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ON MEDIAL B-ALGEBRAS

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ABSTRACT. In this paper we introduce the notion of medial B-algebras, and we obtain a fundamental theorem of B-homomorphism for B-algebras.

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1. Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras [4, 5]. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [2, 3] Q. P. Hu and X. Li introduced a wide class of abstract algebras: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. J. Neggers and H. S. Kim [8] introduced the notion of d-algebras, i.e., (I) $x * x = 0$;(V) $0 * x = 0$; (VI) $x * y = 0$ and $y * x = 0$ implay $x = y$, which is another useful generalization of BCK-algebras, and then they investigated several relations between d-algebras and oriented digraphs. Recently, Y. B. Jun, E. H. Roh and H. S. Kim [6] introduced a new notion, called an BH-algebra, i.e., (I),(II) $x * 0 = 0$ and (IV), which is a generalization of $BCH/BCI/BCK$-algebras. They also defined the notions of ideals and boundedness in BH-algebras, and showed that there is a maximal ideal in bounded BH-algebras. J. Neggers and H. S. Kim [9] introduced and investigated a class of algebras, i.e., the class of B-algebras, which is related to several classed of algebras of interest such as $BCH/BCI/BCK$-algebras and which seems to have rather nice properties without being excessively complicated otherwise. Furthermore, a digraph on algebras defined below demonstrates a rather interesting connection between B-algebras and groups. J. R. Cho and H. S. Kim [1] discussed further relations between B-algebras and other classed of algebras, such as quasigroups.

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J. Neggers and H. S. Kim [10] introduced the notion of normality in B-algebras and obtained a fundamental theorem of B-homomorphism for B-algebras.

In this paper we introduce the notion of medial B-algebras, and we obtain a fundamental theorem of B-homomorphism for B-algebras.

2. Preliminaries

In this section, we introduce some notions and results which have also been discussed in [1, 9]. A B-algebra is a non-empty set X with a constant 0 and a binary operation " $*$ " satisfying the following axioms:

(I) $x * x = 0$,
(II) $x * 0 = x$,
(III) $(x * y) * z = x * (z * (0 * y))$

for all x, y, z in X.

Example 2.1. Let $X := \{0, 1, 2\}$ be a set with the following table:

$*$	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

Then $(X; *, 0)$ is a B-algebra.

Example 2.2 ([9]). Let X be the set of all real numbers except for a negative integer $-n$. Define a binary operation $*$ on X by

$$x * y := \frac{n(x - y)}{n + y}.$$

Then $(X; *, 0)$ is a B-algebra.

Example 2.3. Let $X := \{0, 1, 2, 3, 4, 5\}$ be a set with the following table:

$*$	0	1	2	3	4	5
0	0	2	1	3	4	5
1	1	0	2	4	5	3
2	2	1	0	5	3	4
3	3	4	5	0	2	1
4	4	5	3	1	0	2
5	5	3	4	2	1	0

Then $(X; *, 0)$ is a B-algebra (see[10]).

Example 2.4 ([9]). Let $F < x, y, z >$ be the free group on three elements. Define $u * v := vuv^{-2}$. Thus $u * u = e$ and $u * e = u$. Also $e * u = u^{-1}$. Now, given $a, b, c, \in F < x, y, z >$, let

$$\begin{aligned} w(a,b,c) &= ((a*b)*c)(a*(c*(e*b))^{-1} \\ &= (cbab^{-2}c^{-2})(b^{-1}cb^2a^{-1}cbcb^2)^{-1} \\ &= cbab^{-2}c^{-2}b^{-2}c^{-1}b^{-1}c^{-1}ba^{-1}b^{-2}c^{-1}b. \end{aligned}$$

Let $N(*)$ be the normal subgroup of $F < x, y, z >$ generated by the elements $w(a, b, c)$. Let $G = F < x, y, z > /N(*)$. On G define the operation "·" as usual and define

$$(uN(*)) * (vN(*)) := (u * v)N(*).$$

It follows that $(uN(*)) * (uN(*)) = eN(*)$, $(uN(*)) * (eN(*)) = uN(*)$ and

$$w(aN(*), bN(*), cN(*)) = w(a, b, c)N(*) = eN(*).$$

Hence $(G; *, eN(*))$ is a B-algebra.

