Communications for Statistical Applications and Methods
2014, Vol. 21, No. 5, 363–386

DOI: http://dx.doi.org/10.5351/CSAM.2014.21.5.363
Print ISSN 2287-7843 / Online ISSN 2383-4757

Estimation of Median in the Presence of Three Known Quartiles of an Auxiliary Variable

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Abstract

This paper has improved several ratio type estimators of the population median including their generalization in the presence of three known quartiles of an auxiliary variable. The properties of the improved estimators are discussed and applied. Both the empirical and simulation studies confirm that our new estimators perform efficiently.

Keywords: Population median, variance, auxiliary variable, quartiles.

1. Introduction

The problem of estimating the population median rather than the population mean gets more research attention from the statistics community when variables are highly skewed. Examples of skewed variables include income, expenditure *etc.* While the problem of estimating a population mean in the presence of an auxiliary variable has been widely discussed in the finite population sampling literature, relatively less effort has been devoted to the development of efficient methods for estimating a finite population median. Chambers and Dunstan (1986), Kuk and Mak (1989), Mak and Kuk (1993), Rao *et al.* (1990), Meeden and Vardeman (1991), Meeden (1995), Garcia and Cebrian (2001), Singh *et al.* (2001, 2007), Rueda and Arcos (2002), Allen *et al.* (2002), Singh and Puertas (2003), Singh *et al.* (2003), Singh, Sidhu and Singh (2006), Singh, Singh and Puertas (2006), Singh, Tailor, Singh and Kim (2007), Singh and Solanki (2013) and Sharma and Singh (2014) addressed the importance of estimating the population median in the presence of an auxiliary variable.

Let Y_i and X_i, $i = 1, 2, \ldots, N$, denote the values of the population units for the study variable Y and the auxiliary variable X, respectively. Further let y_i and x_i, $i = 1, 2, \ldots, n$, are the particular values of the units included in a sample S_n of size n drawn by simple random sampling without replacement (SRSWOR). In the absence of the X_i, the sample median $\widehat{M}_y$ (or the sample estimate $\widehat{Q}_{2y}$) is a natural estimator of population median M_y (or the second quartile Q_{2y}) of Y When the values of the auxiliary variable X_i are available, Kuk and Mak (1989) suggested a ratio estimator for M_y as:

$$\widehat{M}^{(0)}_{(R)} = \widehat{M}_y v^{-1}, \tag{1.1}$$

where $v = \widehat{Q}_{2x}/Q_{2x}$ and $\widehat{Q}_{2x}$ is the sample estimate of the second quartile Q_{2x} (or the population median M_x) of the auxiliary variable X. Let Q_{1z} be the first quartile of the variable z, Q_{2z} be the

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Published 30 September 2014 / journal homepage: http://csam.or.kr

Table 1

	$Y \le Q_{1y}$	$Y \le M_y$	$Y \le Q_{3y}$
$X \le Q_{1x}$	$P(Q_{1x}, Q_{1y})$	$P(Q_{1x}, M_y)$	$P(Q_{1x}, Q_{3y})$
$X \le Q_{2x}$	$P(Q_{2x}, Q_{1y})$	$P(Q_{2x}, M_y)$	$P(Q_{2x}, Q_{3y})$
$X \le Q_{3x}$	$P(Q_{3x}, Q_{1y})$	$P(Q_{3x}, M_y)$	$P(Q_{3x}, Q_{3y})$

second quartile of the variable z, Q_{3z} be the third quartile of the variable z with $z = x, y$. Then, we have

$$P(Q_{tx}, Q_{ty}) = P(X \le Q_{tx} \cap Y \le Q_{ty}), \quad t = 1, 2, 3,$$

where these may be given in a table of proportions as: Table 1.

The variance of the natural estimator $\widehat{M}_y$ is:

$$V\left(\widehat{M}_y\right) = \frac{(1-f)}{4n}\left(f_y\left(M_y\right)\right)^{-2}, \tag{1.2}$$

see Gross (1980), where f_y denotes the probability density function of Y and $f = (n/N)$.

Following Kuk and Mak (1989), the variance of the ratio estimator $\widehat{M}_R^{(0)}$ to the first degree of approximation is given by:

$$V\left(\widehat{M}_R^{(0)}\right) = V\left(\widehat{M}_y\right)\left[1 + A_0\left(A_0 - 2\rho_0\right)\right], \tag{1.3}$$

which is less than the variance of the natural estimator $\widehat{M}_y$ if

$$\rho_0 > \frac{A_0}{2}, \tag{1.4}$$

where $A_0 = (M_y/Q_{2x})(f_y(M_y)/f_x(Q_{2x}))$, f_y and f_x denote the probability density functions of Y and X respectively. Here $\rho_0 = (4P_{22} - 1)$ with $P_{22} = P(Q_{2x}, M_y) = P[X \le Q_{2x} \cap Y \le M_y]$ goes from -1 to $+1$ as P_{22} increases from 0 to 0.5.

In this paper, we suggest a new class of ratio-type estimators that is broader than the previous class of estimators suggested by Rueda *et al.* (2004) and Allen *et al.* (2002). We also investigate the properties of a few estimators, that belong to the the class of estimators proposed by Rueda *et al.* (2004), Allen *et al.* (2002) and the proposed broader class of estimators for two new models: a quadratic and a cubic model using extensive simulation study. We find that the estimators belonging to all three classes of estimators perform very well under the new simulation study set-up that is based on quadratic and cubic models.

2. Suggested Ratio-Type Estimators

We compare the following alternative estimators of the population median M_y,

(i) When the first quartile Q_{1x} of X is known:

$$\widehat{M}_R^{(1)} = \widehat{M}_y u^{-1}, \tag{2.1}$$

where $u = \widehat{Q}_{1x}/Q_{1x}$ and $\widehat{Q}_{1x}$ is the sample estimate of Q_{1x}.

(ii) When the third quartile Q_{3x} of X is known:

$$\widehat{M}_R^{(2)} = \widehat{M}_y w^{-1}, \tag{2.2}$$

where $w = \widehat{Q}_{3x}/Q_{3x}$ and $\widehat{Q}_{3x}$ is the sample estimate of Q_{3x}.

(iii) When the median M_x and the first quartile Q_{1x} of X are known:

$$\widehat{M}_R^{(3)} = \widehat{M}_y(uv)^{-1}. \tag{2.3}$$

(iv) When the median M_x and the first quartile Q_{3x} of X are known:

$$\widehat{M}_R^{(4)} = \widehat{M}_y(vw)^{-1}. \tag{2.4}$$

(v) When Q_{1x}, Q_{2x} and Q_{3x} of X are known:

$$\widehat{M}_R^{(5)} = \widehat{M}_y(uvw)^{-1}. \tag{2.5}$$

Note that the first two estimators are special case of the class of estimators proposed by Rueda *et al.* (2004), and the last three estimators are a special cases of the class of estimators by Allen *et al.* (2002).

To obtain the variances of the above estimators, we write:

$$\widehat{M}_y = M_y(1+e_0)\left\{\text{or } \widehat{Q}_{2y} = Q_{2y}(1+e_0)\right\}, \quad \widehat{Q}_{1x} = Q_{1x}(1+e_1),$$
$$\widehat{Q}_{2x} = Q_{2x}(1+e_2)\left\{\text{or } \widehat{M}_x = M_x(1+e_2)\right\} \quad \text{and} \quad \widehat{Q}_{3x} = Q_{3x}(1+e_3)$$

such that

$$E(e_i) \approx 0, \quad \forall i = 1,2,3$$

and

$$E\left(e_0^2\right) \approx \frac{(1-f)}{4n}\frac{1}{M_y^2\left(f_y\left(M_y\right)\right)^2}, \quad E\left(e_1^2\right) \approx \frac{3(1-f)}{16n}\frac{1}{Q_{1x}^2\left(f_x\left(Q_{1x}\right)\right)^2},$$
$$E\left(e_2^2\right) \approx \frac{(1-f)}{4n}\frac{1}{Q_{2x}^2\left(f_x\left(Q_{2x}\right)\right)^2}, \quad E\left(e_3^2\right) \approx \frac{3(1-f)}{16n}\frac{1}{Q_{3x}^2\left(f_x\left(Q_{3x}\right)\right)^2},$$
$$E\left(e_0e_1\right) \approx \frac{(1-f)}{8n}\frac{\rho_1}{M_yQ_{1x}f_y\left(M_y\right)f_x\left(Q_{1x}\right)}, \quad E\left(e_0e_2\right) \approx \frac{(1-f)}{4n}\frac{\rho_0}{M_yQ_{2x}f_x\left(Q_{2x}\right)f_y\left(M_y\right)},$$
$$E\left(e_0e_3\right) \approx \frac{(1-f)}{8n}\frac{\rho_2}{Q_{3x}M_yf_x\left(Q_{3x}\right)f_y\left(M_y\right)}, \quad E\left(e_1e_2\right) \approx \frac{(1-f)}{8n}\frac{1}{Q_{1x}Q_{2x}f_x\left(Q_{1x}\right)f_x\left(Q_{2x}\right)},$$
$$E\left(e_1e_3\right) \approx \frac{(1-f)}{16n}\frac{1}{Q_{1x}Q_{3x}f_x\left(Q_{1x}\right)f_x\left(Q_{3x}\right)}, \quad E\left(e_2e_3\right) \approx \frac{(1-f)}{8n}\frac{1}{Q_{2x}Q_{3x}f_x\left(Q_{2x}\right)f_x\left(Q_{3x}\right)},$$

where $\rho_1 = \{8P(Q_{1x}, M_y) - 1\}$ goes from -1 to $+1$ as $P(Q_{1x}, M_y) = P[X \leq Q_{1x} \cap Y \leq M_y]$ increases from 0 to 0.25, and $\rho_2 = \{8P(Q_{3x}, M_y) - 3\}$ ranges from -1 to $+1$ as $P(Q_{3x}, M_y) = P[X \leq Q_{3x} \cap Y \leq M_y]$ increases from 0.25 to 0.5. Thus to the first degree of approximation, the variances of $\widehat{M}_R^{(i)}$, $i = 1$ to 5 are respectively given by:

$$V\left(\widehat{M}_R^{(1)}\right) \approx V\left(\widehat{M}_y\right)\left[1 + A_1\left(\frac{3}{4}A_1 - \rho_1\right)\right], \tag{2.6}$$

$$V\left(\widehat{M}_R^{(2)}\right) \approx V\left(\widehat{M}_y\right)\left[1 + A_2\left(\frac{3}{4}A_2 - \rho_2\right)\right], \tag{2.7}$$

$$V\left(\widehat{M}_R^{(3)}\right) \approx V\left(\widehat{M}_y\right)\left[1 + A_0\left(A_0 - 2\rho_0\right) + A_1\left(\frac{3}{4}A_1 - \rho_1\right) + A_0A_1\right], \tag{2.8}$$

$$V\left(\widehat{M}_R^{(4)}\right) \approx V\left(\widehat{M}_y\right)\left[1 + A_0(A_0 - 2\rho_0) + A_1\left(\frac{3}{4}A_2 - \rho_2\right) + A_0A_2\right], \tag{2.9}$$

$$V\left(\widehat{M}_R^{(5)}\right) \approx V\left(\widehat{M}_y\right)\left[1+A_0(A_0 - 2\rho_0)+A_1\left(\frac{3}{4}A_1-\rho_1\right)+A_2\left(\frac{3}{4}A_2 - \rho_2\right)+A_0(A_1 + A_2)+\frac{1}{2}A_1A_2\right], \tag{2.10}$$

where $A_1 = (M_y/Q_{1x})(f_y(M_y)/f_x(Q_{1x}))$ and $A_2 = (M_y/Q_{3x})f_y(M_y)/f_x(Q_{3x}))$. We note from (2.6)–(2.10) respectively that:

$$V\left(\widehat{M}_R^{(1)}\right) < V\left(\widehat{M}_y\right) \quad \text{if} \quad \rho_1 > \frac{3}{4}A_1, \tag{2.11}$$

$$V\left(\widehat{M}_R^{(2)}\right) < V\left(\widehat{M}_y\right) \quad \text{if} \quad \rho_2 > \frac{3}{4}A_2, \tag{2.12}$$

$$V\left(\widehat{M}_R^{(3)}\right) < V\left(\widehat{M}_y\right) \quad \text{if} \quad \left[A_0(A_0 + A_1 - 2\rho_0) + A_1\left(\frac{3}{4}A_1 - \rho_1\right)\right] < 0, \tag{2.13}$$

$$V\left(\widehat{M}_R^{(4)}\right) < V\left(\widehat{M}_y\right) \quad \text{if} \quad \left[A_0(A_0 + A_2 - 2\rho_0) + A_2\left(\frac{3}{4}A_2 - \rho_2\right)\right] < 0, \tag{2.14}$$

$$V\left(\widehat{M}_R^{(5)}\right) < V\left(\widehat{M}_y\right) \quad \text{if} \quad \left[A_0\left(A_0+A_1+A_2-2\rho_0\right)+A_1\left(\frac{3}{4}A_1+\frac{1}{2}A_2-\rho_1\right)+A_2\left(\frac{3}{4}A_2-\rho_2\right)\right] < 0. \tag{2.15}$$

