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Orbital Elements Evolution Due to a Perturbing Body in an Inclined Elliptical Orbit

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This paper intends to highlight the effect of the third-body in an inclined orbit on a spacecraft orbiting the primary mass. To achieve this goal, a new origin of coordinate is introduced in the primary and the X-axis toward the node of the spacecraft. The disturbing function is expanded up to the second order using Legendre polynomials. A double-averaged analytical model is exploited to produce the evolutions of mean orbital elements as smooth curves.

Keywords: Third-body perturbations; averaged model; long period perturbations

1. INTRODUCTION

The effect of the third-body perturbation is one of the most important mechanisms in delivering the major Earth orbiting objects into the regions where the atmosphere starts to decay. Several papers have studied the third-body perturbation of the Sun and the Moon on a satellite orbiting the Earth and major related works are listed as follows. In a series of papers (Kozai 1959, 1962, 1965), Kozai developed the secular and long period terms of the disturbing function due to lunisolar perturbations in terms of the orbital elements of a satellite, the Sun and the Moon, and applied those secular perturbations to asteroids with high inclination and eccentricity assuming that Jupiter is the perturbing body. Blitzer (1959) obtained the estimates of secular terms due to lunisolar disturbances. In the sequence, Cook (1962) used the Lagrange's planetary equations to obtain expressions for the variations of elements during the revolution of a satellite and for the rates of variation of the elements. Giacaglia (1974) obtained the disturbing function for the disturbance of the Moon using the ecliptic elements for the Moon and the equatorial elements for a satellite. Hough (1981) studied the effects of lunisolar disturbances in orbits close to critical inclinations from the point of view of the geopotential of the Earth and concluded that the effects are significant in high altitudes. Broucke (2003) and Prado & Costa (1998), truncated the disturbing function of the third body after the second and the fourth order expansion in Legendre polynomials, respectively. After that, Costa (1998) expanded the order of this model to the eighth order. Solórzano & Prado (2007) developed a semi-analytical study of the perturbation caused on a spacecraft by a third body with a single averaged model to eliminate the terms due to the short-lived periodic motion of the spacecraft. Prado (2003) studied the evolution of retrograde orbits around the Earth. The existence of circular, equatorial and frozen orbits was also considered. Costa & Prado (2002) discussed the critical angle of a third-body perturbation that is a value for the inclination such that any near-circular orbit with inclination below this value remains near- circular. Domingos et al. (2008) showed an analytical expansion to study the third-body perturbation for a case where the perturbing body is in an elliptical orbit, based on the expansion of the perturbing function in polynomials of Legendre.

In this paper, the study done by Broucke (2003), Prado (2003) and Domingos et al. (2008) is expanded, including the eccentricity and inclination of the perturbing body. A three-dimensional model is used to study the effects of a third-body gravitational potential on the spacecraft

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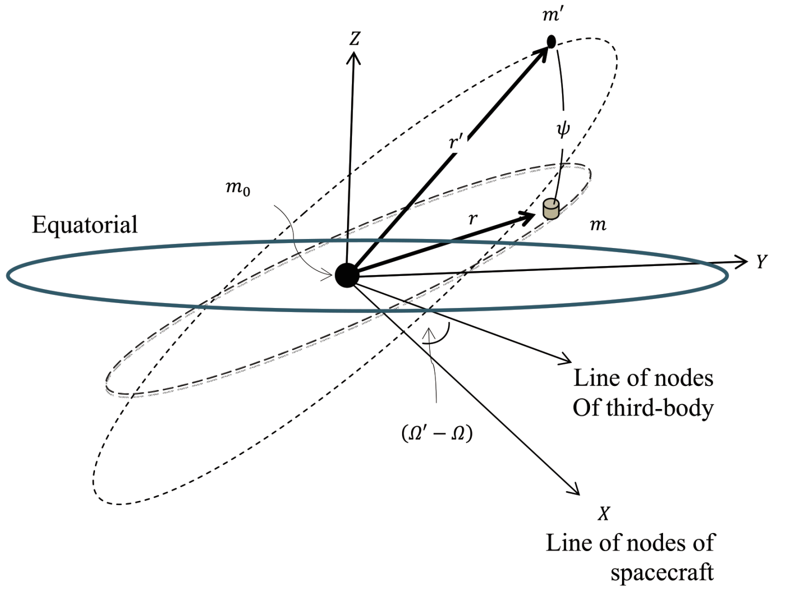

