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Further Results on Characteristic Functions Without Contour Integration

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Abstract

Characteristic functions play an important role in probability and statistics; however, a rigorous derivation of these functions requires contour integration, which is unfamiliar to most statistics students. Without resorting to contour integration, Datta and Ghosh (2007) derived the characteristic functions of normal, Cauchy, and double exponential distributions. Here, we derive the characteristic functions of t, truncated normal, skew-normal, and skew-t distributions. The characteristic functions of normal, cauchy distributions are obtained as a byproduct. The derivations are straightforward and can be presented in statistics masters theory classes.

Keywords: t distribution, truncated normal, skew-normal, skew-t, stochastic representation.

1. Introduction

Characteristic functions play an important role in probability and statistics. However, a rigorous derivation of these functions requires contour integration, which is unfamiliar to most statistics students. Thus, students are typically advised to compute the moment generating function when it is finite and then substitute it for t in the moment generating function to obtain the characteristic function(cf), where $i = \sqrt{-1}$. This technique obviously works in many important cases, but it does not work when the moment generating function is not finite, for example, in the case of the t distribution with finite degrees of freedom. As a result, Datta and Ghosh (2007) derived the cfs of some well-known distributions(the normal, Cauchy, and double exponential distributions), without resort to contour integration. Our objective is to extend their results and develop the cfs of the t, truncated normal, skew-normal, and skew-t distributions, also without using contour integration. Kim and Genton (2011) derived these cfs using rigorous complex integration techniques, however contour integration is not familiar to most statistics students.

This paper is organized as follows. In Section 2, we derive the cfs of the t, truncated normal, skew-normal, and skew-t distributions. Multivariate extensions of these results are discussed in Section 3. Finally, Section 4 provides the conclusions.

2. Univariate Results

We first define the integral representation of the modified Bessel function of the third kind (p.182 of Watson, 1966), as follows.

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Definition 1. *The integral representation of the modified Bessel function of the third kind is defined by*

$$K_\lambda(w) = \frac{1}{2}\int_0^\infty x^{\lambda-1}\exp\left\{-\frac{1}{2}w\left(x+\frac{1}{x}\right)\right\}dx, \quad w > 0 \text{ for } \lambda \in \mathbb{R}.$$

We transform the above definition to obtain Lemma 1, which is crucial to develop the cf of the t distribution in a simpler way.

Lemma 1.

$$\frac{2K_\lambda\left(\sqrt{\nu\xi}\right)}{(\xi/\nu)^{\frac{\lambda}{2}}} = \int_0^\infty y^{\lambda-1}\exp\left\{-\frac{1}{2}\left(\xi y+\frac{\nu}{y}\right)\right\}dy$$

Proof: From the definition of the modified Bessel function of the third kind,

$$K_\lambda\left(\sqrt{\nu\xi}\right) = \frac{1}{2}\int_0^\infty x^{\lambda-1}\exp\left\{-\frac{1}{2}\sqrt{\nu\xi}\left(x+\frac{1}{x}\right)\right\}dx.$$

Let $y = \sqrt{\nu/\xi}x$, then

$$K_\lambda\left(\sqrt{\nu\xi}\right) = \frac{1}{2}\left(\frac{\xi}{\nu}\right)^{\frac{\lambda}{2}}\int_0^\infty y^{\lambda-1}\exp\left\{-\frac{1}{2}\left(\xi y+\frac{\nu}{y}\right)\right\}dy.$$

Rearranging the terms, we have the result. □

Remark 1. If we let $f(x) = (\xi/\nu)^{\lambda/2}\left[2K_\lambda(\sqrt{\nu\xi})\right]^{-1}x^{\lambda-1}\exp\{-(\xi x+\nu/x)/2\}$, $x \in \mathbb{R}^+$, then $f(x)$ is the probability density function(pdf) of the generalized inverse Gaussian distribution with parameters (λ, ν, ξ) (Barndorff-Nielsen et al., 1982). Lemma 1 shows that $f(x)$ is indeed a pdf.