From (1.3) and (2.6)–(2.10), we noticed that:

$$V\left(\widehat{M}_R^{(1)}\right) < V\left(\widehat{M}_R^{(0)}\right) \quad \text{if} \quad A_1\left(\frac{3}{4}A_1 - \rho_1\right) < A_0(A_0 - 2\rho_0), \tag{2.16}$$

$$V\left(\widehat{M}_R^{(2)}\right) < V\left(\widehat{M}_R^{(0)}\right) \quad \text{if} \quad A_2\left(\frac{3}{4}A_2 - \rho_2\right) < A_0(A_0 - 2\rho_0), \tag{2.17}$$

$$V\left(\widehat{M}_R^{(3)}\right) < V\left(\widehat{M}_R^{(0)}\right) \quad \text{if} \quad \rho_1 > \left(A_0 + \frac{3}{4}A_1\right), \tag{2.18}$$

$$V\left(\widehat{M}_R^{(4)}\right) < V\left(\widehat{M}_R^{(0)}\right) \quad \text{if} \quad \rho_2 > \left(A_0 + \frac{3}{4}A_2\right), \tag{2.19}$$

$$V\left(\widehat{M}_R^{(5)}\right) < V\left(\widehat{M}_R^{(0)}\right) \quad \text{if} \quad \left[A_1\left(A_0 + \frac{3}{4}A_2 - \rho_1\right) + A_2\left(A_0 + \frac{1}{2}A_1 + \frac{3}{4}A_2 - \rho_2\right)\right] < 0, \tag{2.20}$$

It follows from (1.4) and (2.16) that the suggested estimator $\widehat{M}_R^{(1)}$ is more efficient than $\widehat{M}_y$ and $\widehat{M}_R^{(0)}$ if

$$\frac{A_0}{2} < \rho_0 < \frac{1}{2}\left[A_0 + \left(\frac{A_1}{A_0}\right)\left(\rho_1 - \frac{3}{4}A_1\right)\right]. \tag{2.21}$$

Note from (1.4) and (2.17) that the proposed estimator $\widehat{M}_R^{(2)}$ is more efficient than $\widehat{M}_y$ and $\widehat{M}_R^{(0)}$ if

$$\frac{A_0}{2} < \rho_0 < \frac{1}{2}\left[A_0 + \left(\frac{A_2}{A_0}\right)\left(\rho_2 - \frac{3}{4}A_2\right)\right]. \tag{2.22}$$

From (2.6) and (2.8) it is observed that $V(\widehat{M}_R^{(3)}) < V(\widehat{M}_R^{(1)})$ if

$$\rho_0 < \frac{1}{2}\sum_{i=0}^{1} A_i. \tag{2.23}$$

From (2.7) and (2.9) we note that $V(\widehat{M}_R^{(4)}) < V(\widehat{M}_R^{(2)})$ if

$$\rho_0 > \frac{A_0 + A_2}{2}. \tag{2.24}$$

From (2.8), (2.9) and (2.10) it is observed that:

$$V\left(\widehat{M}_R^{(5)}\right) < V\left(\widehat{M}_R^{(3)}\right) \quad \text{if} \quad \rho_2 > \left[A_0 + \frac{1}{2}A_1 + \frac{3}{4}A_2\right] \tag{2.25}$$

and

$$V\left(\widehat{M}_R^{(5)}\right) < V\left(\widehat{M}_R^{(4)}\right) \quad \text{if} \quad \rho_1 > \left[A_0 + \frac{3}{4}A_1 + \frac{1}{2}A_2\right]. \tag{2.26}$$

From (2.6), (2.7) and (2.10), we note that:

$$V\left(\widehat{M}_R^{(5)}\right) < V\left(\widehat{M}_R^{(1)}\right) \quad \text{if} \quad (2\rho_0 A_0 + \rho_2 A_2) > \left[A_0 \sum_{i=0}^{2} A_i + A_2\left(\frac{1}{2}A_1 + \frac{3}{4}A_2\right)\right] \tag{2.27}$$

and

$$V\left(\widehat{M}_R^{(5)}\right) < V\left(\widehat{M}_R^{(2)}\right) \quad \text{if} \quad (2\rho_0 A_0 + \rho_1 A_1) > \left[A_0 \sum_{i=0}^{2} A_i + A_1\left(\frac{3}{4}A_1 + \frac{1}{2}A_2\right)\right]. \tag{2.28}$$

Remark 1. Let $Q_x(\beta)$ denote the β-quantile of the auxiliary variable X, then $Q_{1x} = Q_x(0.25)$, $M_x = Q_{2x} = Q_x(0.50)$ and $Q_{3x} = Q_x(0.75)$. Further let $\widehat{Q}_x(\beta)$ be an estimator of $Q_x(\beta)$, then $\widehat{Q}_{1x} = \widehat{Q}_x(0.25)$, $\widehat{M}_x = \widehat{Q}_{2x} = \widehat{Q}_x(0.50)$ and $\widehat{Q}_{3x} = \widehat{Q}_x(0.75)$.

It is assumed that $Q_x(\beta)$ is known. Thus motivated by Rueda *et al.* (2005) and Arcos *et al.* (2005) one may define the following estimator:

$$\widehat{M}_R^{(\beta)}\Big) = \widehat{M}_y a^{-1}, \tag{2.29}$$

for the population median M_y of Y, where $a = Q_x(\beta)/\widehat{Q}_x(\beta)$. It can be easily shown that the estimators $\widehat{M}_R^{(1)}$, $\widehat{M}_R^{(0)}$ and $\widehat{M}_R^{(2)}$ are members of the estimators for $\beta = 0.25, 0.50$ and 0.75 respectively. To the first degree of approximation, the variance of $\widehat{M}_R^{(\beta)}$ is given by:

$$V\left\{\widehat{M}_R^{(\beta)}\Big)\right\} \approx V\left(\widehat{M}_y\right)\left[1 + 4\beta A_x(\beta)\left\{(1-\beta)A_x(\beta) - \rho_{xy}(\beta)\right\}\right], \tag{2.30}$$

where $A_x(\beta) = \{M_y f_y(M_y)\}/\{Q_x(\beta) f_x(Q_x(\beta))\}$, $P\{Q_x(\beta), M_y\} = P[X \leq Q_x(\beta) \cap Y \leq M_y]$ and $\rho_{xy}(\beta) = [(2/\beta)P\{Q_x(\beta), M_y\} - 1]$ goes from -1 to $+1$ as $P\{Q_x(\beta), M_y\}$ increases from 0 to β.

It is observed from (2.30) that $V(\widehat{M}_R^{(\beta)}) < V(\widehat{M}_y)$ if

$$\rho_{xy}(\beta) > (1-\beta)A_x(\beta). \tag{2.31}$$

We note that inequalities (1.4), (2.11) and (2.12) can be easily obtained from (2.31) just by putting $\beta = 0.50, 0.25, 0.75$ respectively.

3. General Class of Estimators

Let the first quartile Q_{1x}, the second quartile Q_{2x} and the third quartile Q_{3x} of the auxiliary variable X be known. With $u = \widehat{Q}_{1x}/Q_{1x}$, $v = \widehat{Q}_{2x}/Q_{2x}$ and $w = \widehat{Q}_{3x}/Q_{3x}$, along the lines of Srivastava and Jhajj (1981) we define a class of estimators of the population median M_y as

$$\widehat{M}_g^{(s)} = S\left(\widehat{M}_y, u, v, w\right), \tag{3.1}$$

where $S(\bullet)$ is a function of $(\widehat{M}_y, u, v, w)$ such that $S(P) = M_y \implies S_{1000}(P) = \partial S(\bullet)/\partial \widehat{M}_y|_P = 1$ with $P = (M_y, 1, 1, 1)$ and satisfying the following conditions:

(i) Whatever be the sample chosen, $(\widehat{M}_y, u, v, w)$ assume values in a bounded closed convex subset, U, of the four dimensional real space containing the point P.

(ii) In U, the function $S(\widehat{M}_y, u, v, w)$ is continuous and bounded.

(iii) The first, second and third partial derivatives of $S(\widehat{M}_y, u, v, w)$ exist and are continuous and bounded in U.

Under the conditions (i) and (ii) the bias and variance of the class of estimators, $\widehat{M}_g^{(S)}$, exist since there are only a finite number of possible samples. Expanding the function $S(\widehat{M}_y, u, v, w)$ about the point $(\widehat{M}_y, u, v, w) = (M_y, 1, 1, 1) = P$ in a third order Taylor's series, we obtain

$$\begin{aligned}
\widehat{M}_g^{(S)} &= S(P) + \left(\widehat{M}_y - M_y\right) S_{1000}(P) + (u-1)S_{0100}(P) + (v-1)S_{0010}(P) + (w-1)S_{0001}(P) \\
&+ \frac{1}{2}\Big\{\left(\widehat{M}_y - M_y\right)^2 S_{2000}(P) + (u-1)^2 S_{0200}(P) + (v-1)^2 S_{0020}(P) + (w-1)^2 S_{0002}(P) \\
&+ 2\left(\widehat{M}_y - M_y\right)(u-1)S_{1100}(P) + 2\left(\widehat{M}_y - M_y\right)(v-1)S_{1010}(P) + 2\left(\widehat{M}_y - M_y\right)(w-1)S_{1001}(P) \\
&+ 2(u-1)(v-1)S_{0110}(P) + 2(u-1)(w-1)S_{0101}(P) + 2(v-1)(w-1)S_{0011}(P)\Big\} \\
&+ \frac{1}{3!}\left\{\left(\widehat{M}_y - M_y\right)\frac{\partial}{\partial \widehat{M}_y} + (u-1)\frac{\partial}{\partial v} + (w-1)\frac{\partial}{\partial w}\right\}^3 S\left(\widehat{M}_y^*, u^*, v^*, w^*\right),
\end{aligned} \tag{3.2}$$

where $\widehat{M}_y^* = \{M_y + \eta(\widehat{M}_y - M_y)\}$, $u^* = \{1+\eta(u-1)\}$, $v^* = \{1+\eta(v-1)\}$, $w^* = \{1+\eta(w-1)\}$, $0 < \eta < 1$ and $S_{0100}(P) = \partial S(\bullet)/\partial u|_P$, $S_{0010}(P) = \partial S(\bullet)/\partial v|_P$, $S_{0001}(P) = \partial S(\bullet)/\partial w|_P$, $S_{2000}(P) = \partial^2 S(\bullet)/\partial \hat{M}_y^2|_P$, $S_{0200}(P) = \partial^2 S(\bullet)/\partial u^2|_P$, $S_{0020}(P) = \partial^2 S(\bullet)/\partial v^2|_P$, $S_{0002}(P) = \partial^2 S(\bullet)/\partial w^2|_P$, $S_{1100}(P) = \partial^2 S(\bullet)/\partial u \partial \hat{M}_y|_P$, $S_{1010}(P) = \partial^2 S(\bullet)/\partial v \partial \hat{M}_y|_P$, $S_{1001}(P) = \partial^2 S(\bullet)/\partial w \partial \hat{M}_y|_P$, $S_{0110}(P) = \partial^2 S(\bullet)/\partial v \partial u|_P$, $S_{0101}(P) = \partial^2 S(\bullet)/\partial w \partial u|_P$, $S_{0011}(P) = \partial^2 S(\bullet)/\partial w \partial v|_P$.