Fig. 1. Relation between the angle ψ and the orbital elements of any two objects with mass *m* and *m'*.

that orbits near a celestial body including the eccentricity and the inclination of the third-body in the formulation. There is no assumption in our study on the perturbed and perturbing orbits except the usual assumption that there are only three bodies involved in the system: the first body with mass, m_0, fixed at the origin of the reference. The double-average is taken over the mean motion of the spacecraft and that of the disturbing body after expansion of the disturbing function in Legendre polynomials.

This paper is organized as follows. In Section 2, we formulate the disturbing function (using the traditional expansion in Legendre polynomials) due to a third body truncated up to the second-order term. In Section 3, The Lagrange's planetary equations for the variations of spacecraft's orbital elements are shown. In Section 4, we simulate the study assuming a spacecraft in an elliptical inclined orbit around the Moon with its motion perturbed by the Earth.

2. ELLIPTICAL AND INCLINED DISTURBING POTENTIAL

About notations through this paper, we use the well-known Keplerian elements: the semi-major axis, a, the eccentricity, e, the inclination, i, the right ascension of ascending node, Ω, the argument of periapsis, w, f is the true anomaly and the mean anomaly, M. Also, the same notations with prime will be used for the third body. It is assumed that the main body with mass m_0 is fixed in the center of the reference system. The perturbing body, with mass m' is in an elliptic inclined orbit with mean motion n', given by the expression $n'^2a'^3=g(m_0+m')$, where r and r' are the radius vectors of the spacecraft and m' (assuming $r' \gg r_0$), and ψ is the angle between these radius vectors. The disturbing function (using the traditional expansion in Legendre polynomials; P_n) due to the third body is given by

$$R = \frac{\mu' g(m_0 + m')}{r'} \sum_{n=2}^{\infty} \left(\frac{r}{r'}\right)^n P_n(\cos(\psi)) \tag{1}$$

where $\mu' = \frac{m'}{m_0+m'}$, g is the gravitational constant and ψ is the angle between the line that connects the massive central body and the perturbed body (the spacecraft) and the line that connects the massive central body and the perturbing body (the third body).

For the models used in this research, it is necessary to calculate the parts of the disturbing function due to R_2,

$$R_2 = \frac{\mu' n'^2 a^2}{2} \left(\frac{a'}{r'}\right)^3 \left(\frac{r}{a}\right)^2 [3cos^2(\psi) - 1] \tag{2}$$

The magnitude of the effect of the disturbing body on the orbit of a spacecraft depends on the position of the disturbing body in its orbit. The next step is to express cos ψ in terms of the orbital elements. It is clear from Fig. 1 that the base vectors along $\vec{r}$ and $\vec{r}'$ may be defined as

$$\vec{r} = r\begin{pmatrix} \cos u \\ \cos i\ \sin u \\ \sin i \sin u \end{pmatrix}, \qquad \vec{r}' = r'\begin{pmatrix} \cos \Delta\Omega \cos u' - \cos i' \sin \Delta\Omega \sin u' \\ \sin \Delta\Omega \cos u' + \cos i' \cos \Delta\Omega \sin u' \\ \sin i' \sin u' \end{pmatrix},$$

where r and r' are the moduli of vectors $\vec{r}$ and $\vec{r}'$ from which

$$\begin{aligned} \cos\psi = \frac{\vec{r}.\vec{r}'}{rr'} &= \cos u\,(\cos \Delta\Omega \cos u' - \cos i' \sin \Delta\Omega \sin u') \\ &+ \cos i \sin u\,(\sin \Delta\Omega \cos u' + \cos i' \cos \Delta\Omega \sin u') \\ &+ \sin i \sin u \sin i' \sin u' \end{aligned}$$

where $u=f+\omega$ is usually known as the argument of latitude and $\Delta\Omega=\Omega'-\Omega$ is the difference of the nodal longitudes. In the subsequent developments the adopted reference frame is an equatorial system with the positive *X-axis* towards the node of the orbit of m, *Z-axis* towards the north pole of the equator of the primary, and the *Y-axis* completing a right handed system.