2.1. *t* distribution

The standard t distribution can be expressed as scale mixtures of normal distribution as follows.

$$X = W^{-\frac{1}{2}}Z, \tag{2.1}$$

where $Z \sim N(0, 1)$ independent of W. Here $W \sim \Gamma(\nu/2, \nu/2)$ and its pdf is given by

$$f_W(w) = \frac{(\nu/2)^{\frac{\nu}{2}}}{\Gamma(\nu/2)}w^{\frac{\nu}{2}-1}\exp\left(-\frac{\nu w}{2}\right), \quad w > 0.$$

To clarify this, the pdf of (2.1) can be obtained as follows. Let $\phi(\cdot)$ and $f_W(\cdot)$ denote the pdfs of $N(0, 1)$ and $\Gamma(\nu/2, \nu/2)$, respectively, then

$$\begin{aligned} f_X(x) &= \int_0^\infty f_{X|w}(x)f_W(w)dw = \int_0^\infty \sqrt{w}\phi\left(\sqrt{w}x\right)f_W(w)dw \\ &= \frac{(\nu/2)^{\frac{\nu}{2}}}{\Gamma(\nu/2)\sqrt{2\pi}}\int_0^\infty w^{\frac{\nu+1}{2}-1}e^{-\frac{(x^2+\nu)w}{2}}dw \\ &= \frac{\Gamma((\nu+1)/2)}{\Gamma(\nu/2)\sqrt{\nu\pi}}\left(1+\frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}}, \end{aligned}$$

where $x \in \mathbb{R}$ and $\nu > 0$. Now we can obtain the cf of the standard t distribution.

Theorem 1. *Let $X \sim t(\nu)$, then the cf of X is*

$$\psi_X(t) = \frac{K_{\frac{\nu}{2}}\left(\sqrt{\nu}|t|\right)\left(\sqrt{\nu}|t|\right)^{\frac{\nu}{2}}}{\Gamma(\nu/2)2^{\frac{\nu}{2}-1}}, \quad t \in \mathbb{R} \text{ and } \nu > 0.$$

Proof:

$$\begin{aligned}\psi_X(t) = E\left[e^{itX}\right] &= E\left[E\left(e^{itX}|W\right)\right] = E\left[\exp\left(-\frac{t^2}{2W}\right)\right] \\ &= \int_0^\infty \frac{\exp(-t^2/(2w))}{\Gamma(\nu/2)(2/\nu)^{\frac{\nu}{2}}} w^{\frac{\nu}{2}-1}\exp\left(-\frac{\nu w}{2}\right)dw \\ &= \frac{\nu^{\frac{\nu}{2}}}{\Gamma(\nu/2)2^{\frac{\nu}{2}}}\int_0^\infty w^{\frac{\nu}{2}-1}\exp\left[-\frac{1}{2}\left(\nu w + \frac{t^2}{w}\right)\right]dw \\ &= \frac{\nu^{\frac{\nu}{2}}}{\Gamma(\nu/2)2^{\frac{\nu}{2}}}\frac{2K_{\frac{\nu}{2}}\left(\sqrt{\nu}|t|\right)}{(\nu/t^2)^{\frac{\nu}{4}}} \quad \text{by Lemma 1.}\end{aligned}$$

Thus we have the result. □

2.2. Skew-normal and truncated normal distributions

A random variable Z has a skew-normal distribution developed by Azzalini (1985) if its pdf is

$$f_Z(z) = 2\phi(z)\Phi(\alpha z), \quad z \in \mathbb{R}, \tag{2.2}$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ denote the pdf and cumulative distribution function(cdf) of the standard normal $N(0,1)$ distribution, respectively. The parameter $\alpha \in \mathbb{R}$ controls the skewness (shape) of the distribution. When Z has the pdf (2.2), we write $Z \sim SN(\alpha)$. If $\alpha = 0$, then (2.2) reduces to the $N(0,1)$ pdf. Using a differential equations approach, Pewsey (2000) obtained the cf of $SN(\alpha)$ as follows.

