J. Appl. Math. & Informatics Vol. **32***(2014), No. 5 - 6, pp. 843 - 848*
http://dx.doi.org/10.14317/jami.2014.843

GLOBAL DYNAMICS OF A NON-AUTONOMOUS RATIONAL DIFFERENCE EQUATION

ÖZKAN ÖCALAN

Abstract. In this paper, we investigate the boundedness character, the periodic character and the global behavior of positive solutions of the difference equation

$$x_{n+1} = p_n + \frac{x_n}{x_{n-1}}, \quad n = 0, 1, \cdots,$$

where $\{p_n\}$ is a two periodic sequence of nonnegative real numbers and the initial conditions x_{-1}, x_0 are arbitrary positive real numbers.

AMS Mathematics Subject Classification: 39A10.
Key words and phrases : Boundedness character, dynamics, periodic solution, global stability.

1. Introduction

Recently, there has been an increasing interest in the study of the qualitative analyses of rational difference equations. For example, see $[1-8]$ and the references cited therein.

This work studies the boundedness character and the global asymptotic stability for the positive solutions of the difference equation

$$x_{n+1} = p_n + \frac{x_n}{x_{n-1}}, \quad n = 0, 1, \cdots, \tag{1.1}$$

where $\{p_n\}$ is a two periodic sequence of nonnegative real numbers and the initial conditions x_{-1}, x_0 are arbitrary positive numbers.

As far as we can examine, this is the first work devote to the investigation of the type Eq.(1.1).

Now, we assume $p_{2n} = \alpha$ and $p_{2n+1} = \beta$ in Eq.(1.1). Then we have

$$x_{2n+1} = \alpha + \frac{x_{2n}}{x_{2n-1}}, \quad n = 0, 1, \cdots \tag{1.2}$$

Received February 15, 2014. Revised June 15, 2014. Accepted June 19, 2014.

and

$$x_{2n+2} = \beta + \frac{x_{2n+1}}{x_{2n}}, \quad n = 0, 1, \cdots . \tag{1.3}$$

The autonomous case of Eq.(1.1) is

$$x_{n+1} = p + \frac{x_n}{x_{n-1}}, \quad n = 0, 1, \cdots , \tag{1.4}$$

where $p > 0$ and the initial conditions x_{-1}, x_0 are arbitrary positive numbers. We now consider the local asymptotic stability of the unique equilibrium $\overline{x} = p + 1$ of Eq.(1.4).

The linearized equation for Eq. (1.4) about the positive equilibrium $\overline{x} = p+1$ is

$$x_{n+1} - \frac{1}{p+1} x_n + \frac{1}{p+1} x_{n-1} = 0.$$

The following theorem is given in [1].

Theorem A. *Consider Eq.* (1.4) *and assume that* $x_{-1}, x_0, p \in (0, \infty)$. *Then the unique positive equilibrium* $\overline{x} = p + 1$ *of Eq.* (1.4) *is globally asymptotically stable.*

2. Boundedness Character of Eq. (1.1)

In this section, we investigate the boundedness character of Eq. (1.1). So, we have the following result.

Theorem 2.1. *Suppose that* $\alpha > 1$ *and* $\beta > 1$ *with* $\alpha \neq \beta$, *then every positive solution of Eq.*(1.1) *is bounded.*

Proof. It is clear from Eq. (1.2) and (1.3) that

$$x_{2n} > \beta \text{ and } x_{2n-1} > \alpha, \qquad \text{for every } n \geq 1. \tag{2.1}$$

Then, from (1.2) and (2.1) we obtain

$$x_{2n+1} = \alpha + \frac{x_{2n}}{x_{2n-1}} < \alpha + \frac{x_{2n}}{\alpha} \tag{2.3}$$

and from (1.3) and (2.1) we obtain

$$x_{2n} = \beta + \frac{x_{2n-1}}{x_{2n-2}} < \beta + \frac{x_{2n-1}}{\beta}. \tag{2.4}$$

From (2.3), (2.4) using induction we get

$$\begin{aligned} x_{2n+1} &< \alpha + \frac{\beta}{\alpha} + \frac{1}{\beta}\left(1 + \frac{1}{\alpha\beta} + \frac{1}{\alpha^2\beta^2} + \cdots\right) + \frac{1}{\alpha^2}\left(1 + \frac{1}{\alpha\beta} + \frac{1}{\alpha^2\beta^2} + \cdots\right) + x_{-1} \\ &= \alpha + \frac{\beta}{\alpha} + \frac{1}{\beta}\left(\frac{\alpha\beta}{\alpha\beta - 1}\right) + \frac{1}{\alpha^2}\left(\frac{\alpha\beta}{\alpha\beta - 1}\right) + x_{-1} \\ &= \alpha + \frac{\beta}{\alpha} + \frac{\alpha}{\alpha\beta - 1} + \frac{1}{\alpha}\left(\frac{\beta}{\alpha\beta - 1}\right) + x_{-1} \\ &= \left(\alpha + \frac{\beta}{\alpha}\right)\frac{\alpha\beta}{(\alpha\beta - 1)} + x_{-1}, \end{aligned}$$