Lemma 2.5 ([9]). *If $(X; *, 0)$ is a B-algebra, then $y * z = y * (0 * (0 * z))$ for any $y, z \in X$.*

Proposition 2.6 ([9]). *If $(X; *; 0)$ is a B-algebra, then*

$$x * (y * z) = (x * (0 * z)) * y$$

for any $x, y, z \in X$.

Lemma 2.7 ([1]). *Let $(X; *, 0)$ be a B-algebra. Then we have the following statements.*

(i) *if $x * y = 0$ then $x = y$ for any $x, y \in X$;*
(ii) *if $0 * x = 0 * y$ then $x = y$ for any $x, y \in X$;*
(iii) *$0 * (0 * x) = x$ for any $x \in X$.*

Let $(X; *, 0_X)$ and $(Y; \bullet, 0_Y)$ be B-algebras. A mapping $\varphi : X \longrightarrow Y$ is called a *B-homomorphism*[10] if $\varphi(x * y) = \varphi(x) \bullet \varphi(y)$ for any $x, y \in X$.

Example 2.8 ([10]). Let $X := \{0, 1, 2, 3\}$ be a set with the following table:

$*$	0	1	2	3
0	0	2	1	3
1	1	0	3	2
2	2	3	0	1
3	3	1	2	0

Then $(X; *, 0)$ is a B-algebra[1]. If we define $\varphi(0) = 0, \varphi(1) = 3, \varphi(2) = 3$ and $\varphi(3) = 0$, then $\varphi : X \longrightarrow Y$ is a B-homomorphism.

A B-homomorphism $\varphi : X \longrightarrow Y$ is called a *B-isomorphism*[10] if φ is a bijection, and denote it by $X \cong Y$. Note that if $\varphi : X \longrightarrow Y$ is a B-isomorphism then $\varphi^{-1} : Y \longrightarrow X$ is also a B-isomorphism. If we define $\varphi(0) = 0, \varphi(1) = 2, \varphi(2) = 1$ and $\varphi(3) = 3$ in Example 2.8, then $\varphi : X \longrightarrow Y$ is a B-isomorphism. Let $\varphi : X \longrightarrow Y$ be a B-homomorphism. Then the subset $\{x \in X \mid \varphi(x) = 0_Y\}$ of X is called the *kernel* of the B-homomorphism φ, and denote it by $Ker\varphi$

Definition 2.9 ([10])**.** Let $(X; *, 0)$ be a B-algebra. A non-empty subset N of X is called a *subalgebra* of X if $x * y \in N$, for any $x, y \in N$.

In Example 2.8, $N_1 := \{0, 3\}$ is a subalgebra of X, while $N_2 := \{0, 1\}$ is not a subalgebra of X, since $0 * 1 = 2 \notin N_2$. Note that any subalgebra of a B-algebra is also a B-algebra.

Theorem 2.10 ([10])**.** *Let $(X; *, 0)$ be a B-algebra and $\emptyset \neq N \subseteq X$. Then the following are equivalent:*

(a) *N is a subalgebra of X.*
(b) *$x * (0 * y)$, $0 * y \in N$, for any $x, y \in N$.*

Note that any kernel of a B-homomorphism is a subalgebra of X.

3. Medial B-algebras

Let $(X; *, 0)$ be a B-algebra and let N be a subalgebra of X. The set X(resp., N) is said to be *medial* if $(x * n_1) * (y * n_2) = (x * y) * (n_1 * n_2)$ for any $x, y, n_1, n_2 \in X$(resp., for any $x, y, n_1, n_2 \in N$).