$$\psi_Z(t) = e^{-\frac{t^2}{2}}\left[1 + i\tau(\delta t)\right], \quad t \in \mathbb{R}, \tag{2.3}$$

where $\tau(x) = \int_0^x be^{v^2/2}dv$, $x > 0$, $b = \sqrt{2/\pi}$, and $\delta = \alpha/\sqrt{1+\alpha^2}$. Here $\tau(-x) = -\tau(x)$ for $x > 0$. Kim and Genton (2011) showed that (2.3) is equal to

$$\psi_Z(t) = 2e^{-\frac{t^2}{2}}\Phi(i\delta t), \quad t \in \mathbb{R}$$

using rigorous complex contour integration. Instead of the two previous approaches, we introduce a physics proof (p.93 of Durrett, 1996) to derive the cf of the skew-normal distribution. First, note that the probabilistic representation of the skew-normal distribution (Henze, 1986) is

$$Z = \frac{\alpha}{\sqrt{1+\alpha^2}}|U| + \frac{1}{\sqrt{1+\alpha^2}}V,$$

where U and V are independent $N(0,1)$. Using this representation, we derive the cf of Z. Before the derivation, we need to prove a simple Lemma.

Lemma 2.

$$\int_0^\infty e^{\frac{t^2}{2}} e^{itu}\phi(u)du = \Phi(it).$$

Proof:

$$\int_0^\infty e^{\frac{t^2}{2}} e^{itu}\phi(u)du = \frac{1}{\sqrt{2\pi}}\int_0^\infty \exp\left\{\frac{-(u-it)^2}{2}\right\} du.$$

Let $u - it = w$, then the last integral becomes $\frac{1}{\sqrt{2\pi}}\int_{-it}^{\infty} e^{-w^2/2}dw = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{it} e^{-w^2/2}dw = \Phi(it)$. □

We can now derive the cf of a skew-normal distribution.

Theorem 2. *Let $Z \sim SN(\alpha)$, then*

$$\psi_Z(t) = 2e^{-\frac{t^2}{2}}\Phi(i\delta t), \quad t \in \mathbb{R}. \tag{2.4}$$

Proof: From $Z = \frac{\alpha}{\sqrt{1+\alpha^2}}|U| + \frac{1}{\sqrt{1+\alpha^2}}V$, we know that $Z||U| = u \sim N(\delta u, 1-\delta^2)$. Thus,

$$\psi_Z(t) = E\left[e^{itZ}\right] = E\left[E\left(e^{itZ}||U| = u\right)\right] = \int_0^\infty E\left(e^{itZ}||U| = u\right) f_{|U|}(u)du,$$

where $f_{|U|}(u) = 2\phi(u),\ 0 < u < \infty$. Hence

$$\psi_Z(t) = \int_0^\infty \exp\left\{i\delta ut - \frac{1-\delta^2}{2}t^2\right\} 2\phi(u)du = 2e^{-\frac{t^2}{2}}\int_0^\infty e^{\frac{\delta^2 t^2}{2}} e^{i\delta ut}\phi(u)du.$$

By Lemma 2, we have the result. □

Remark 2. $|U|$ is a truncated normal random variable with a pdf $2\phi(u), 0 < u < \infty$. From Lemma 2, we can easily obtain the cf of a truncated normal distribution by multiplying both sides by $2e^{-t^2/2}$, as follows:

$$\psi_{|U|}(t) = 2e^{-\frac{t^2}{2}}\Phi(it), \quad t \in \mathbb{R}.$$

Now, we show that the cf in (2.4) is equal to that of Pewsey (2000), shown in (2.3).