Noting that $S(P) = M_y$, $S_{(1000)}(P) = 1$, $S_{(2000)}(P) = 0$, expressing $(\widehat{M}_y - M_y)$, $(u-1)$, $(v-1)$ and $(w-1)$ in terms of e's and neglecting terms of e's having power greater than two in (3.2), we have

$$\left\{\widehat{M}_g^{(S)} - M_y\right\} \approx M_y e_0 + e_1 S_{0100}(P) + e_2 S_{0010}(P) + e_3 S_{0001}(P) + \frac{1}{2}\left\{e_1^2 S_{0200}(P) + e_2^2 S_{0020}(P) + e_3^2 S_{0002}(P) + 2M_y e_0 e_1 S_{1100}(P) + 2M_y e_0 e_2 S_{1010}(P) + 2M_y e_0 e_3 S_{1001}(P) + 2e_1 e_2 S_{0110}(P) + 2e_1 e_3 S_{0101}(P) + 2e_2 e_3 S_{0011}(P)\right\}. \tag{3.3}$$

Taking expectation of both sides of (3.3) we get the bias of $\widehat{M}_g^{(S)}$ to the first degree of approximation as

$$B\left\{\widehat{M}_g^{(S)}\right\} \approx \frac{(1-f)}{8n\left\{f_y\left(M_y\right)\right\}^2}\left[\left(\frac{A_1}{M_y}\right)\left\{\frac{3}{4}\left(\frac{A_1}{M_y}\right)S_{0200}(P) + \left(\frac{A_0}{M_y}\right)S_{0110}(P) + \frac{1}{2}\left(\frac{A_2}{M_y}\right)S_{0101}(P) + \rho_1 S_{1100}(P)\right\} + \frac{A_2}{M_y}\left\{\frac{3}{4}\left(\frac{A_2}{M_y}\right)S_{0002}(P) + \rho_2 S_{1001}(P) + \left(\frac{A_0}{M_y}\right)S_{0011}(P)\right\} + \left(\frac{A_0}{M_y}\right)\left\{\left(\frac{A_0}{M_y}\right)S_{0020}(P) + 2\rho_0 S_{1010}(P)\right\}\right], \tag{3.4}$$

Squaring both sides of (3.3) and neglecting terms of e's having power greater than two we have:

$$\begin{aligned}\left\{\widehat{M}_g^{(S)} - M_y\right\}^2 &\approx \left[M_y e_0 + e_1 S_{0100}(P) + e_2 S_{0010}(P) + e_3 S_{0001}(P)\right]^2 \\ &= \Big[M_y^2 e_0^2 + e_1^2 S_{0100}^2(P) + e_2^2 S_{0010}^2(P) + e_3^2 S_{0001}^2(P) \\ &\quad + 2M_y\left\{e_0 e_1 S_{0100}(P) + e_0 e_2 S_{0010}(P) + e_0 e_3 S_{0001}(P)\right\} \\ &\quad + 2\left\{e_1 e_2 S_{0100}(P) S_{0010}(P) + e_1 e_3 S_{0100}(P) S_{0001}(P)\right\} \\ &\quad + 2e_2 e_3 S_{0010}(P) S_{0001}(P)\Big].\end{aligned} \tag{3.5}$$

Taking expectation of both sides of (3.5) we get the variance of $\widehat{M}_g^{(S)}$ to the first degree of approximation as:

$$\begin{aligned}V\left\{\widehat{M}_g^{(S)}\right\} \approx V\left\{\widehat{M}_y\right\}\Bigg[1 &+ \left(\frac{A_1}{M_y}\right)S_{0100}(P)\left\{\frac{3}{4}\left(\frac{A_1}{M_y}\right)S_{0100}(P) + \left(\frac{A_0}{M_y}\right)S_{0010}(P) + \frac{1}{2}\left(\frac{A_2}{M_y}\right)S_{0001}(P) + \rho_1\right\} + \left(\frac{A_0}{M_y}\right)S_{0010}(P)\left\{\left(\frac{A_0}{M_y}\right)S_{0010}(P) + 2\rho_0\right\} \\ &+ \left(\frac{A_2}{M_y}\right)S_{0001}(P)\left\{\frac{3}{4}\left(\frac{A_2}{M_y}\right)S_{0001}(P) + \rho_2 + \left(\frac{A_0}{M_y}\right)S_{0010}(P)\right\}\Bigg].\end{aligned} \tag{3.6}$$

The variance at (3.6) is minimized for:

$$\begin{aligned}S_{0100}(P) &= (\rho_0 - \rho_1)\frac{Q_{1x} f_x\left(Q_{1x}\right)}{f_y\left(M_y\right)}, \\ S_{0010}(P) &= -\frac{1}{2}(4\rho_0 - \rho_1 - \rho_2)\frac{Q_{2x} f_x(Q_{2x})}{f_y\left(M_y\right)}, \\ S_{0001}(P) &= (\rho_0 - \rho_2)\frac{Q_{3x} f_x(Q_{3x})}{f_y\left(M_y\right)}.\end{aligned} \tag{3.7}$$

with the resulting minimum variance of $\widehat{M}_g^{(S)}$ given by:

$$\text{Min. } V\left\{\widehat{M}_g^{(S)}\right\} = V\left\{\widehat{M}_y\right\}\left[1 - \frac{1}{2}(\rho_0 - \rho_1)^2 - \frac{1}{2}(\rho_0 - \rho_2)^2 - \rho_0^2\right]. \tag{3.8}$$

Thus we state the following theorem:

Theorem 1. *Up to terms of order n^{-1},*

$$V\left\{\widehat{M}_g^{(S)}\right\} \geq V\left\{\widehat{M}_y\right\}\left[1 - \frac{1}{2}(\rho_0 - \rho_1)^2 - \frac{1}{2}(\rho_0 - \rho_2)^2 - \rho_0^2\right]$$

with equality holding if:
$S_{0100}(P) = (\rho_0 - \rho_1)\{Q_{1x} f_x(Q_{1x})\}/f_y(M_y)$, $S_{0010}(P) = -1/2(4\rho_0 - \rho_1 - \rho_2)\{Q_{2x} f_x(Q_{2x})\}/f_y(M_y)$, $S_{0001}(P) = (\rho_0 - \rho_2)\{Q_{3x} f_x(Q_{3x})\}/f_y(M_y)$.

Any parametric function $S(\widehat{M}_y, u, v, w)$, satisfying the regularity conditions, can generate asymptotically acceptable estimator. The following estimators:

$$\begin{aligned}
\widehat{M}_{g(1)}^{(s)} &= \widehat{M}_y u^{\alpha_1} v^{\alpha_2} w^{\alpha_3},\\
\widehat{M}_{g(2)}^{(s)} &= \widehat{M}_y \left[1 + \alpha\left(u^{\alpha_1} v^{\alpha_2} w^{\alpha_3} - 1\right)\right]^{-1},\\
\widehat{M}_{g(3)}^{(s)} &= \widehat{M}_y \left[w_1 u^{\alpha_1} + w_2 v^{\alpha_2} + w_3 w^{\alpha_3}\right], \quad \sum_{i=1}^{3} w_i = 1,\\
\widehat{M}_{g(4)}^{(s)} &= \widehat{M}_y \left[1 + \alpha_1(u-1) + \alpha_2(v-1) + \alpha_3(w-1)\right],\\
\widehat{M}_{g(5)}^{(s)} &= \widehat{M}_y \left[1 - \alpha_1(u-1) - \alpha_2(v-1) - \alpha_3(w-1)\right]^{-1},\\
\widehat{M}_{g(6)}^{(s)} &= \widehat{M}_y \exp\left[\alpha_1(u-1) + \alpha_2(v-1) + \alpha_3(w-1)\right],\\
\widehat{M}_{g(7)}^{(s)} &= \widehat{M}_y + \alpha_1(u-1) + \alpha_2(v-1) + \alpha_3(w-1),
\end{aligned}$$

etc. of M_y are members of the class $\widehat{M}_g^{(s)}$, where $(\alpha_1, \alpha_2, \alpha_3)$ are real constants. The bias and variance expressions of the estimators $\widehat{M}_g^{(s)}$; $j = 1, 2, \ldots, 7$ can be obtained from (3.4) and (3.6) just by putting the suitable values of the derivatives of the function $S(\hat{M}_y, u, v, w)$ about the point $P = (M_y, 1, 1, 1)$. The optimum values of the constants α_1, α_2 and α_3 are obtained by the right hand sides of (3.7) and the resulting estimators will have the same minimum variance given by (3.8).

Remark 2. The following classes of estimators of the population median M_y:

$$\widehat{M}_g^{(1)} = S\left(\widehat{M}, u\right) \equiv H\left(\widehat{M}_y, u\right) = \widehat{M}_g^{(h)}(\text{say}), \quad (\text{only } Q_{1x} \text{ of } X \text{ is known}) \tag{3.9}$$

$$\widehat{M}_g^{(2)} = S\left(\widehat{M}_y, v\right) \equiv F\left(\widehat{M}_y, v\right) = \widehat{M}_g^{(f)}(\text{say}), \quad (\text{only } Q_{2x} \text{ of } X \text{ is known}) \tag{3.10}$$

$$\widehat{M}_g^{(3)} = S\left(\widehat{M}_y, w\right) \equiv T\left(\widehat{M}_y, w\right) = \widehat{M}_g^{(t)}(\text{say}), \quad (\text{only } Q_{3x} \text{ of } X \text{ is known}) \tag{3.11}$$

$$\widehat{M}_g^{(4)} = S\left(\widehat{M}_y, u, v\right) \equiv D\left(\widehat{M}_y, u, v\right) = \widehat{M}_g^{(d)}(\text{say}), \quad (\text{only } Q_{1x} \text{ and } Q_{2x} \text{ are known}) \tag{3.12}$$

$$\widehat{M}_g^{(5)} = S\left(\widehat{M}_y, v, w\right) \equiv G\left(\widehat{M}_y, v, w\right) = \widehat{M}_g^{(g)}(\text{say}), \quad (\text{only } Q_{2x} \text{ and } Q_{3x} \text{ are known}) \tag{3.13}$$

$$\widehat{M}_g^{(6)} = S\left(\widehat{M}_y, u, w\right) \equiv J\left(\widehat{M}_y, u, w\right) = \widehat{M}_g^{(J)}(\text{say}), \quad (\text{only } Q_{1x} \text{ and } Q_{3x} \text{ are known}) \tag{3.14}$$

Table 2: Values of derivatives

Estimator	Values of derivatives					
	$S_{0100}(P)$	$S_{0010}(P)$	$S_{0001}(P)$	$S_{1100}(P)$	$S_{1010}(P)$	$S_{1001}(P)$
$\widehat{M}_g^{(h)}$	$H_{01}(M_y, 1)$	00	00	$H_{11}(M_y, 1)$	00	00
$\widehat{M}_g^{(f)}$	00	$F_{01}(M_y, 1)$	00	00	$F_{11}(M_y, 1)$	00
$\widehat{M}_g^{(t)}$	00	00	$T_{01}(M_y, 1)$	00	00	$T_{11}(M_y, 1)$
$\widehat{M}_g^{(d)}$	$D_{010}(L)$	$D_{001}(L)$	00	$D_{110}(L)$	$D_{101}(L)$	00
$\widehat{M}_g^{(g)}$	00	$G_{010}(L)$	$G_{001}(L)$	00	$G_{110}(L)$	$G_{101}(L)$
$\widehat{M}_g^{(J)}$	$J_{010}(L)$	00	$J_{001}(L)$	$J_{110}(L)$	00	$J_{101}(L)$
	$S_{0110}(P)$	$S_{0101}(P)$	$S_{0011}(P)$	$S_{0200}(P)$	$S_{0020}(P)$	$S_{0002}(P)$
$\widehat{M}_g^{(h)}$	00	00	00	$H_{02}(M_y, 1)$	00	00
$\widehat{M}_g^{(f)}$	00	00	00	00	$F_{02}(M_y, 1)$	00
$\widehat{M}_g^{(t)}$	00	00	00	00	00	$T_{02}(M_y, 1)$
$\widehat{M}_g^{(d)}$	$D_{011}(L)$	00	00	$D_{020}(L)$	$D_{002}(L)$	00
$\widehat{M}_g^{(g)}$	00	00	$G_{011}(L)$	00	$G_{020}(L)$	$G_{002}(L)$
$\widehat{M}_g^{(J)}$	00	$J_{011}(L)$	00	$J_{020}(L)$	00	$J_{002}(L)$

are the members of the proposed class of estimators $\widehat{M}_g^{(S)}$, where $H(\widehat{M}_y, u)$, $F(\widehat{M}_y, v)$, $T(\widehat{M}_y, w)$, $D(\widehat{M}_y, u, v)$, $G(\widehat{M}_y, v, w)$ and $J(\widehat{M}_y, u, w)$ are the functions of $(\widehat{M}_y, u)$, $(\widehat{M}_y, v)$, $(\widehat{M}_y, w)$, $(\widehat{M}_y, u, v)$, $(\widehat{M}_y, v, w)$ and $(\widehat{M}_y, u, w)$ respectively such that:

$$\begin{aligned} H(M_y, 1) = M_y &\Longrightarrow H_{10}(M_y, 1) = 1; \quad F(M_y, 1) = M_y \Longrightarrow F_{10}(\widehat{M}_y, 1) = 1, \\ T(M_y, 1) = M_y &\Longrightarrow T_{10}(M_y, 1) = 1; \quad D(L) = M_y \Longrightarrow D_{10}(L) = 1, \\ G(L) = M_y &\Longrightarrow G_{10}(L) = 1; \quad J(L) = M_y \Longrightarrow J_{10}(L) = 1, \end{aligned}$$

where $L = (M_y, 1, 1)$; and $H_{10}(M_y, 1)$, $F_{10}(\widehat{M}_y, 1)$, $T_{10}(M_y, 1)$, $D_{10}(L)$, $G_{10}(L)$ and $J_{10}(L)$ denote the first order partial derivatives of the functions $H(\widehat{M}_y, u)$, $F(\widehat{M}_y, v)$, $T(\widehat{M}_y, w)$, $D(\widehat{M}_y, u, v)$, $G(\widehat{M}_y, v, w)$ and $J(\widehat{M}_y, u, w)$ respectively about the point $(M_y, 1)$ and L. The bias and variance expressions of the class of estimators $\widehat{M}_g^{(r)}$, $r = h, f, t, d, g, J$; can be obtained from (3.4) and (3.6) by putting suitable values of the derivatives as shown in Table 2.