Example 3.1. The B-algebra in Example 2.8, is medial. The B-algebra in Example 2.3, is not medial, since $(5 * 2) * (4 * 3) = 4 * 1 = 5 \neq 3 = 1 * 5 = (5 * 4) * (2 * 3)$.

J. Neggers and H. S. Kim[10] introduced the notion of a normal subalgebra in B-algebras. A nonempty subset N of X is said to be *normal* (or *normal subalgebra*) of X if $(x * a) * (y * b) \in N$ for any $x * a, y * b \in N$.

Example 3.2. The subalgebra $N_1 = \{0, 3\}$ is both a normal and a medial subalgebra of X in Example 2.8, while the subalgebra $N_2 = \{0, 3\}$ in Example 2.3 is medial, but not normal.

Example 3.3. Let $X := \{0, 1, 2, 3\}$ be a set with the following table:

$*$	0	1	2	3
0	0	3	2	1
1	1	0	3	2
2	2	1	0	3
3	3	2	1	0

Then $(X; *, 0)$ is a B-algebra and the subalgebra $N_3 = \{0, 2\}$ is a medial subalgebra of X.

Let $(X;*,0)$ be a B-algebra and let N be a subalgebra of X. Define a relation $\sim_N$ on X by $x \sim_N y$ if and only if $x*N = y*N$, where $x, y \in X$. Then it is easy to show that $\sim_N$ is an equivalence relation on X. Assume X is medial (or N is a medial subalgebra of X). If $x \sim_N y$ and $a \sim_N b$, where $x, y, a, b \in N$, then $x*N = y*N$ and $a*N = b*N$ and hence $x = y*n_1, a = b*n_2$ for some $n_1, n_2 \in N$. Hence $x*a = (y*n_1)*(b*n_2) = (y*b)*(n_1*n_2) \in (y*b)*N$, since X(resp., N) is medial. For any $(x*a)*n_3 \in (x*a)*N$, we have

$$\begin{aligned}(x*a)*n_3 &= ((y*b)*(n_1*n_2))*n_3\\ &= (y*b)*(n_3*(0*(n_1*n_2))) \text{ [by (III)]}\\ &\in (y*b)*N \text{ [by Thereom 2.10]}\end{aligned}$$

Hence $(x*a)*N \subseteq (y*b)*N$. Similarly, we obtain $(y*b)*N \subseteq (x*a)*N$. This means that $x*a \sim_N y*b$, i.e., $\sim_N$ is a congruence relation on X. Denote the equivalence class containing x by $[x]_N$, i.e., $[x]_N = \{y \in X \mid x \sim_N y\}$ and let $X/N := \{[x]_N \mid x \in X\}$. We show that X/N is a B-algebra.

Theorem 3.4. *Let X be a medial B-algebra and let N be a subalgebra of X. Then X/N is a medial B-algebra with $N = [0]_N$.*

Proof. If we define $[x]_N*[y]_N := [x*y]_N$ then the operation "$*$" is well-defined, since $\sim_N$ is a congruence relation on X. We claim that $[0]_N = N$. If $x \in [0]_N$, then $x*N = 0*N$, and hence by (II) $x = x*0 \in x*N = 0*N$, i.e., $x = 0*n$ for some $n \in N$. Since N is a subalgebra and $0 \in N$, $x = 0*n \in N$. Hence $[0]_N \subseteq N$.