Lemma 3.

$$\psi_Z(t) = 2e^{-\frac{t^2}{2}}\Phi(i\delta t) = e^{-\frac{t^2}{2}}\left[1 + i\tau(\delta t)\right], \quad t \in \mathbb{R}.$$

Proof: Let $w = iv$, then

$$\begin{aligned}
\psi_Z(t) &= 2e^{-\frac{t^2}{2}}\Phi(i\delta t)\\
&= 2e^{-\frac{t^2}{2}}\left[\int_{-\infty}^0 \frac{1}{\sqrt{2\pi}}e^{-\frac{w^2}{2}}dw + \int_0^{i\delta t}\frac{1}{\sqrt{2\pi}}e^{-\frac{w^2}{2}}dw\right]\\
&= 2e^{-\frac{t^2}{2}}\left[\frac{1}{2} + \int_0^{i\delta t}\frac{1}{\sqrt{2\pi}}e^{-\frac{w^2}{2}}dw\right]\\
&= e^{-\frac{t^2}{2}}\left[1 + i\int_0^{\delta t}\frac{\sqrt{2}}{\sqrt{\pi}}e^{-\frac{(iv)^2}{2}}dv\right]\\
&= e^{-\frac{t^2}{2}}\left[1 + i\int_0^{\delta t} be^{\frac{v^2}{2}}dv\right].
\end{aligned}$$

Thus, from the definition of $\tau(x)$, we have the result. □

Extending these results to location-scale parameters is straightforward using the property of the cf. For a rigorous proof of the previous results, let $f(z) = \exp(-z^2/2)$, defined on the complex domain. Then, $f(z)$ is an entire function (*i.e.*, it has no singularity on the whole complex domain), and $\oint_R f(z)dz = 0$ by the Cauchy integral theorem. After some complex integration techniques, we have the result (Kim and Genton, 2011).

2.3. Skew-*t* distribution

The cf of the skew-*t* distribution is obtained using scale mixtures of skew-normal distributions. Similar to the *t* distribution, the skew-*t* distribution of a random variable, $X \sim St(\alpha, \nu)$, is related to the skew-normal distribution. That is,

$$X = W^{-\frac{1}{2}} Z,$$

where $Z \sim SN(\alpha)$ is independent of $W \sim \Gamma(\nu/2, \nu/2)$. A particular case of the skew-*t* distribution is the skew-Cauchy distribution when $\nu = 1$. In addition, when $\nu \to \infty$, we obtain the skew-normal distribution as the limiting case.

Theorem 3. *Let $X \sim St(\alpha, \nu)$, then the cf of X is:*

$$\psi_X(t) = \psi_T(t) + i\tau^* \left(\frac{\delta t}{\sqrt{w}} \right),$$

where $\psi_T(t)$ is given in Theorem 1,

$$\tau^* \left(\frac{\delta t}{\sqrt{w}} \right) = \int_0^\infty \exp\left(-\frac{t^2}{2w} \right) \tau \left(\frac{\delta t}{\sqrt{w}} \right) dH(w), \quad \text{for } \delta t > 0, \tag{2.5}$$

and $\tau^*(-x) = -\tau^*(x)$ for $x > 0$. Here $H(w)$ is the cdf of $W \sim \Gamma(\nu/2, \nu/2)$.

Proof: The conditional distribution of X, given $W = w$, follows a skew-normal distribution; that is, $X|W = w \sim SN(0, w^{-1}, \alpha)$. Then, the cf of X is

$$\begin{aligned}
\psi_X(t) &= \int_0^\infty \int_{\mathbb{R}} \exp(itx) f(x|w) dx dH(w) = \int_0^\infty \psi_{X|w}(t) dH(w) \\
&= \int_0^\infty \exp\left(-\frac{t^2}{2w} \right) \left\{ 1 + i\tau \left(\frac{\delta t}{\sqrt{w}} \right) \right\} dH(w) \\
&= \psi_T(t) + i \int_0^\infty \exp\left(-\frac{t^2}{2w} \right) \tau \left(\frac{\delta t}{\sqrt{w}} \right) dH(w),
\end{aligned}$$

where $\psi_T(t)$ is the cf of the $t(\nu)$ distribution given in Theorem 1. The integrand of (2.5), without the constant $2/\sqrt{\pi}$, becomes Dawson's integral when $\delta = 1$. □

When $\nu = 1$, the cf of the skew-Cauchy distribution is given by

$$\psi_X(t) = \exp(-|t|) + i\tau^* \left(\frac{\delta t}{\sqrt{w}} \right), \quad t \in \mathbb{R}$$

using $K_{1/2}(r) = \sqrt{\pi/2r} e^{-r}$ (Kotz and Nadarajah, 2004) and $W \sim \chi_1^2$.