The last relation can be rewritten in a compact form as below:

$$\cos\psi = \sum_{i,j=-1}^{1} A_{ij} \cos\left(f+\omega + i(f'+\omega') + j(\Omega'-\Omega)\right) \tag{3}$$

where the non-vanishing coefficients are

$$\begin{aligned} A_{-1-1} &= \frac{1}{4}(1+\cos i + \cos i' + \cos i \cos i') \\ A_{-10} &= \frac{1}{2}\sin i \sin i' \\ A_{-11} &= \frac{1}{4}(1-\cos i - \cos i' + \cos i \cos i') \\ A_{1-1} &= \frac{1}{4}(1+\cos i - \cos i' - \cos i \cos i') \\ A_{10} &= -\frac{1}{2}\sin i \sin i' \\ A_{11} &= \tfrac{1}{4}(1-\cos i + \cos i' - \cos i \cos i') \end{aligned}$$

$\vec{r}$

The next step is to average expression (2), using Eq. (3), over the short period of the spacecraft as well as with respect to the distant perturbing body. The standard definition for the average can be written as:

$$\langle f \rangle = \frac{1}{2\pi}\int_0^{2\pi} f\, dM$$

This yields

$$\begin{aligned} \langle\langle R_2 \rangle\rangle = \frac{\mu' n'^2 a^2}{4\left(1-e'^2\right)^{3/2}} \Big\{ & (2+3e^2)\Big[-1 + \frac{3}{2}(A_{-1-1}^2 + A_{-10}^2 + A_{-11}^2 + A_{1-1}^2 + A_{10}^2 + A_{11}^2) \\ & + 3(A_{-1-1}A_{-10} + A_{-10}A_{-11} + A_{1-1}A_{10} + A_{10}A_{11})\cos(\Omega'-\Omega) + 3(A_{-1-1}A_{-11} + A_{1-1}A_{11})\cos\left(2(\Omega'-\Omega)\right)\Big] \\ & + 15e^2\Big[(A_{-11}A_{1-1} + A_{-10}A_{10} + A_{-1-1}A_{11})\cos(2\omega) \\ & + (A_{-1-1}A_{1-1})\cos\left(2\omega - 2(\Omega'-\Omega)\right) + (A_{-10}A_{1-1} + A_{-1-1}A_{10})\cos\left(2\omega - (\Omega'-\Omega)\right) \\ & + (A_{-11}A_{10} + A_{-10}A_{11})\cos\left(2\omega + (\Omega'-\Omega)\right) + (A_{-11}A_{11})\cos\left(2\omega + 2(\Omega'-\Omega)\right)\Big]\Big\} \end{aligned}$$

Rewriting above equation gives

$$\langle\langle R_2 \rangle\rangle = \mu' {n'}^2 a^2 \sum_{j,k=-2}^{2} \left\{\alpha_{jk}(e,e')\beta_{jk}(i,i')\ \cos\left(j\omega + k(\Omega'-\Omega)\right)\right\} \tag{4}$$