For any $x \in N$, since N is subalgebra of X, $0*x \in N$, say $n_1 = 0*x$. By applying Lemma 2.7-(iii), $x = 0*(0*x) \in 0*N$. We show that $x*N = 0*N$. For any $x*n \in x*N$,

$$\begin{aligned}x*n &= (0*(0*x))*n \quad \text{[by Lemma 2.7-(iii)]}\\ &= (0*(0*x))*(n*0)\\ &= (0*n)*(0*n)*((0*x)*0) \quad [X : \text{medial}]\\ &= (0*n)*(0*x)\\ &= (0*n)*n_1 \quad [n_1 = 0*x]\\ &= 0*(n_1*(0*n)) \quad \text{[by (III)]}\\ &\in 0*N \quad \text{[by Theorem 2.10]}\end{aligned}$$

Hence $x*N \subseteq 0*N$. If $y \in 0*N$, then $y = 0*n_2$ for some $n_2 \in N$. Hence $y = 0*n_2 = (x*x)*n_2 = x*(n_2*(0*x))$. Since $x \in N$, by Theorem 2.10, $n_2*(0*x) \in N$. Hence $y \in x*N$, i.e., $0*N \subseteq x*N$. Thus $x*N = 0*N$, i.e., $x \sim_N 0$. Hence $x \in [0]_N$, proving $N \subseteq [0]_N$. Checking three axioms and mediality is trivial and we omit the proof. □

Theorem 3.4 can be replaced by the following statement:

Theorem 3.4′. *Let X be a B-algebra and N be a medial subalgebra of X. Then X/N is a medial B-algebra with $N = [0]_N$.*

The B-algebra X/N discussed in Theorems 3.4 and 3.4′ is called the *quotient* B-algebra of X by N.

Proposition 3.5. *Let N be a medial subalgebra of the B-algebra $(X;*,0)$. Then the mapping $\gamma : X \longrightarrow X/N$, given by $\gamma(x) := [x]_N$, is a surjective B-homomorphism, and $Ker\gamma = N$.*

Proof. The mapping γ is obviously surjective. For all $x, y \in X$, $\gamma(x * y) = [x * y]_N = [x]_N * [y]_N = \gamma(x) * \gamma(y)$. Hence γ is a B-homomorphism. We claim that $\{x \in X \,|\, [x]_N = [0]_N\} = N$. For any $n \in N$, we show that $n * N = 0 * N$. If $n_1 \in N$, by Lemma 2.7-(iii), $n * n_1 = (0 * (0 * n)) * n_1 = 0 * (n_1 * (0 * (0 * n))) = 0 * (n_1 * n) \in 0 * N$, i.e., $n * N \subseteq 0 * N$. For any $0 * n_2 \in 0 * N$, $0 * n_2 = (n * n) * n_2 = n * (n_2 * (0 * n)) \in n * N$, i.e., $0 * N \subseteq n * N$. This proves $0 * N = n * N$, i.e., $[n]_N = [0]_N$. If $[x]_N = [0]_N$, then $x * N = 0 * N$, i.e., $x = 0 * n_1$ for some $n_1 \in N$. Since N is a subalgebra of X, $x = 0 * n_1 \in N$. Hence

$$\begin{aligned} Ker\gamma &= \{x \in X \,|\, \gamma(x) = N\} \\ &= \{x \in X \,|\, [x]_N = N\} \\ &= \{x \in X \,|\, [x]_N = [0]_N\} \\ &= N, \end{aligned}$$

proving the proposition. □

The mapping γ discussed in Proposition 3.5 is called the *natural*(or *canonical*) B-homomorphism of X onto X/N.

Proposition 3.6. *Let X be a medial B-algebra. If $\varphi : X \longrightarrow Y$ is a B-homomorphism, then the kernel $Ker\varphi$ is a medial subalgebra of X.*

Proof. Straightforward. □

By Theorem 3.4 and Proposition 3.6, if $\varphi : X \longrightarrow Y$ is a B-homomorphism, then $X/Ker\varphi$ is a B-algebra.

A B-algebra $(X;*,0)$ is said to be *commutative*[9] if $a * (0 * b) = b * (0 * a)$ for any $a, b \in X$. The B-algebra in Example 2.1 is commutative, while the B-algebra in Example 2.3 is not commutative, since $3 * (0 * 4) = 2 \neq 1 = 4 * (0 * 3)$.