3. Multivariate Results

In this section, we extend univariate results to multivariate cases.

3.1. Multivariate *t* distribution

Similar to the t distribution, the multivariate t distribution can be expressed as scale mixtures of multivariate normal distribution as follows. Let $\mathbf{X} = W^{-1/2}\mathbf{Z}$, where $W \sim \Gamma(\nu/2, \nu/2)$ independently of $\mathbf{Z} \sim N_p(\mathbf{0}, I_p)$. Then the p-dimensional random vector $\mathbf{X}$ follows the multivariate t distribution. To see this, let $\phi_p(\mathbf{x})$ and $f_W(w)$ denote the pdfs of $N_p(\mathbf{0}, I_p)$ and $\Gamma(\nu/2, \nu/2)$, respectively, then

$$
\begin{aligned}
f_{\mathbf{X}}(\mathbf{x}) &= \int_0^\infty f_{\mathbf{X}|w}(\mathbf{x}) f_W(w) dw = \int_0^\infty w^{\frac{p}{2}} \phi_p(\sqrt{w}\mathbf{x}) f_W(w) dw \\
&= \frac{(\nu/2)^{\frac{\nu}{2}}}{\Gamma(\nu/2)(2\pi)^{\frac{p}{2}}} \int_0^\infty w^{\frac{\nu+p}{2}-1} \exp\left(-\frac{(\mathbf{x}^\top\mathbf{x}+\nu)w}{2}\right) dw \\
&= \frac{(\nu/2)^{\frac{\nu}{2}}}{\Gamma(\nu/2)(2\pi)^{p/2}} \frac{\Gamma((\nu+p)/2)}{((\mathbf{x}^\top\mathbf{x}+\nu)/2)^{\frac{\nu+p}{2}}} \\
&= \frac{\Gamma((\nu+p)/2)}{\Gamma(\nu/2)(\nu\pi)^{\frac{p}{2}}} \left(1 + \frac{\mathbf{x}^\top\mathbf{x}}{\nu}\right)^{-\frac{\nu+p}{2}}, \quad \mathbf{x} \in \mathbb{R}^p,\ \nu > 0.
\end{aligned}
$$

We can now derive the cf of the multivariate t distribution as follows.

Theorem 4. *Let $\mathbf{X} \sim t_p(\nu)$, then the cf of the multivariate t distribution is*

$$
\psi_{\mathbf{X}}(\mathbf{t}) = \frac{K_{\nu/2}(\| \sqrt{\nu}\mathbf{t} \|)(\| \sqrt{\nu}\mathbf{t} \|)^{\frac{\nu}{2}}}{\Gamma(\nu/2)2^{\frac{\nu}{2}-1}}, \quad \mathbf{t} \in \mathbb{R}^p \text{ and } \nu > 0,
$$

where $\| \mathbf{t} \| = \sqrt{\mathbf{t}^\top\mathbf{t}}$.

Proof:

$$
\begin{aligned}
\psi_{\mathbf{X}}(\mathbf{t}) &= E\left[e^{i\mathbf{t}^\top\mathbf{X}}\right] = E\left[E\left(e^{i\mathbf{t}^\top\mathbf{X}}|W\right)\right] = E\left[\exp\left(-\frac{\mathbf{t}^\top\mathbf{t}}{2W}\right)\right] \\
&= \int_0^\infty \exp\left(-\frac{\mathbf{t}^\top\mathbf{t}}{2w}\right) f_W(w) dw \\
&= \frac{\nu^{\frac{\nu}{2}}}{\Gamma(\nu/2)2^{\frac{\nu}{2}}} \int_0^\infty w^{\frac{\nu}{2}-1} \exp\left[-\frac{1}{2}\left(\nu w + \frac{\mathbf{t}^\top\mathbf{t}}{w}\right)\right] dw \\
&= \frac{\nu^{\frac{\nu}{2}}}{\Gamma(\nu/2)2^{\frac{\nu}{2}}} \frac{2K_{\frac{\nu}{2}}\left[\sqrt{\nu}(\mathbf{t}^\top\mathbf{t})^{\frac{1}{2}}\right]}{(\nu/\mathbf{t}^\top\mathbf{t})^{\frac{\nu}{4}}} \quad \text{by Lemma 1.}
\end{aligned}
$$