Thus the variances of the estimators $\widehat{M}_g^{(h)}$, $\widehat{M}_g^{(f)}$, $\widehat{M}_g^{(t)}$, $\widehat{M}_g^{(d)}$, $\widehat{M}_g^{(g)}$ and $\widehat{M}_g^{(J)}$ to the first degree of approximation are respectively given by

$$V\left\{\widehat{M}_g^{(h)}\right\} \approx V\left\{\widehat{M}_y\right\}\left[1 + \left\{\frac{H_{01}(M_y, 1)}{M_y}\right\} A_1 \left(\frac{3}{4}\left\{\frac{H_{01}(M_y, 1)}{M_y}\right\} A_1 + \rho_1\right)\right] \tag{3.15}$$

$$V\left\{\widehat{M}_g^{(f)}\right\} \approx V\left\{\widehat{M}_y\right\}\left[1 + \left\{\frac{F_{01}(M_y, 1)}{M_y}\right\} A_0 \left(\left\{\frac{F_{01}(M_y, 1)}{M_y}\right\} A_0 + 2\rho_0\right)\right] \tag{3.16}$$

$$V\left\{\widehat{M}_g^{(t)}\right\} \approx V\left\{\widehat{M}_y\right\}\left[1 + \left\{\frac{T_{01}(M_y, 1)}{M_y}\right\} A_2 \left(\left\{\frac{T_{01}(M_y, 1)}{M_y}\right\} A_2 + 2\rho_2\right)\right] \tag{3.17}$$

$$\begin{aligned} V\left\{\widehat{M}_g^{(d)}\right\} \approx V\left\{\widehat{M}_y\right\}\Bigg[1 &+ \left(\frac{A_1 D_{010}(L)}{M_{yu}}\right)\left\{\frac{3}{4}\left(\frac{A_1 D_{010}(L)}{M_y}\right) + \frac{D_{001}(L)}{M_y} A_0 + \rho_1\right\} \\ &+ \left(\frac{A_0 D_{001}(L)}{M_y}\right)\left\{\frac{A_0 D_{001}(L)}{M_y} + 2\rho_0\right\}\Bigg], \end{aligned} \tag{3.18}$$

$$V\left\{\widehat{M}_g^{(g)}\right\} \approx V\left\{\widehat{M}_y\right\}\left[1+\left(\frac{A_2G_{001}(L)}{M_y}\right)\left\{\frac{3}{4}\left(\frac{A_2G_{001}(L)}{M_y}\right)+\frac{G_{010}(L)}{M_y}A_0+\rho_2\right\}\right.$$
$$\left.+\left(\frac{A_0G_{010}(L)}{M_y}\right)\left\{\frac{A_0G_{010}(L)}{M_y}+2\rho_0\right\}\right], \tag{3.19}$$

$$V\left\{\widehat{M}_g^{(J)}\right\} \approx V\left\{\widehat{M}_y\right\}\left[1+\left(\frac{A_1}{M_y}\right)J_{010}(L)\left\{\frac{3}{4}\left(\frac{A_1}{M_y}\right)J_{010}(L)+\rho_1\right\}\right.$$
$$\left.+\left(\frac{A_2}{M_y}\right)J_{001}(L)\left\{\frac{3}{4}\left(\frac{A_2}{M_y}\right)J_{001}(L)+\rho_2\right\}+\frac{1}{2}\left(\frac{A_1A_2}{M_y^2}\right)J_{010}(L)J_{001}(L)\right], \tag{3.20}$$

which are minimized, respectively, for:

$$H_{01}(M_y,1) = -\frac{2}{3}\frac{Q_{1x}f_x(Q_{1x})}{f_y(M_y)}\rho_1, \tag{3.21}$$

$$F_{01}(M_y,1) = -\frac{Q_{2x}f_x(Q_{2x})}{f_y(M_y)}\rho_0, \tag{3.22}$$

$$T_{01}(M_y,1) = -\frac{2}{3}\frac{Q_{3x}f_x(Q_{3x})}{f_y(M_y)}\rho_2, \tag{3.23}$$

$$D_{010}(L) = \frac{Q_{1x}f_x(Q_{1x})}{f_y(M_y)}(\rho_0-\rho_1) \quad \text{and} \quad D_{001}(L) = \frac{Q_{2x}f_x(Q_{2x})}{2f_y(M_y)}(\rho_1-3\rho_0), \tag{3.24}$$

$$G_{010}(L) = \frac{Q_{2x}f_x(Q_{2x})}{2f_y(M_y)}(\rho_2-3\rho_0) \quad \text{and} \quad G_{001}(L) = \frac{Q_{3x}f_x(Q_{3x})}{f_y(M_y)}(\rho_0-\rho_2), \tag{3.25}$$

$$J_{010}(L) = \frac{Q_{1x}f_x(Q_{1x})}{f_y(M_y)}(\rho_2-3\rho_1) \quad \text{and} \quad J_{001}(L) = \frac{Q_{3x}f_x(Q_{3x})}{f_y(M_y)}(\rho_1-3\rho_2), \tag{3.26}$$

where $H_{01}(M_y,1)$, $F_{01}(M_y,1)$, $T_{01}(M_y,1)$, $D_{010}(L)$, $D_{001}(L)$, $G_{010}(L)$, $G_{001}(L)$, $J_{010}(L)$ and $J_{001}(L)$ denote the first order partial derivatives of the functions $H(\widehat{M}_y,u)$, $F(\widehat{M}_y,v)$, $T(\widehat{M}_y,w)$, $D(\widehat{M}_y,u,v)$, $G(\widehat{M}_y,v,w)$ and $J(\widehat{M}_y,u,w)$ respectively about the point $(M_y,1)$ and L.

Thus the resulting minimum variances of $\widehat{M}_g^{(h)}$, $\widehat{M}_g^{(f)}$, $\widehat{M}_g^{(t)}$, $\widehat{M}_g^{(d)}$, $\widehat{M}_g^{(g)}$ and $\widehat{M}_g^{(J)}$ are respectively given by:

$$\text{Min.}V\left(\widehat{M}_g^{(h)}\right) \approx V\left(\widehat{M}_y\right)\left(1-\frac{1}{3}\rho_1^2\right), \tag{3.27}$$

$$\text{Min.}V\left(\widehat{M}_g^{(f)}\right) \approx V\left(\widehat{M}_y\right)\left(1-\rho_0^2\right), \tag{3.28}$$

$$\text{Min.}V\left(\widehat{M}_g^{(t)}\right) \approx V\left(\widehat{M}_y\right)\left(1-\frac{1}{3}\rho_2^2\right), \tag{3.29}$$

$$\text{Min.}V\left(\widehat{M}_g^{(d)}\right) \approx V\left(\widehat{M}_y\right)\left[1-\frac{3}{2}\rho_0^2-\frac{1}{2}\rho_1^2+\rho_0\rho_1\right], \tag{3.30}$$

$$\text{Min.}V\left(\widehat{M}_g^{(g)}\right) \approx V\left(\widehat{M}_y\right)\left[1-\frac{3}{2}\rho_0^2-\frac{1}{2}\rho_2^2+\rho_0\rho_2\right], \tag{3.31}$$

$$\text{Min.}V\left(\widehat{M}_g^{(J)}\right) \approx V\left(\widehat{M}_y\right)\left[1-\frac{1}{8}\left(3\rho_1^2+3\rho_2^2-2\rho_1\rho_2\right)\right]. \tag{3.32}$$

Now we established the following theorem:

Theorem 2. *Up to terms of order n^{-1},*

(i) $V(\widehat{M}_g^{(h)}) \geq V(\widehat{M}_y)(1-1/3\rho_1^2)$ *with equality holding if* $H_{01}(M_y, 1) = -(2/3)\{Q_{1x}f_x(Q_{1x})\}/f_y(M_y)\rho_1$,

(ii) $Min.V(\widehat{M}_g^{(f)}) \geq V(\widehat{M}_y)(1-\rho_0^2)$ *with equality holding if* $F_{01}(M_y, 1) = -\{Q_{2x}f_x(Q_{2x})\}/f_y(M_y)\rho_0$,

(iii) $Min.V(\widehat{M}_g^{(t)}) \geq V(\widehat{M}_y)\left(1-1/3\rho_2^2\right)$ *with equality holding if* $T_{01}(M_y, 1) = -(2/3)\{Q_{3x}f_x(Q_{3x})\}/f_y(M_y)\rho_2$,

(iv) $Min.V(\widehat{M}_g^{(d)}) \geq V(\widehat{M}_y)[1-3/2\rho_0^2-1/2\rho_1^2+\rho_0\rho_1]$ *with equality holding if*

$$D_{010}(L) = \frac{Q_{1x}f_x(Q_{1x})}{f_y\left(M_y\right)}(\rho_0-\rho_1) \text{ and } D_{001}(L) = \frac{Q_{2x}f_x(Q_{2x})}{2f_y\left(M_y\right)}(\rho_1-3\rho_0),$$

(v) $Min.V(\widehat{M}_g^{(g)}) \geq V(\widehat{M}_y)(1-3/2\rho_0^2-1/2\rho_0^2+\rho_0\rho_2)$ *with equality holding if*

$$G_{010}(L) = \frac{Q_{2x}f_x(Q_{2x})}{2f_y\left(M_y\right)}(\rho_2-3\rho_0) \text{ and } G_{001}(L) = \frac{Q_{3x}f_x(Q_{3x})}{f_y(M_y)}(\rho_0-\rho_2),$$

(vi) $Min.V(\widehat{M}_g^{(J)}) \geq V(\widehat{M}_y)[1-1/8(3\rho_1^2+3\rho_2^2-2\rho_1\rho_2)]$ *with equality holding if*

$$J_{010}(L) = \frac{Q_{1x}f_x(Q_{1x})}{f_y\left(M_y\right)}(\rho_2-3\rho_1) \text{ and } J_{001}(L) = \frac{Q_{3x}f_x(Q_{3x})}{f_y\left(M_y\right)}(\rho_1-3\rho_2).$$

Remark 3. Motivated by Srivastava and Jhajj (1981), Rueda *et al.* (2005) and Arcos *et al.* (2005), one may also define a class of estimators for the population median M_y as:

$$\widehat{M}_\theta = \theta\left(\widehat{M}_y, a\right), \tag{3.33}$$

where $a = \{\widehat{Q}_x(\beta)/Q_x(\beta)\}$ and $\theta(\widehat{M}_y, a)$ is a function of $(\widehat{M}_y, a)$ such that $\theta(M_y, 1) = M_y \Longrightarrow \theta_{10}(M_y, 1) = 1$ and satisfies conditions similar to those given for $\widehat{M}_g^{(S)}$ at (3.1), where $\theta_{10}(M_y, 1)$ is a first order partial derivative of the function $\theta_{10}(\widehat{M}_y, a)$ with respect to $\widehat{M}_y$ about the point $(M_y, 1)$.