where the non-vanishing coefficients are

$$\alpha_{00} = \frac{(2+3e^2)}{4(1-e'^2)^{3/2}}$$
$$\alpha_{01} = \alpha_{02} = \frac{3(2+3e^2)}{4(1-e'^2)^{3/2}}$$
$$\alpha_{20} = \alpha_{2-2} = \alpha_{2-1} = \alpha_{21} = \alpha_{22} = \frac{15e^2(2+3e^2)}{4(1-e'^2)^{3/2}}$$
$$\beta_{00} = -1 + \frac{3}{2}(A^2_{-1-1} + A^2_{-10} + A^2_{-11} + A^2_{1-1} + A^2_{10} + A^2_{11})$$
$$\beta_{01} = A_{-1-1}A_{-10} + A_{-10}A_{-11} + A_{1-1}A_{10} + A_{10}A_{11}$$
$$\beta_{02} = A_{-1-1}A_{-11} + A_{1-1}A_{11}$$
$$\beta_{20} = A_{-11}A_{1-1} + A_{-10}A_{10} + A_{-1-1}A_{11}$$
$$\beta_{2-2} = A_{-1-1}A_{1-1}$$
$$\beta_{2-1} = A_{-10}A_{1-1} + A_{-1-1}A_{10}$$
$$\beta_{21} = A_{-11}A_{10} + A_{-10}A_{11}$$
$$\beta_{22} = A_{-11}A_{11}$$

3. LAGRANGE'S PERTURBATION EQUATIONS

Using Lagrange planetary equations, the variation of the orbital elements due to the averaging disturbing potential «R_2» is established as, e.g. Brouwer & Clemence (1961):

$$\frac{da}{dt} = 0,$$
$$\frac{de}{dt} = \frac{\mu' n'^2\sqrt{1-e^2}}{n\, e}\sum_{j,k=-2}^{2} j\{\alpha_{jk}(e,e')\beta_{jk}(i,i')\sin(j\omega + k\Delta\Omega)\},$$
$$\frac{di}{dt} = \frac{-\mu' n'^2}{n\sqrt{1-e^2}}\sum_{j,k=-2}^{2}\left\{\alpha_{jk}(e,e')\beta_{jk}(i,i')\left(j\cot i + \frac{k}{\sin i}\right)\right\}\sin(j\omega + k\Delta\Omega),$$
$$\frac{d\omega}{dt} = \frac{\mu' n'^2}{n\sqrt{1-e^2}}\sum_{j,k=-2}^{2}\left\{\frac{1-e^2}{e}\alpha'_{jk}(e,e')\beta_{jk}(i,i') - \cot i\,\alpha_{jk}(e,e')\beta'_{jk}(i,i')\right\}\cos(j\omega + k\Delta\Omega),$$
$$\frac{d\Omega}{dt} = \frac{\mu' n'^2}{n\sqrt{1-e^2}\sin i}\sum_{j,k=-2}^{2}\left\{\alpha_{jk}(e,e')\beta'_{jk}(i,i')\cos(j\omega + k\Delta\Omega)\right\},$$
$$\frac{dM}{dt} = -\frac{\mu' n'^2}{n}\sum_{j,k=-2}^{2}\left\{\frac{(1-e^2)}{e}\alpha'_{jk}(e,e') + 2a\alpha_{jk}(e,e')\right\}\beta_{jk}(i,i')\cos(j\omega + k\Delta\Omega),$$

where

$$\alpha'_{jk}(e,e') = \frac{\partial\alpha_{jk}}{\partial e}, \beta'_{jk}(i,i') = \frac{\partial\beta_{jk}}{\partial i}$$

4. NUMERICAL SIMULATIONS

In this section, we suppose a spacecraft in an elliptical inclined orbit around the Moon and its motion perturbed by the Earth for a duration time of 1,000 days. The initial conditions found in Prado (2003) and Domingos et al. (2008) ; a lunar satellite with a_0=0.01 (3,844km), e_0=0.01, i_0=15°,60° and ω_0=Ω_0=0° were used to show the behavior of the evolutions of the spacecraft orbital elements with time. The perturbing body was in elliptical (e'=0.1) inclined (10°≤i'≤80°). From the graphs shown in Fig. 2 and Fig. 3, it is revealed that the evolutions of argument of pericenter (ω) keeps increasing for a fixed third body's inclination (i') but as (i') goes to higher angle, the rate of increase in ω gets lower.