Thus, we have the result. □

3.2. Multivariate skew-normal distribution

A p-dimensional random vector, $\mathbf{Z}$, is said to have a multivariate skew-normal distribution if it is continuous with pdf

$$f_{\mathbf{Z}}(\mathbf{z}) = 2\phi_p\left(\mathbf{z};\Omega_{\mathbf{z}}\right)\Phi\left(\boldsymbol{\alpha}^\top\mathbf{z}\right), \quad \mathbf{z}\in\mathbb{R}^p,$$

where $\phi_p(\mathbf{z};\Omega_{\mathbf{z}})$ is the p-dimensional normal pdf with zero mean and correlation matrix $\Omega_{\mathbf{z}}$, and $\boldsymbol{\alpha}$ is a p-dimensional vector controlling the skewness(shape). We write this as $\mathbf{Z}\sim SN_p(\Omega_{\mathbf{z}},\boldsymbol{\alpha})$. Similarly to the standardization of multivariate normal distribution, there is a sort of canonical form of a multivariate skew-normal distribution (proposition 4 of Azzalini and Capitanio, 1999). We rephrase this as a Lemma without proof.

Lemma 4. *For a variable $\mathbf{Z}\sim SN_p(\Omega_{\mathbf{z}},\boldsymbol{\alpha})$, there exists a linear transform $\mathbf{Z}^* = A^*\mathbf{Z}$ such that $\mathbf{Z}^*\sim SN_p(I_p,\boldsymbol{\alpha}^*)$ where at most one component of $\boldsymbol{\alpha}^*$ is not zero.*

Here, $A^* = (C^\top P)^{-1}$, $\boldsymbol{\alpha}^* = P^\top C\boldsymbol{\alpha}$, $\Omega_{\mathbf{z}} = C^\top C$ and an orthogonal matrix P with one column on the same direction of $C\boldsymbol{\alpha}$. Hence the density of $\mathbf{Z}^*$ is of the form

$$2\prod_{i=1}^{p}\phi\left(z_i^*\right)\Phi\left(\alpha_m^* z_m^*\right),$$

where $\alpha_m^* = \sqrt{\boldsymbol{\alpha}^\top\Omega_{\mathbf{z}}\boldsymbol{\alpha}}$ is the only non-zero component of $\boldsymbol{\alpha}^*$. Using Lemma 4, we have the cf of multivariate skew-normal distribution.

Theorem 5. *Let $\mathbf{Z}\sim SN_p(\Omega_{\mathbf{z}},\boldsymbol{\alpha})$, then the cf of $\mathbf{Z}$ is*

$$\begin{aligned}\psi_{\mathbf{Z}}(\mathbf{t}) &= 2\exp\left(-\frac{1}{2}\mathbf{t}^\top\Omega_{\mathbf{z}}\mathbf{t}\right)\Phi\left(i\boldsymbol{\delta}^\top\mathbf{t}\right)\\ &= \exp\left(-\frac{1}{2}\mathbf{t}^\top\Omega_{\mathbf{z}}\mathbf{t}\right)\left(1+i\tau\left(\boldsymbol{\delta}^\top\mathbf{t}\right)\right).\end{aligned}$$

Proof: We first obtain the cf of $\mathbf{Z}^*$ and then calculate that of $\mathbf{Z}$.