Theorem 3.7. *Let X be a commutative medial B-algebra and let $\varphi : X \longrightarrow Y$ be a B-homomorphism. Then $X/Ker\varphi \cong Im\varphi$. In particular, if φ is surjective, then $X/Ker\varphi \cong Y$.*

Proof. Let $K := Ker\varphi$. If we define $\Psi : X/K \longrightarrow Im\varphi$ by $\Psi([x]_K) := \varphi(x)$, then Ψ is well-defined. In fact, suppose that $[x]_K = [y]_K$. Then $x \sim_K y$ and $x * K = y * K$, i.e., $x = y * k_1, y = x * k_2$ for some $k_1, k_2 \in K$. Hence $\varphi(x) = \varphi(y * k_1) = \varphi(y) * \varphi(k_1) = \varphi(y) * 0 = \varphi(y)$, i.e., $\Psi([x]_K) = \Psi([y]_K)$. Suppose that $\Psi([x]_K) = \Psi([y]_K)$, where $[x]_K, [y]_K \in X/K$. Then $\varphi(x) = \varphi(y)$. If $\alpha \in [x]_K$, then $\alpha \sim_K x$ and $\alpha * K = x * K$. This means that $\alpha = x * k_1, x = \alpha * k_2$

for some $k_1, k_2 \in K$. Hence $\varphi(\alpha) = \varphi(x * k_1) = \varphi(x) * \varphi(k_1) = \varphi(x) = \varphi(y)$, which implies $\varphi(\alpha * y) = \varphi(\alpha) * \varphi(y) = 0$. Hence $\alpha * y \in Ker\varphi = K$, i.e., $\alpha * y = k_3$ for some $k_3 \in K$. Similarly, $\varphi(y) * \varphi(\alpha) = 0$ implies $y * \alpha = k_4$ for some $k_4 \in K$. Sice X is commutative,

$$\begin{aligned}\alpha &= \alpha * 0\\ &= \alpha * (y * y)\\ &= (\alpha * (0 * y)) * y\\ &= (y * (0 * \alpha)) * y \quad [X\text{:commutative}]\\ &= y * (y * \alpha)\\ &= y * k_4.\end{aligned}$$

For any $\alpha * k_4 \in \alpha * K, \alpha * k = (y * k_4) * k = y * (k * (0 * k_4)) \in y * K$. Hence $\alpha * K \subseteq y * K$. Conversely, we have

$$\begin{aligned}y &= y * 0\\ &= y * (\alpha * \alpha)\\ &= (\alpha * (0 * y)) * \alpha\\ &= \alpha * (\alpha * y)\\ &= \alpha * k_3 \in \alpha * K,\end{aligned}$$

proving $y * K \subseteq \alpha * K$. Hence $\alpha * K = y * K$, i.e., $\alpha \sim_K y$. This proves $\alpha \in [y]_K$. Similarly, $[y]_K \subseteq [x]_K$. Thus $[x]_K = [y]_K$, proving that Ψ is injective. Obviously Ψ is surjective. Since $\Psi([x]_K * [y]_K) = \Psi([x * y]_K) = \varphi(x * y) = \varphi(x) * \varphi(y) = \Psi([x]_K) * \Psi([y]_K)$, Ψ is a B-homomorphism. Hence $X/Ker\varphi \cong Im\varphi$. $\square$

Example 3.8. In Example 2.8, since $K = Ker\varphi = \{0, 3\}$, we have $[0]_K = \{0, 3\}$ and $[1]_K = \{x \in X \mid x * 1 \in K\} = \{1, 2\}$. Hence $X/Ker\varphi = \{[0]_K, [1]_K\}$ and $X/Ker\varphi \cong Im\varphi$ by defining $\Psi([0]_K) = \varphi(0)$ and $\Psi([1]_K) = \varphi(1)$.

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