To the first degree of approximation, the bias and variance of $\widehat{M}_\theta$ are, respectively, given by

$$B\left(\widehat{M}_\theta\right) \approx \frac{\beta(1-f)}{2n}\left[\frac{\rho_{xy}(\beta)\theta_{11}\left(M_y, 1\right)}{Q_x(\beta)f_x\{Q_x(\beta)\}f_y\left(M_y\right)} + \frac{(1-\beta)\theta_{02}\left(M_y, 1\right)}{\{Q_x(\beta)\}^2\{f_x\left(Q_x(\beta)\right)\}^2}\right], \tag{3.34}$$

$$V\left(\widehat{M}_\theta\right) \approx V\left(\widehat{M}_y\right)\left[1+4\beta\left\{\frac{\theta_{01}\left(M_y, 1\right)}{M_y}\right\}A_x(\beta)\left\{(1-\beta)\left(\frac{\theta_{01}\left(M_y, 1\right)}{M_y}\right)A_x(\beta)+\rho_{xy}(\beta)\right\}\right], \tag{3.35}$$

where $\theta_{01}(M_y, 1) = \{\partial\theta(\widehat{M}_y, a)\}/\partial a|_{(M_y,1)}$, $\theta_{11}(M_y, 1) = \{\partial\theta(\widehat{M}_y, a)\}/\{\partial\{Q_x(\beta)\}\partial\widehat{M}_y\}|_{(M_y,1)}$ and $\theta_{02}(M_y, 1) = \{\partial^2\theta(\widehat{M}_y, a)\}/\partial\{Q_x(\beta)\}^2|_{(M_y,1)}$.

The variance of $\widehat{M}_\theta$ at (3.34) is minimized for

$$\theta_{01}\left(M_y, 1\right) = -\frac{\rho_{xy}\beta}{2(1-\beta)} \frac{Q_x(\beta) f_x\{Q_x(\beta)\}}{f_y\left(M_y\right)}. \tag{3.36}$$

Substitution of (3.36) in (3.35) yields the minimum variance of $\widehat{M}_\theta$ as

$$\text{Min.}V\left\{\widehat{M}_\theta\right\} = V\left(\widehat{M}_y\right)\left[1 - \frac{\beta}{(1-\beta)}\rho_{xy}^2(\beta)\right]. \tag{3.37}$$

Thus we state the following theorem.

Theorem 3. *Up to terms of order n^{-1},*

$$V\left(\widehat{M}_\theta\right) \geq V\left(\widehat{M}_y\right)\left[1 - \frac{\beta}{(1-\beta)}\rho_{xy}^2(\beta)\right].$$

with equality holding if

$$\theta_{01}\left(M_y, 1\right) = -\frac{\rho_{xy}(\beta)}{2(1-\beta)} \frac{Q_x(\beta) f_x\{Q_x(\beta)\}}{f_y(M_y)}.$$

It can be easily seen from (1.2), (2.30) and (3.37) that the proposed class of estimators $\widehat{M}_\theta$ is more efficient than the conventional estimator $\widehat{M}_y$ and the ratio-type estimator $\widehat{M}_R^{(\beta)}$. The classes of estimators $\widehat{M}_g^{(h)}$, $\widehat{M}_g^{(f)}$ and $\widehat{M}_g^{(t)}$ are members of the class $\widehat{M}_\theta$. The bias and variance of these estimators can be easily obtained from (3.34) and (3.35) just by putting $\beta = 0.25, 0.50, 0.75$.

Remark 4. It is to be mentioned that the optimum values of the parameters involved in the estimators depend on unknown population values such as $f_x(Q_{1x})$, $f_x(Q_{2x})$, $f_x(Q_{3x})$, ρ_0, ρ_0, ρ_0 and $f_y(M_y)$. Thus, to use such an estimator one has to use guessed or estimated values of these population values which can be obtained from either past data or experience. If the guessed values of these population values are not known then it is advisable to use sample data at hand to estimate these parameters. Following the procedure as outlined in Singh *et al.* (2007) and Silverman (1986), it can be shown that the class of estimators based on estimated optimum values has the same variance to the first degree of approximation as that of the estimators based on exact optimum values.

4. Efficiency Comparison

From (1.2), (3.8), (3.27)–(3.32), we show that:

$$V\left(M_y\right) - \text{Min.}V\left\{\widehat{M}_g^{(h)}\right\} = \frac{1}{3}V\left(\widehat{M}_y\right)\rho_1^2 > 0, \tag{4.1}$$

$$V\left(M_y\right) - \text{Min.}V\left\{\widehat{M}_g^{(f)}\right\} = V\left(\widehat{M}_y\right)\rho_0^2 > 0, \tag{4.2}$$

$$V\left(M_y\right) - \text{Min.}V\left\{\widehat{M}_g^{(t)}\right\} = \frac{1}{3}V\left(\widehat{M}_y\right)\rho_2^2 > 0, \tag{4.3}$$

$$\text{Min.}\left\{\widehat{M}_g^{(h)}\right\} - \text{Min.}V\left\{\widehat{M}_g^{(d)}\right\} = \frac{1}{6}V\left(\widehat{M}_y\right)(3\rho_0 - \rho_1)^2, \tag{4.4}$$

$$\text{Min.}\left\{\widehat{M}_g^{(h)}\right\} - \text{Min.}V\left\{\widehat{M}_g^{(g)}\right\} = \frac{1}{6}V\left(\widehat{M}_y\right)\left[3\left(3\rho_0^2+\rho_2^2-2\rho_0\rho_2\right)-2\rho_1^2\right] > 0, \tag{4.5}$$

$$\text{if } \left(3\rho_0^2+\rho_2^2-2\rho_0\rho_2\right) > \frac{2}{3}\rho_1^2, \tag{4.6}$$

$$\text{Min.}\left\{\widehat{M}_g^{(h)}\right\} - \text{Min.}V\left\{\widehat{M}_g^{(J)}\right\} = \frac{1}{24}V\left(\widehat{M}_y\right)(\rho_1-3\rho_2)^2 > 0, \tag{4.7}$$

$$\text{Min.}\left\{\widehat{M}_g^{(f)}\right\} - \text{Min.}V\left\{\widehat{M}_g^{(d)}\right\} = \frac{1}{2}V\left(\widehat{M}_y\right)(\rho_0-\rho_1)^2 > 0, \tag{4.8}$$

$$\text{Min.}\left\{\widehat{M}_g^{(f)}\right\} - \text{Min.}V\left\{\widehat{M}_g^{(g)}\right\} = \frac{1}{2}V\left(\widehat{M}_y\right)(\rho_0-\rho_2)^2 > 0, \tag{4.9}$$

$$\text{Min.}\left\{\widehat{M}_g^{(f)}\right\} - \text{Min.}V\left\{\widehat{M}_g^{(j)}\right\} = \frac{1}{8}V\left(\widehat{M}_y\right)\left[3\rho_1^2+3\rho_2^2-2\rho_1\rho_2-8\rho_0^2\right] > 0, \tag{4.10}$$

$$\text{if } \left(3\rho_1^2+3\rho_2^2-2\rho_1\rho_2\right) > 8\rho_0^2, \tag{4.11}$$

$$\text{Min.}\left\{\widehat{M}_g^{(t)}\right\} - \text{Min.}V\left\{\widehat{M}_g^{(d)}\right\} = \frac{1}{6}V\left(\widehat{M}_y\right)\left[3\left(3\rho_0^2+\rho_1^2-2\rho_0\rho_1\right)-2\rho_2^2\right] > 0, \tag{4.12}$$

$$\text{if } \left(3\rho_0^2+\rho_1^2-2\rho_0\rho_1\right) > \frac{2}{3}\rho_2^2, \tag{4.13}$$

$$\text{Min.}\left\{\widehat{M}_g^{(t)}\right\} - \text{Min.}V\left\{\widehat{M}_g^{(g)}\right\} = \frac{1}{6}V\left(\widehat{M}_y\right)(3\rho_0-\rho_2)^2 > 0, \tag{4.14}$$

$$\text{Min.}\left\{\widehat{M}_g^{(t)}\right\} - \text{Min.}V\left\{\widehat{M}_g^{(j)}\right\} = \frac{1}{24}V\left(\widehat{M}_y\right)(3\rho_1-\rho_2)^2 > 0, \tag{4.15}$$

$$\text{Min.}\left\{\widehat{M}_g^{(d)}\right\} - \text{Min.}V\left\{\widehat{M}_g^{(s)}\right\} = \frac{1}{2}V\left(\widehat{M}_y\right)(\rho_0-\rho_2)^2 > 0, \tag{4.16}$$

$$\text{Min.}\left\{\widehat{M}_g^{(g)}\right\} - \text{Min.}V\left\{\widehat{M}_g^{(s)}\right\} = \frac{1}{2}V\left(\widehat{M}_y\right)(\rho_0-\rho_1)^2 > 0, \tag{4.17}$$

$$\text{Min.}\left\{\widehat{M}_g^{(j)}\right\} - \text{Min.}V\left\{\widehat{M}_g^{(s)}\right\} = \frac{1}{8}V\left(\widehat{M}_y\right)(4\rho_0-\rho_1-\rho_2)^2 > 0, \tag{4.18}$$

From (4.1)–(4.4), (4.7)–(4.9) and (4.14)–(4.18) we have the following inequalities:

$$\text{Min.}\left\{\widehat{M}_g^{(s)}\right\} \le \text{Min.}V\left\{\widehat{M}_g^{(d)}\right\} \le \text{Min.}\left\{\widehat{M}_g^{(h)}\right\} \le \text{Min.}V\left\{\widehat{M}_y\right\}, \tag{4.19}$$

$$\text{Min.}\left\{\widehat{M}_g^{(s)}\right\} \le \text{Min.}V\left\{\widehat{M}_g^{(j)}\right\} \le \text{Min.}\left\{\widehat{M}_g^{(h)}\right\} \le \text{Min.}V\left\{\widehat{M}_y\right\}, \tag{4.20}$$

$$\text{Min.}\left\{\widehat{M}_g^{(s)}\right\} \le \text{Min.}V\left\{\widehat{M}_g^{(d)}\right\} \le \text{Min.}\left\{\widehat{M}_g^{(f)}\right\} \le \text{Min.}V\left\{\widehat{M}_y\right\}, \tag{4.21}$$

$$\text{Min.}\left\{\widehat{M}_g^{(s)}\right\} \le \text{Min.}V\left\{\widehat{M}_g^{(g)}\right\} \le \text{Min.}\left\{\widehat{M}_g^{(f)}\right\} \le \text{Min.}V\left\{\widehat{M}_y\right\}, \tag{4.22}$$

$$\text{Min.}\left\{\widehat{M}_g^{(s)}\right\} \le \text{Min.}V\left\{\widehat{M}_g^{(g)}\right\} \le \text{Min.}\left\{\widehat{M}_g^{(t)}\right\} \le \text{Min.}V\left\{\widehat{M}_y\right\}, \tag{4.23}$$

$$\text{Min.}\left\{\widehat{M}_g^{(s)}\right\} \le \text{Min.}V\left\{\widehat{M}_g^{(j)}\right\} \le \text{Min.}\left\{\widehat{M}_g^{(t)}\right\} \le \text{Min.}V\left\{\widehat{M}_y\right\}. \tag{4.24}$$

Thus, we see that the class of estimators $\widehat{M}_g^{(s)}$ is the best (in the sense of having least variance) among all discussed estimators here.

