses the heat transfer rate, so the problem which needs slow cooling process may consider the higher values of the magnetic interaction parameter ($M^2$). The velocity power index $m = 1$ represents the flat sheet case, but physically stretching surfaces like metal extrusion sheet, hot rolling and continuous casting of metals need not to be flat, it may have some thickness variations. So it is meaningful to consider $m \neq 1$ which governs the uneven surfaces and wall thickness parameter ($\alpha$) to govern such thickness variations. The temperature along such surfaces may also vary. So the role of temperature index parameter($r$) becomes vital for the physical problems depends on cooling.

## 6. CONCLUSION

The problem of steady, nonlinear, hydromagnetic flow over a stretching sheet with variable thickness and variable surface temperature has been analyzed. A parametric study on dimensionless velocity, temperature, skin friction coefficient and heat transfer rate are carried out. In the absence of Magnetic interaction parameter, when $M^2 = 0$, $r = 0$ and $\lambda = 0$ the results are in excellent agreement with that of Khader and Megahed [18]. In the light of the present investigation the following conclusions are drawn:

- Dimensionless velocity gets decelerated for increasing magnetic field strength, whereas it gets accelerated for increasing velocity power index and decreasing wall thickness.

- In both the cases of air ($Pr = 0.71$) and water ($Pr = 7.02$), increasing magnetic field and velocity power index has the influence to enhance the temperature distribution, but for increasing wall thickness parameter and temperature index parameter has different effects to suppress the dimensionless temperature.
- Non dimensional skin friction coefficient has increased in magnitude for increasing magnetic field and wall thickness. For increasing velocity power index parameter, the skin friction coefficient gets increased for $(\alpha > 0.25)$, where as it gets decreased for $(\alpha \leq 0.25)$.
- Wall thickness parameter and temperature power index are to enhance the non dimensional rate of heat transfer. The magnetic field strength and the velocity power index pull down the non dimensional heat transfer rate.
- Thickening of the boundary layer happens for increasing values of velocity power index, whereas it get thinner for increase in magnetic field strength and wall thickness parameter.
- Thermal boundary layers were enriched by magnetic field and velocity power index, thinner thermal boundary layers were obtained for increasing wall thickness and temperature index parameter.
- Water ($Pr = 7.02$) has more heat transfer rate than that the air ($Pr = 0.71$), especially for larger values of wall thickness parameter.

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