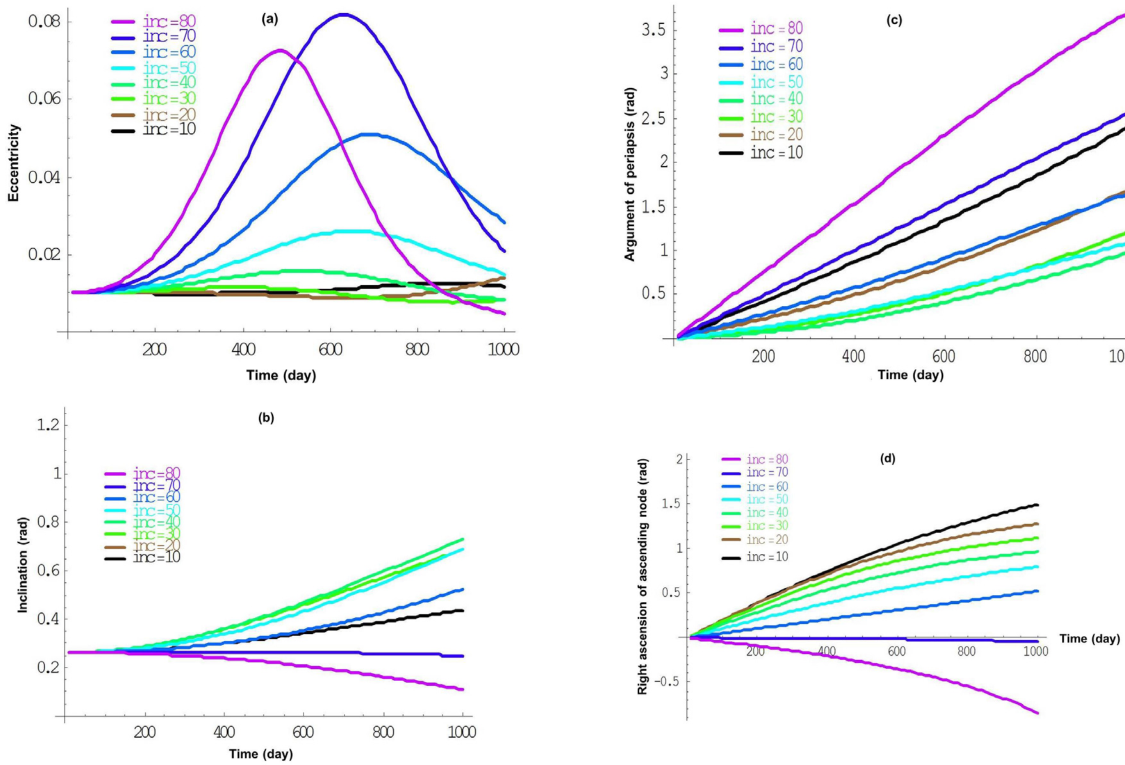

Fig. 2. Evolutions of orbital elements for spacecraft's over 1,000 days with initial conditions a_0=0.01 (3,844*km*), e_0=0.01, i_0=15°, and ω_0=Ω_0=0°. It is supposed that the initial conditions of the third body are a'= (384,400*km*), e'=0.1, and ω'=Ω'=0°, and 10° ≤ i' ≤ 80°, inc= i'. (a) Eccentricity, (b) Inclination (rad), (c) Argument of periapsis (rad), (d) Right ascension of ascending node (rad)