5. Empirical Study

In order to study the gain in magnitude of the relative efficiency of the class of estimators $\widehat{M}_g^{(s)}$ making use of three known quartiles Q_{1x}, Q_{2x} and Q_{3x} with respect to the other six classes of estimators $\widehat{M}_g^{(h)}$

, $\widehat{M}_g^{(f)}$, $\widehat{M}_g^{(t)}$, $\widehat{M}_g^{(d)}$, $\widehat{M}_g^{(g)}$ and $\widehat{M}_g^{(j)}$, we calculate the percent relative efficiencies. They are:

$$\text{RE}(h, s) = \frac{2\left(3-\rho_1^2\right)}{6\left(1-\rho_0^2\right)-3(\rho_0-\rho_1)^2-3(\rho_0-\rho_2)^2} \times 100, \tag{5.1}$$

$$\text{RE}(f, s) = \frac{2\left(1-\rho_0^2\right)}{2\left(1-\rho_0^2\right)-(\rho_0-\rho_1)^2-(\rho_0-\rho_2)^2} \times 100, \tag{5.2}$$

$$\text{RE}(t, s) = \frac{2\left(3-\rho_2^2\right)}{6\left(1-\rho_0^2\right)-3(\rho_0-\rho_1)^2-3(\rho_0-\rho_2)^2} \times 100, \tag{5.3}$$

$$\text{RE}(d, s) = \frac{\left(2-3\rho_0^2-\rho_1^2+2\rho_0\rho_1\right)}{2\left(1-\rho_0^2\right)-(\rho_0-\rho_1)^2-(\rho_0-\rho_2)^2} \times 100, \tag{5.4}$$

$$\text{RE}(g, s) = \frac{\left(2-3\rho_0^2-\rho_2^2+2\rho_0\rho_2\right)}{2\left(1-\rho_0^2\right)-(\rho_0-\rho_1)^2-(\rho_0-\rho_2)^2} \times 100, \tag{5.5}$$

$$\text{RE}(j, s) = \frac{8-\left(3\rho_1^2+3\rho_2^2-2\rho_1\rho_2\right)}{8\left(1-\rho_0^2\right)-4(\rho_0-\rho_1)^2-4(\rho_0-\rho_2)^2} \times 100, \tag{5.6}$$

It is noticed that the relative efficiency expressions (5.1) to (5.6) depend only upon three unknown parameters ρ_0, ρ_1 and ρ_2. The values of ρ_j, $j = 0, 1, 2$ behave as correlation coefficient between -1 to $+1$ as shown earlier. To study the pattern of relative efficiencies, we have considered all possible combinations $\rho_j = 0.1, 0.3, 0.5, 0.7, 0.9$ for $j = 0, 1, 2$. Out of the total 125 possible combinations, Table 3 shows that class of estimators $\widehat{M}_g^{(S)}$, which makes use of three known quartiles, is 111% times remains at least as better as the other six classes of the estimators based on the use of one or two known quartiles. The class $\widehat{M}_g^{(S)}$ shows relative efficiency 100%. In Table 3, two or more values of ρ_j, $j = 0, 1, 2$ are equivalent. For example, if $\rho_0 = \rho_1$, the relative efficiency RE(g, s) is 100% irrespective of the value of ρ_2 . Remember that the class of estimators $\widehat{M}_g^{(g)}$ depends upon the second and third quartiles. The relation $\rho_0 = \rho_1$ indicates that $P(Q_{2x}, M_y) = 2P(Q_{1x}, M_y)$ which requires the use of known second quartile Q_{2x} and the first quartile Q_{1x}. In other words, the use of the first quartile Q_{1x}, in addition to Q_{2x} and Q_{3x}, in the class $\widehat{M}_g^{(S)}$ is not gaining anything over $\widehat{M}_g^{(g)}$ under such situations. In practical situations, the values of ρ_j, $j = 0, 1, 2$ may be different. Thus, we observe that in most of the practical situations the values ρ_j, $j = 0, 1, 2$ are different. The use of three known quartiles shows moderate gain over the other situations discussed in the present investigation.

The results of empirical study are also depicted through graphical representation for better understanding.

6. Simulation Study

In this section, we consider a random number based true simulation study while comparing the proposed five ratio type estimators of median from the bias and the simulated mean squared errors points of view. We generated different populations of size $N = 2001$ units as follows. We used the IMSL Subroutine RNGAM(NP, AX, XP) to generate random variables from the gamma distribution with the shape parameter $\alpha = 3.5$ and then scaled then with $\beta = 2.0$ by using the IMSL Subroutine SSCAL(NP, BX, XP, 1). Thus, we have obtained a random variable $x_i \sim \text{Gamma}(\alpha, \beta)$, $i = 1, 2, \ldots, N$.

Table 3: Expected gains with the use of three known quartiles over one or two known quartiles

ρ_0	ρ_1	ρ_2	RE(h,s)	RE(f,s)	RE(t,s)	RE(d,s)	RE(g,s)	RE(j,s)
0.1	0.1	0.1	100.67	100.00	100.67	100.00	100.00	100.51
		0.3	102.75	102.06	100.00	102.06	100.00	100.00
		0.5	109.52	108.79	100.73	108.79	100.00	100.55
		0.7	123.05	122.22	103.29	122.22	100.00	102.47
		0.9	148.76	147.76	108.96	147.76	100.00	106.72
	0.3	0.1	100.00	102.06	102.75	100.00	102.06	100.00
		0.3	102.11	104.21	102.11	102.11	102.11	100.53
		0.5	108.99	111.24	103.00	108.99	102.25	102.25
		0.7	122.78	125.32	105.91	122.78	102.53	105.70
		0.9	149.23	152.31	112.31	149.23	103.08	112.31
	0.5	0.1	100.73	108.79	109.52	100.00	108.79	100.55
		0.3	103.00	111.24	108.99	102.25	108.99	102.25
		0.5	110.44	119.28	110.44	109.64	109.64	105.42
		0.7	125.57	135.62	114.61	124.66	110.96	110.96
		0.9	155.37	167.80	123.73	154.24	113.56	121.19
	0.7	0.1	103.29	122.22	123.05	100.00	122.22	102.47
		0.3	105.91	125.32	122.78	102.53	122.78	105.70
		0.5	114.61	135.62	125.57	110.96	124.66	110.96
		0.7	132.80	157.14	132.80	128.57	128.57	119.84
		0.9	170.75	202.04	148.98	165.31	136.73	136.73
	0.9	0.1	108.96	147.76	148.76	100.00	147.76	106.72
		0.3	112.31	152.31	149.23	103.08	149.23	112.31
		0.5	123.73	167.80	155.37	113.56	154.24	121.19
		0.7	148.98	202.04	170.75	136.73	165.31	136.73
		0.9	208.57	282.86	208.57	191.43	191.43	170.00
0.3	0.1	0.1	114.56	104.60	114.56	102.30	102.30	114.37
		0.3	111.99	102.25	108.99	100.00	102.25	108.99
		0.5	114.56	104.60	105.36	102.30	102.30	105.17
		0.7	123.05	112.35	103.29	109.88	102.47	102.47
		0.9	140.38	128.17	102.82	125.35	102.82	100.70
	0.3	0.1	108.99	102.25	111.99	102.25	100.00	108.99
		0.3	106.59	100.00	106.59	100.00	100.00	104.95
		0.5	108.99	102.25	103.00	102.25	100.00	102.25
		0.7	116.87	109.64	100.80	109.64	100.00	100.60
		0.9	132.88	124.66	100.00	124.66	100.00	100.00
	0.5	0.1	105.36	104.60	114.56	102.30	102.30	105.17
		0.3	103.00	102.25	108.99	100.00	102.25	102.25
		0.5	105.36	104.60	105.36	102.30	102.30	100.57
		0.7	113.17	112.35	103.29	109.88	102.47	100.00
		0.9	129.11	128.17	102.82	125.35	102.82	100.70
	0.7	0.1	103.29	112.35	123.05	102.47	109.88	102.47
		0.3	100.80	109.64	116.87	100.00	109.64	100.60
		0.5	103.29	112.35	113.17	102.47	109.88	100.00
		0.7	111.56	121.33	111.56	110.67	110.67	100.67
		0.9	128.72	140.00	112.31	127.69	112.31	103.08
	0.9	0.1	102.82	128.17	140.38	102.82	125.35	100.70
		0.3	100.00	124.66	132.88	100.00	124.66	100.00
		0.5	102.82	128.17	129.11	102.82	125.35	100.70
		0.7	112.31	140.00	128.72	112.31	127.69	103.08
		0.9	132.73	165.45	132.73	132.73	132.73	108.18

Then we generated a random variable $z_i \sim N(0,1)$, for $i = 1, 2, \ldots, N$, by using the ISML Subroutine RNNOR(NP,Z). Using these random variables, we generate populations from two models as:

$$\text{Quadratic Model: } y_i = 2 - 0.6x_i + 0.8x_i^2 + z_i x_i^g \tag{6.1}$$

ρ_0	ρ_1	ρ_2	RE(h,s)	RE(f,s)	RE(t,s)	RE(d,s)	RE(g,s)	RE(j,s)
0.5	0.1	0.1	168.93	127.12	168.93	113.56	113.56	168.64
		0.3	153.33	115.38	149.23	103.08	112.31	149.23
		0.5	148.76	111.94	136.82	100.00	111.94	136.57
		0.7	153.33	115.38	128.72	103.08	112.31	127.69
		0.9	168.93	127.12	123.73	113.56	113.56	121.19
	0.3	0.1	149.23	115.38	153.33	112.31	103.08	149.23
		0.3	136.62	105.63	136.62	102.82	102.82	134.51
		0.5	132.88	102.74	125.57	100.00	102.74	124.66
		0.7	136.62	105.63	117.84	102.82	102.82	117.61
		0.9	149.23	115.38	112.31	112.31	103.08	112.31
	0.5	0.1	136.82	111.94	148.76	111.94	100.00	136.57
		0.3	125.57	102.74	132.88	102.74	100.00	124.66
		0.5	122.22	100.00	122.22	100.00	100.00	116.67
		0.7	125.57	102.74	114.61	102.74	100.00	110.96
		0.9	136.82	111.94	108.96	111.94	100.00	106.72
	0.7	0.1	128.72	115.38	153.33	112.31	103.08	127.69
		0.3	117.84	105.63	136.62	102.82	102.82	117.61
		0.5	114.61	102.74	125.57	100.00	102.74	110.96
		0.7	117.84	105.63	117.84	102.82	102.82	106.34
		0.9	128.72	115.38	112.31	112.31	103.08	103.08
	0.9	0.1	123.73	127.12	168.93	113.56	113.56	121.19
		0.3	112.31	115.38	149.23	103.08	112.31	112.31
		0.5	108.96	111.94	136.82	100.00	111.94	106.72
		0.7	112.31	115.38	128.72	103.08	112.31	103.08
		0.9	123.73	127.12	123.73	113.56	113.56	100.85
0.7	0.1	0.1	664.44	340.00	664.44	220.00	220.00	663.33
		0.3	398.67	204.00	388.00	132.00	172.00	388.00
		0.5	321.51	164.52	295.70	106.45	158.06	295.16
		0.7	302.02	154.55	253.54	100.00	154.55	251.52
		0.9	321.51	164.52	235.48	106.45	158.06	230.65
	0.3	0.1	388.00	204.00	398.67	172.00	132.00	388.00
		0.3	277.14	145.71	277.14	122.86	122.86	272.86
		0.5	236.59	124.39	223.58	104.88	119.51	221.95
		0.7	225.58	118.60	194.57	100.00	118.60	194.19
		0.9	236.59	124.39	178.05	104.88	119.51	178.05
	0.5	0.1	295.70	164.52	321.51	158.06	106.45	295.16
		0.3	223.58	124.39	236.59	119.51	104.88	221.95
		0.5	195.04	108.51	195.04	104.26	104.26	186.17
		0.7	187.07	104.08	170.75	100.00	104.08	165.31
		0.9	195.04	108.51	155.32	104.26	104.26	152.13
	0.7	0.1	253.54	154.55	302.02	154.55	100.00	251.52
		0.3	194.57	118.60	225.58	118.60	100.00	194.19
		0.5	170.75	104.08	187.07	104.08	100.00	165.31
		0.7	164.05	100.00	164.05	100.00	100.00	148.04
		0.9	170.75	104.08	148.98	104.08	100.00	136.73
	0.9	0.1	235.48	164.52	321.51	158.06	106.45	230.65
		0.3	178.05	124.39	236.59	119.51	104.88	178.05
		0.5	155.32	108.51	195.04	104.26	104.26	152.13
		0.7	148.98	104.08	170.75	100.00	104.08	136.73
		0.9	155.32	108.51	155.32	104.26	104.26	126.60

and

$$\text{Cubic Model: } y_i = 2 + 0.9x_i - 0.8x_i^2 + 0.6x_i^3 + z_i x_i^g \quad (6.2)$$

with $g = 0.5, 1.0, 1.5, 2.0$.