$$\begin{aligned}\psi_{\mathbf{Z}^*}(t) &= E\left[e^{i\mathbf{t}^\top\mathbf{z}^*}\right] = E\left[e^{i\sum t_i z_i^*}\right] = \prod_{i=1}^{p}E\left[e^{it_i z_i^*}\right] \quad \text{by independence}\\ &= \left(\prod_{i=1, i\neq m}^{p} e^{-\frac{t_i^2}{2}}\right)2e^{-\frac{t_m^2}{2}}\Phi\left(i\delta_m^* t_m\right), \quad \text{where } \delta_m^* = \frac{\alpha_m^*}{\sqrt{1+\alpha_m^{*\,2}}}\\ &= 2\exp\left(-\frac{1}{2}\mathbf{t}^\top\mathbf{t}\right)\Phi\left(i\delta_m^* t_m\right).\end{aligned}$$

Using $\mathbf{Z} = C^\top P\mathbf{Z}^*$ and $\delta_m^* t_m = \boldsymbol{\delta}^{*\top}\mathbf{t}$, where $\boldsymbol{\delta}^* = \boldsymbol{\alpha}^*/\sqrt{1+\boldsymbol{\alpha}^\top\Omega_{\mathbf{z}}\boldsymbol{\alpha}}$ by the construction of $\boldsymbol{\alpha}^*$, we have

$$\begin{aligned}\psi_{\mathbf{Z}}(\mathbf{t}) &= E\left[e^{i\mathbf{t}^\top(C^\top P\mathbf{Z}^*)}\right] = E\left[e^{i(P^\top C\mathbf{t})^\top\mathbf{Z}^*}\right]\\ &= 2\exp\left(-\frac{1}{2}\mathbf{t}^\top C^\top PP^\top C\mathbf{t}\right)\Phi\left(i\boldsymbol{\delta}^{*\top}P^\top C\mathbf{t}\right).\end{aligned}$$

Since P is an orthogonal matrix, $\Omega_{\mathbf{z}} = C^\top C$, $\boldsymbol{\alpha}^* = P^\top C\boldsymbol{\alpha}$, and

$$\begin{aligned}\boldsymbol{\delta}^{*\top} P^\top C\mathbf{t} &= \frac{\boldsymbol{\alpha}^{*\top}}{\sqrt{1+\boldsymbol{\alpha}^\top\Omega_{\mathbf{z}}\boldsymbol{\alpha}}} P^\top C\mathbf{t} = \frac{(P^\top C\boldsymbol{\alpha})^\top}{\sqrt{1+\boldsymbol{\alpha}^\top\Omega_{\mathbf{z}}\boldsymbol{\alpha}}} P^\top C\mathbf{t} \\ &= \frac{\boldsymbol{\alpha}^\top C^\top P P^\top C\mathbf{t}}{\sqrt{1+\boldsymbol{\alpha}^\top\Omega_{\mathbf{z}}\boldsymbol{\alpha}}} = \frac{\boldsymbol{\alpha}^\top \Omega_{\mathbf{z}}\mathbf{t}}{\sqrt{1+\boldsymbol{\alpha}^\top\Omega_{\mathbf{z}}\boldsymbol{\alpha}}} = \boldsymbol{\delta}^\top\mathbf{t},\end{aligned}$$

the cf of $\mathbf{Z}$ is $\psi_{\mathbf{Z}}(\mathbf{t}) = 2\exp\left(-\frac{1}{2}\mathbf{t}^\top\Omega_{\mathbf{Z}}\mathbf{t}\right)\Phi(i\boldsymbol{\delta}^\top\mathbf{t})$, $\mathbf{t} \in \mathbb{R}^p$. If we substitute $e^{-t_m^2/2}(1 + i\tau(\delta_m^* t_m))$ for $2e^{-t_m^2/2}\Phi(i\delta_m^* t_m)$, we have the second equation. □

3.3. Multivariate skew-*t* distribution

The cf of the multivariate skew-*t* distribution can be obtained using scale mixtures of multivariate skew-normal distributions. The multivariate skew-*t* random vector, $\mathbf{X} \sim St_p(\Omega_{\mathbf{z}}, \boldsymbol{\alpha}, \nu)$, is related to the multivariate skew-normal random vector, $\mathbf{Z}$, by the following stochastic equation:

$$\mathbf{X} = W^{-\frac{1}{2}}\mathbf{Z}, \tag{3.1}$$

where $\mathbf{Z} \sim SN_p(\Omega_{\mathbf{z}}, \boldsymbol{\alpha})$ independent of $W \sim \Gamma(\nu/2, \nu/2)$.