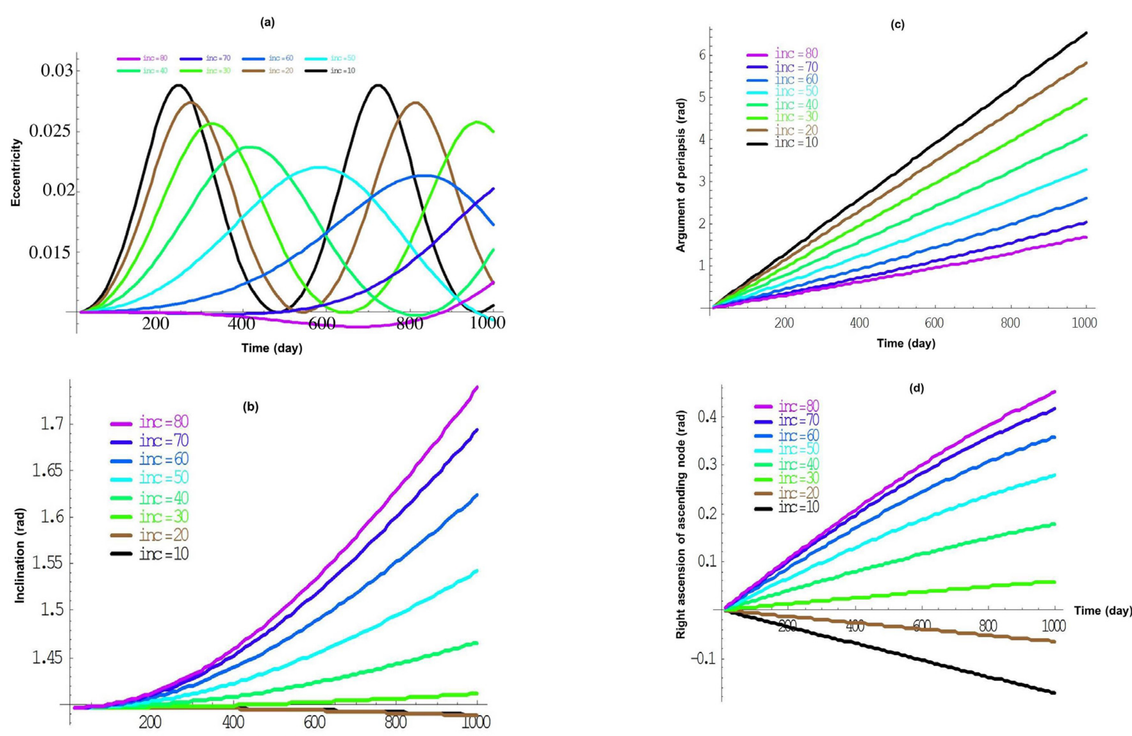

Fig. 3. Evolutions of orbital elements for spacecraft's over 1,000 days with initial conditions a_0=0.01 (3,844*km*), e_0=0.01, i_0=80°, and ω_0=Ω_0=0°. It is supposed that the initial conditions of the third body are a'= (384,400*km*), e'=0.1, and ω'=Ω'=0°, and 10° ≤ i' ≤ 80°, inc= i'. (a) Eccentricity, (b) Inclination (rad), (c) Argument of periapsis (rad), (d) Right ascension of ascending node (rad)

On another hand, the evolutions of right ascension of the ascending node (Ω) and inclination (i) keep decreasing with fixed i' up to the certain angle but as i' changes upward, they increase while the rate of increase in Ω and i decreases. The evolution eccentricity (e) shows periodical behavior with the time even if the amplitude depends strongly on the (i').

5. DISCUSSION AND CONCLUSION

Beginning this study, the basic question is; what is the effect of inclined third body on the spacecraft orbital elements? For the answer, the effects of wide range of the inclinations of the third body (10°-80°) are presented. Using the initial conditions found in Prado (2003) and Domingos et al. (2008), the results of changing the third body's inclinations over (10°-80°) on the model indicated that the effect of the third body's inclinations cannot be neglected to be consistent with many previous studies.

In this paper, we have used three-dimensional model to analyze the perturbations of the third body potential on a spacecraft that orbits close to a heavenly body including the eccentricity and the inclination of the third-body of the perturbing body within the formulation.

All curves turned out to be smooth through the well-known double-average model based on the mean motion of the spacecraft and the perturbing body which plays an important role in the analysis of spacecrafts long term dynamics enabling the short-lived periodic terms to be eliminated.

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