ρ_0	ρ_1	ρ_2	RE(h,s)	RE(f,s)	RE(t,s)	RE(d,s)	RE(g,s)	RE(j,s)
	0.3	0.9	9699.92	1899.99	7299.94	100.00	1899.99	7299.94
		0.5	3055.55	633.33	3055.55	366.67	366.67	2916.66
	0.5	0.7	1018.52	211.11	929.63	122.22	188.89	900.00
		0.9	833.33	172.73	663.64	100.00	172.73	650.00
		0.5	929.63	211.11	1018.52	188.89	122.22	900.00
0.9	0.7	0.7	557.78	126.67	557.78	113.33	113.33	503.33
		0.9	492.16	111.76	429.41	100.00	111.76	394.12
		0.3	7299.94	1899.99	9699.92	1899.99	100.00	7299.94
	0.9	0.5	663.64	172.73	833.33	172.73	100.00	650.00
		0.7	429.41	111.76	492.16	111.76	100.00	394.12
		0.9	384.21	100.00	384.21	100.00	100.00	313.16

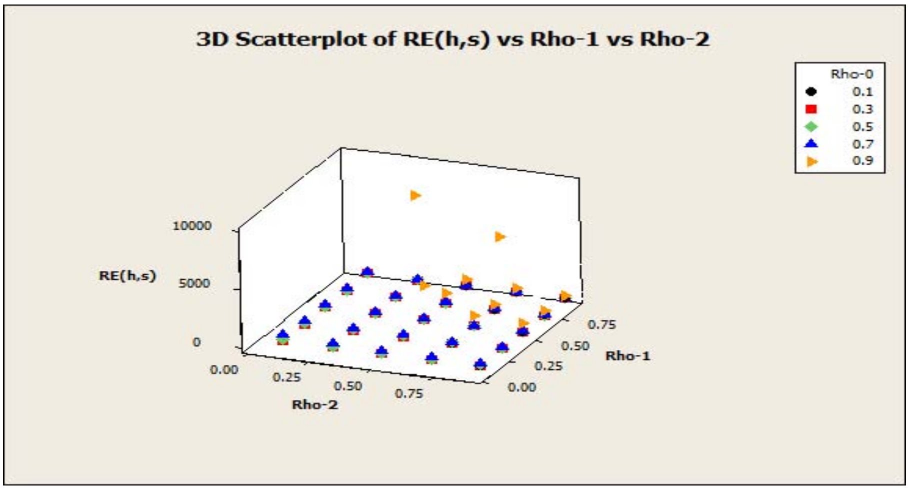

Figure 1: *The values of RE* (h, s) *for different values of* ρ_0*(Rho-0),* ρ_1*(Rho-1) and* ρ_2*(Rho-2).*

Then we used the ISML Subroutine ORDST to find the population medians of the study and auxiliary variables M_y and M_x respectively. We also computed the population first and third quartiles from the same subroutine. From a given population of $N = 2001$ units, we selected NITR=100,000 samples each of size n (as listed in the table), and computed the three different sample quartiles for the study and auxiliary variables. From the k^{th} sample, we computed seven different estimators:

$$\left.\widehat{M}_R^{(-1)}\right|_k = \widehat{M}_y, \quad \left.\widehat{M}_R^{(0)}\right|_k = \widehat{M}_y v^{-1}, \quad \left.\widehat{M}_R^{(1)}\right|_k = \hat{M}_y u^{-1}, \quad \left.\widehat{M}_R^{(2)}\right|_k = \hat{M}_y w^{-1},$$
$$\left.\widehat{M}_R^{(3)}\right|_k = \widehat{M}_y (uv)^{-1}, \quad \left.\widehat{M}_R^{(4)}\right|_k = \widehat{M}_y (vw)^{-1} \text{ and } \left.\widehat{M}_R^{(5)}\right|_k = \widehat{M}_y (uvw)^{-1}$$

for $k = 1, 2, \ldots,$ NITR. Note that the last six estimators are defined in (1.1), and (2.1) to (2.5), but are redefined here for the convenience of readers.

Then, we computed the percent empirical relative bias in the above seven estimators as:

$$\text{RB}\left(\widehat{M}_R^{(\text{est})}\right) = \frac{\left\{1/\text{NITR} \sum_{k=1}^{\text{NITR}} \left. M_R^{(\text{est})}\right|_k - M_y\right\} \times 100\%}{M_y} \tag{6.3}$$

for est $= -1, 0, 1, 2, 3, 4, 5.$

Table 4: Values of the population quartiles while doing simulation.

Model	g	Q_{1y}	Q_{2y}	Q_{3y}	Q_{1x}	Q_{2x}	Q_{3x}
Quadratic	0.5	13.87	30.78	60.34	4.24	6.39	8.94
	1	13.46	30.55	59.79	4.24	6.39	8.94
	1.5	10.68	28.11	61.02	4.24	6.39	8.94
	2	1.41	21.31	69.20	4.24	6.39	8.94
Cubic	0.5	36.79	132.39	374.69	4.24	6.39	8.94
	1	37.19	131.67	375.84	4.24	6.39	8.94
	1.5	37.28	131.05	372.92	4.24	6.39	8.94
	2	34.14	128.12	364.09	4.24	6.39	8.94

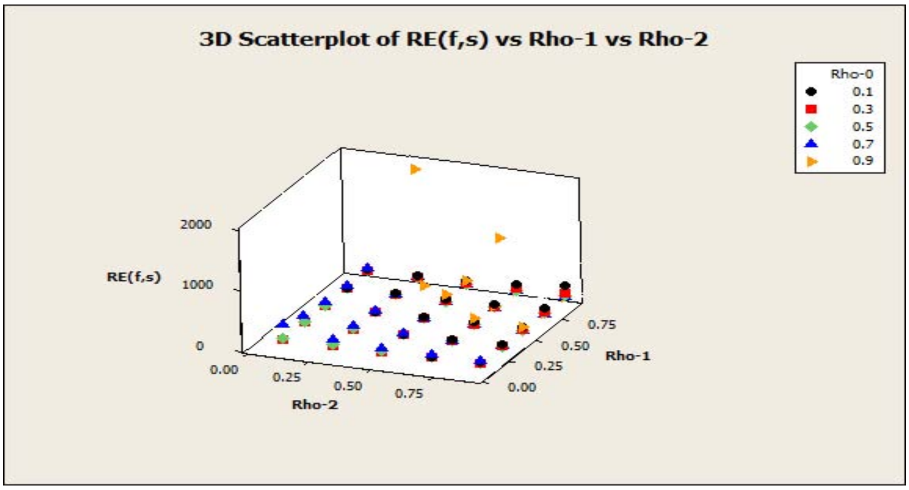

Figure 2: *The values of* $RE(f,s)$ *for different values of* ρ_0*(Rho-0),* ρ_1*(Rho-1) and* ρ_2*(Rho-2).*

Also we computed the percent empirical relative efficiency of the last six ratio type estimators with respect to the sample median estimator $\widehat{M}_y$ as:

$$\text{RB}\left(\widehat{M}_R^{(\text{est})}\right) = \frac{\sum_{k=1}^{\text{NITR}}\left[M_R^{(-1)}\big|_k - M_y\right]^2}{\sum_{k=1}^{\text{NITR}}\left[M_R^{(est)}\big|_k - M_y\right]^2} \times 100\%, \quad \text{for est} = 0,1,2,3,4,5. \tag{6.4}$$

The FORTRAN code used in the simulation study is available on request from the authors and also made available online. The results obtained from the simulation study are given in Table 4 and Table 6. Table 4 shows the values of the three population quartiles of the study variable and auxiliary variable. Note that the values of the three population quartiles(Q_{1x}, Q_{2x} and Q_{3x}) of the auxiliary variable remain the same, because we generated the auxiliary variable as $x_i \sim \text{Gamma}(\alpha, \beta)$. The values of the three population quartiles(Q_{1y}, Q_{2y} and Q_{3y}) for the study variable change for every value of g and type of the model used.

Table 5 has been devoted to investigating the empirical percent relative bias in the seven estimators. In it interesting to note that for the situations considered in the study, the absolute value of empirical percent relative bias always remains much less than 10% which is quite acceptable by following Cochran (1963). From Table 5, one can conclude that the empirical percent relative bias is negligible in the estimators considered.

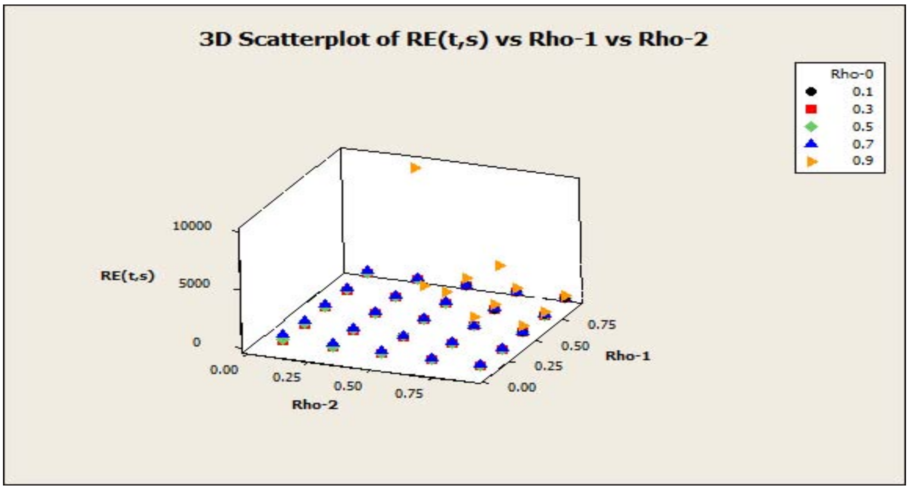

Figure 3: *The values of RE(t, s) for different values of ρ_0(Rho-0), ρ_1(Rho-1) and ρ_2(Rho-2).*

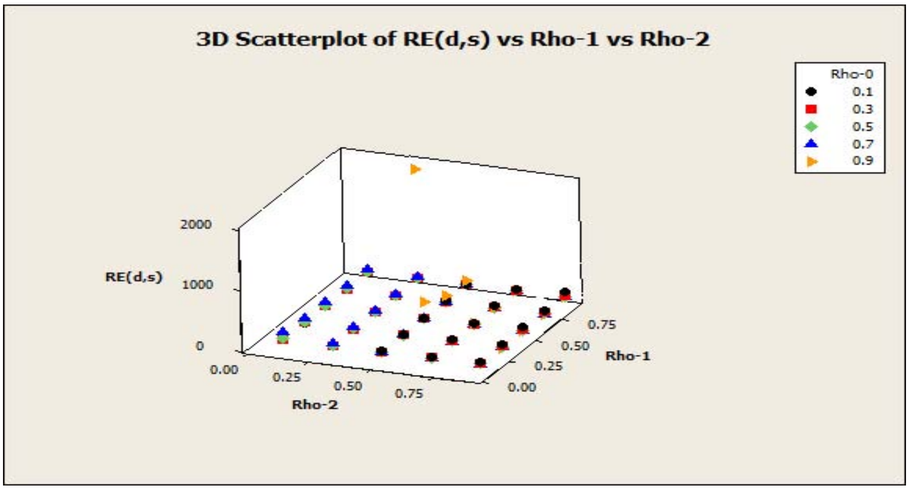

Figure 4: *The values of RE(d, s) for different values of ρ_0(Rho-0), ρ_1(Rho-1) and ρ_2(Rho-2).*

Table 6 shows the empirical percent relative efficiencies of the six estimators $\widehat{M}_R^{(\text{est})}$, est $= 0, 1, 2, 3, 4, 5$ with respect to the control estimator $\widehat{M}_R^{(-1)}$. For the quadratic model, it is interesting to note that the estimator $\widehat{M}_R^{(4)}$ remains more efficient than the other competitors. No doubt the ratio estimator $\widehat{M}_R^{(5)}$, which makes use of all three of the known quartiles also remains less efficient than $\widehat{M}_R^{(4)}$. This shows that, for the case of a quadratic model, the use of the known third quartile of an auxiliary character may not help while making ratio type estimator. It is also to be noted that the estimator $\widehat{M}_R^{(5)}$ remains always more efficient than the other competitors considered in the case of the cubic model. This simulation study also shows that the $\widehat{M}_R^{(5)}$ estimator will be more efficient under different conditions, and $\widehat{M}_R^{(4)}$ will be more efficient under different conditions as derived in section 2 for different estimators.