Theorem 6. *Let* $\mathbf{X}$ *follow the multivariate skew-t distribution defined by* (3.1). *Then the cf of* $\mathbf{X}$ *is:*

$$\psi_{\mathbf{X}}(\mathbf{t}) = \psi_{T_p}\left(\Omega_{\mathbf{z}}^{\frac{1}{2}}\mathbf{t}\right) + i\tau^+\left(\frac{\boldsymbol{\delta}^\top\mathbf{t}}{\sqrt{w}}\right),$$

where $\psi_{T_p}(\mathbf{t})$ *is given in Theorem 4,*

$$\tau^+\left(\frac{\boldsymbol{\delta}^\top\mathbf{t}}{\sqrt{w}}\right) = \int_0^\infty \exp\left(-\frac{\mathbf{t}^\top\Omega_{\mathbf{z}}\mathbf{t}}{2w}\right)\tau\left(\frac{\boldsymbol{\delta}^\top\mathbf{t}}{\sqrt{w}}\right)dH(w)$$

for $\boldsymbol{\delta}^\top\mathbf{t} > 0$ *and* $\tau^+(-x) = -\tau^+(x)$ *for* $x > 0$.

Proof: The conditional distribution of $\mathbf{X}$, given $W = w$, follows a multivariate skew-normal distribution; that is, $\mathbf{X}|W = w \sim SN_p(w^{-1}\Omega_{\mathbf{z}}, \boldsymbol{\alpha})$. Then the cf of $\mathbf{X}$ is

$$\begin{aligned}\psi_{\mathbf{X}}(\mathbf{t}) &= \int_0^\infty \int_{\mathbb{R}^p} \exp\left(i\mathbf{t}^\top\mathbf{x}\right) f(\mathbf{x}|w)\, d\mathbf{x}\, dH(w) = \int_0^\infty \psi_{\mathbf{X}|w}(\mathbf{t})dH(w) \\ &= \int_0^\infty \exp\left(-\frac{\mathbf{t}^\top\Omega_{\mathbf{z}}\mathbf{t}}{2w}\right)\left\{1 + i\tau\left(\frac{\boldsymbol{\delta}^\top\mathbf{t}}{\sqrt{w}}\right)\right\}dH(w) \\ &= \psi_{T_p}\left(\Omega_{\mathbf{z}}^{\frac{1}{2}}\mathbf{t}\right) + i\int_0^\infty \exp\left(-\frac{\mathbf{t}^\top\Omega_{\mathbf{z}}\mathbf{t}}{2w}\right)\tau\left(\frac{\boldsymbol{\delta}^\top\mathbf{t}}{\sqrt{w}}\right)dH(w),\end{aligned}$$

where $\psi_{T_p}(\mathbf{t})$ is given in Theorem 4, which is the cf of the multivariate $t_p(\nu)$ distribution. The remaining calculations are straightforward and the result follows. □

A particular case of the multivariate skew-*t* distribution is the multivariate skew-Cauchy distribution, when $\nu = 1$. In addition, when $\nu \to \infty$, we obtain the multivariate skew-normal distribution as the limiting case. Hence, we have computed the cfs of the multivariate skew-Cauchy and skew-normal distributions. For the other versions of the cfs of the multivariate *t* distribution, see Kotz and Nadarajah (2004). Extending these results to location-scale parameters is straightforward.

4. Discussion

Characteristic functions play an important role in probability and statistics. However, the derivation of these functions needs the theory of complex analysis which is unfamiliar to most statistics students. Therefore, we derived the cfs of the *t*, truncated normal, skew-normal, and skew-*t* distributions in univariate and multivariate cases without using contour integration. We wish to extend the current results in a future study to some other skewed distributions appearing in Azzalini and Capitanio (2014).

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