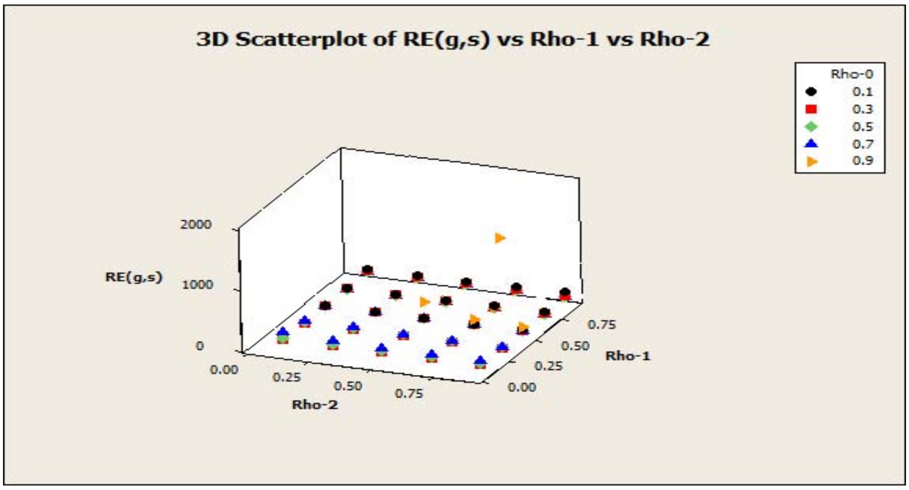

Figure 5: *The values of RE(g, s) for different values of ρ_0(Rho-0), ρ_1(Rho-1) and ρ_2(Rho-2).*

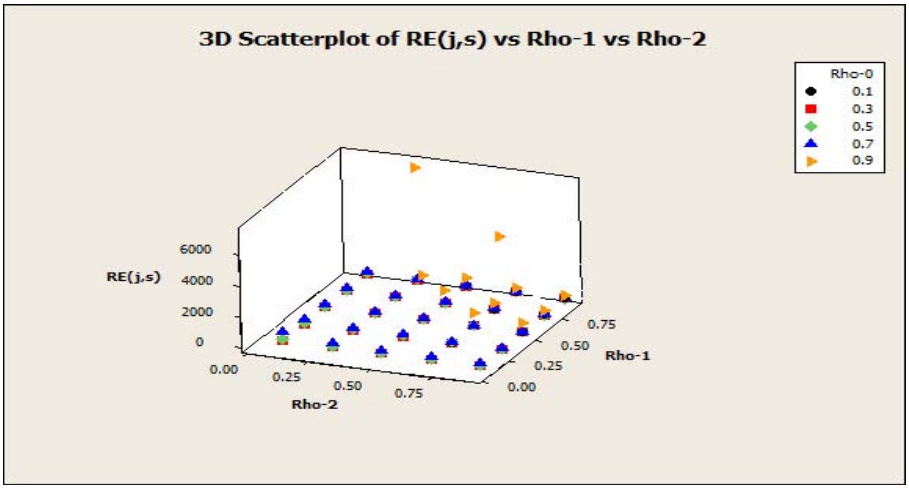

Figure 6: *The values of RE(j, s) for different values of ρ_0(Rho-0), ρ_1(Rho-1) and ρ_2(Rho-2).*

Although Rueda and Arcos (2002) have shown that a small amount of efficiency can be gained by applying ratio estimates to the construction of intervals, but the present simulation study shows a quite significant gain in relative efficiency. This indicates that the proper use of auxiliary information may also lead to significant improvements in the Rueda and Arcos (2002) simulation results. Thus based on our simulation study, we conclude that the empirical relative bias in the ratio type estimators of median is negligible and the percent relative efficiencies show that the use of known quartiles of the auxiliary variable can be used to improve the estimation of the population median of the study variable.

Table 5: Simulated percent relative bias: RB($\widehat{M}_R^{(est)}$).

Model	g	n	est						
			−1	0	1	2	3	4	5
Quadratic	0.5	51	1.6159	0.5517	1.6827	0.3098	1.2639	−0.2337	0.8213
		61	1.4355	0.5337	3.0702	1.5546	2.6877	1.0856	3.5386
		71	1.1467	0.4236	1.1737	0.2057	0.9190	−0.1441	0.5975
		81	1.0760	0.4204	2.2778	1.1263	2.0187	0.7947	2.6207
		91	0.9461	0.3945	0.9538	0.1970	0.7663	−0.0614	0.5044
		101	0.9580	0.4122	1.9475	0.9975	1.7237	0.7138	2.2082
	1	51	1.6722	0.7705	1.7534	0.3719	1.5021	−0.0083	1.0704
		61	1.4593	0.6968	3.1061	1.5859	2.8704	1.2601	3.7349
		71	1.2148	0.6122	1.2561	0.2782	1.1280	0.0539	0.8165
		81	1.1323	0.5860	2.3495	1.1904	2.2072	0.9731	2.8233
		91	0.9965	0.5422	1.0153	0.2552	0.9318	0.0978	0.6820
		101	1.0009	0.5433	2.0028	1.0482	1.8737	0.8592	2.3711
	1.5	51	2.3634	1.6810	2.3351	1.0354	2.3190	0.8770	1.8613
		61	2.0119	1.4324	3.5728	2.1310	3.5351	1.9943	4.3988
		71	1.7391	1.3047	1.7084	0.7934	1.7559	0.7382	1.4389
		81	1.6499	1.2492	2.8077	1.7004	2.8207	1.6337	3.4336
		91	1.4135	1.0995	1.3851	0.6657	1.4443	0.6482	1.1901
		101	1.4438	1.1047	2.3973	1.4899	2.3923	1.4215	2.8940
	2	51	2.2162	1.6562	1.7264	1.0008	1.8269	0.9571	1.4800
		61	1.6953	1.2297	2.8525	1.9204	2.9239	1.8899	3.8868
		71	1.1972	0.8598	0.8457	0.3380	0.9837	0.3701	0.7440
		81	1.0097	0.6827	1.8528	1.1290	1.9304	1.1259	2.6019
		91	0.7559	0.5206	0.4752	0.0783	0.6040	0.1299	0.4113
		101	0.7163	0.4394	1.4072	0.8231	1.4575	0.8077	2.0073
Cubic	0.5	51	3.0682	0.8289	2.3933	1.1479	0.8061	−0.5370	−0.2161
		61	2.4980	0.6052	3.5278	2.1049	2.1512	0.6629	2.4946
		71	1.8859	0.3071	1.3734	0.5051	0.2716	−0.6764	−0.4650
		81	1.6311	0.2297	2.3693	1.2963	1.3666	0.2342	1.5906
		91	1.2812	0.0628	0.8708	0.1918	0.0258	−0.7165	−0.5588
		101	1.2165	0.0712	1.8294	0.9463	1.0110	0.0770	1.1917
	1	51	3.6835	1.4597	3.0134	1.7605	1.4378	0.0841	0.4089
		61	3.1136	1.2385	4.1592	2.7272	2.7949	1.2967	3.1402
		71	2.5016	0.9403	1.9928	1.1218	0.9055	−0.0498	0.1632
		81	2.2411	0.8546	2.9913	1.9107	1.9969	0.8580	2.2207
		91	1.9003	0.6972	1.4922	0.8104	0.6559	−0.0902	0.0647
		101	1.8220	0.6894	2.4437	1.5559	1.6327	0.6913	1.8129
	1.5	51	4.1207	1.9803	3.4593	2.2000	1.9685	0.6049	0.9395
		61	3.5768	1.7754	4.6414	3.1994	3.3488	1.8387	3.7007
		71	2.9435	1.4454	2.4396	1.5657	1.4101	0.4524	0.6667
		81	2.6813	1.3508	3.4432	2.3586	2.5028	1.3554	2.7278
		91	2.3766	1.2196	1.9711	1.2873	1.1787	0.4287	0.5832
		101	2.3033	1.2124	2.9327	2.0418	2.1594	1.2157	2.3378
	2	51	4.1564	2.2398	3.4965	2.2317	2.2412	0.8657	1.2136
		61	3.5446	1.9364	4.6118	3.1674	3.5263	2.0064	3.8870
		71	2.9656	1.6358	2.4716	1.5865	1.6208	0.6479	0.8828
		81	2.6707	1.4908	3.4409	2.3503	2.6642	1.5067	2.9029
		91	2.3382	1.3158	1.9391	1.2530	1.2915	0.5369	0.7077
		101	2.2513	1.2866	2.8894	1.9962	2.2503	1.3029	2.4450

7. Conclusion

In this article, we have suggested a general procedure for estimating the population median of the study variable Y in the presence of three known quartiles of an auxiliary variable X. An asymptotically optimum estimator(AOE), in the proposed class of estimators, is obtained along with its mean squared

Table 6: Simulated percent relative efficiencies: RB($\widehat{M}_R^{(est)}$).

Model	g	n	est					
			0	1	2	3	4	5
Quadratic	0.5	51	362.43	141.96	154.38	318.85	453.61	229.35
		61	359.49	131.89	150.07	288.57	441.63	204.80
		71	355.56	144.96	152.17	326.80	442.29	235.18
		81	351.58	136.55	149.41	299.47	432.56	212.34
		91	348.45	146.22	151.35	332.01	435.85	238.65
		101	346.86	138.99	149.68	307.03	434.31	220.28
	1	51	289.18	141.83	155.19	251.10	342.96	189.48
		61	288.97	131.78	150.85	230.48	332.93	169.48
		71	287.83	144.47	153.24	256.30	337.82	191.88
		81	287.12	135.83	150.27	236.55	328.60	172.49
		91	285.23	146.12	152.17	261.07	333.08	193.17
		101	285.86	138.72	150.41	243.32	330.28	177.89
	1.5	51	180.33	141.25	144.38	178.04	202.25	152.31
		61	180.23	132.13	138.97	166.75	192.96	136.55
		71	179.86	143.51	142.57	180.04	198.36	151.70
		81	178.48	135.19	139.39	168.38	191.63	137.98
		91	178.28	144.33	142.29	180.01	195.81	150.52
		101	179.75	138.56	139.48	173.75	192.82	141.39
	2	51	116.93	121.23	112.05	125.56	118.94	118.58
		61	116.07	116.14	108.07	120.14	114.05	109.82
		71	115.44	120.23	110.43	123.66	116.29	116.23
		81	114.94	116.83	108.03	120.42	113.54	111.06
		91	114.12	119.72	109.51	122.52	114.46	115.02
		101	114.71	118.03	107.80	121.65	113.24	112.23
Cubic	0.5	51	229.10	139.51	143.04	327.64	348.43	436.67
		61	227.59	131.89	138.36	306.49	338.94	398.79
		71	225.47	140.40	140.45	327.56	337.48	433.01
		81	224.16	134.63	137.59	311.31	332.14	404.26
		91	222.89	140.35	139.05	325.76	331.40	428.85
		101	222.41	135.99	137.17	313.64	329.76	410.12
	1	51	227.05	139.37	143.18	322.60	344.88	429.60
		61	225.41	131.42	138.22	299.67	333.36	386.02
		71	223.24	140.37	140.64	322.51	333.79	426.10
		81	221.47	134.12	137.39	303.56	325.63	390.45
		91	220.13	140.38	139.26	320.28	327.05	422.18
		101	219.41	135.42	136.99	305.11	322.75	395.22
	1.5	51	219.48	139.17	143.25	305.75	328.66	400.55
		61	217.64	130.97	138.07	282.49	315.78	355.88
		71	215.17	140.28	140.68	305.16	317.22	397.64
		81	213.30	133.72	137.21	285.68	307.74	360.16
		91	211.64	140.24	139.27	302.23	309.82	393.29
		101	210.87	134.98	136.76	286.84	304.41	364.91
	2	51	200.14	139.53	143.85	268.53	290.44	341.71
		61	199.13	131.48	138.76	250.11	279.80	305.74
		71	197.55	140.53	141.50	269.19	283.08	339.95
		81	196.61	134.02	138.00	253.62	275.12	309.11
		91	195.19	140.79	139.92	268.46	276.98	336.42
		101	195.20	135.76	137.52	256.73	273.17	314.51

error formula. It has been shown that the proposed class of estimators is more efficient than the usual unbiased estimator $\widehat{M}_y$ envisaged by Gross (1980), ratio estimator $\widehat{M}_R^{(0)}$ due to Kuk and Mak (1989) and other estimators of the population median M_y. An empirical study has been carried out in support of the present study. The empirical as well as simulation study supports the theoretical development of the use of the suggested class of estimators in estimating the median in survey sampling.

Acknowledgement

The authors are thankful to the Editor Prof. Sung Keuk Ahn, and two learned referees for very valuable comments on the original manuscript. The authors are also thankful to Professor Stephen A. Sedory, Department of Mathematics, Texas A&M University-Kingsville for his help in editing the revised manuscript.

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Received October 17, 2013; Revised August 13, 2014; Accepted September 4, 2014