$$\begin{aligned} x_{2n+2} &< \beta + \frac{\alpha}{\beta} + \frac{1}{\alpha}\left(1 + \frac{1}{\alpha\beta} + \frac{1}{\alpha^2\beta^2} + \cdots\right) + \frac{1}{\beta^2}\left(1 + \frac{1}{\alpha\beta} + \frac{1}{\alpha^2\beta^2} + \cdots\right) + x_0 \\ &= \beta + \frac{\alpha}{\beta} + \frac{1}{\alpha}\left(\frac{\alpha\beta}{\alpha\beta - 1}\right) + \frac{1}{\beta^2}\left(\frac{\alpha\beta}{\alpha\beta - 1}\right) + x_0 \\ &= \beta + \frac{\alpha}{\beta} + \frac{\beta}{\alpha\beta - 1} + \frac{1}{\beta}\left(\frac{\alpha}{\alpha\beta - 1}\right) + x_0 \\ &= \left(\beta + \frac{\alpha}{\beta}\right)\frac{\alpha\beta}{(\alpha\beta - 1)} + x_0. \end{aligned}$$

The result now follows. □

3. Stability and Periodicity for Eq. (1.1)

In this section, we investigate the periodicity and stability character of positive solutions of Eq. (1.1). Now, we have the following result.

Proposition 3.1. *Consider Eq.* (1.1) *when the case* $\alpha \neq \beta$ *and assume that* $\alpha, \beta \in (0, \infty)$. *Then there exist prime two periodic solutions of Eq.* (1.1).

Proof. In order Eq. (1.1) to possess a periodic solution $\{x_n\}$ of prime period 2, we must find positive numbers x_{-1}, x_0 such that

$$x_{-1} = x_1 = \alpha + \frac{x_0}{x_{-1}}, \qquad x_0 = x_2 = \beta + \frac{x_{-1}}{x_0}. \tag{3.1}$$

Let $x_{-1} = x$, $x_0 = y$, then from (3.1) we obtain the system of equations

$$x = \alpha + \frac{y}{x}, \qquad y = \beta + \frac{x}{y}. \tag{3.2}$$

We prove that (3.2) has a solution $(\overline{x}, \overline{y})$, $\overline{x} > 0$, $\overline{y} > 0$. From the first relation of (3.2) we have

$$y = (x - \alpha)x. \tag{3.3}$$

From (3.3) and the second relation of (3.2) we obtain

$$x(x-\alpha) = \beta + \frac{x}{x(x-\alpha)} = \beta + \frac{1}{(x-\alpha)} \quad \text{and} \quad x(x-\alpha)^2 - \beta(x-\alpha) - 1 = 0.$$

Now we consider the function

$$f(x) = x(x-\alpha)^2 - \beta(x-\alpha) - 1, \quad x > \alpha. \tag{3.4}$$

Then from (3.4) we get

$$\lim_{x\to\alpha^+} f(x) = -1, \qquad \lim_{x\to\infty} f(x) = \infty. \tag{3.5}$$

Hence Eq. (3.4) has a solution $\overline{x} > \alpha$. Then if $\overline{y} = (\overline{x} - \alpha)\overline{x}$, we have that the solution $\overline{x}_n$ of Eq. (1.1) with initial values $x_{-1} = \overline{x}$, $x_0 = \overline{y}$ is a periodic solution of period two. □

Theorem 3.2. *Consider Eq.* (1.1) *when the case* $\alpha \neq \beta$ *and assume that* $\alpha, \beta \in (0, \infty)$. *Suppose that*

$$\frac{\alpha}{\beta^2} + \frac{1}{\alpha\beta} + \frac{1}{\alpha^3} < 1. \tag{3.6}$$

Then the two periodic solutions of Eq. (1.1) *are locally asymptotically stable.*

Proof. From equations (1.3), (1.4) and Proposition 3.1 there exist $\overline{x}, \overline{y}$ such that

$$\overline{x} = \alpha + \frac{\overline{y}}{\overline{x}}, \qquad \overline{y} = \beta + \frac{\overline{x}}{\overline{y}}. \tag{3.7}$$

We set $x_{2n-1} = u_n, \;\; x_{2n} = v_n$ in equations (1.3), (1.4) and so we have

$$u_{n+1} = \alpha + \frac{v_n}{u_n}, \;\; v_{n+1} = \beta + \frac{u_{n+1}}{v_n} = \beta + \frac{\alpha + \frac{v_n}{u_n}}{v_n} = \beta + \frac{\alpha u_n + v_n}{u_n v_n}. \tag{3.8}$$

Then $(\overline{x}, \overline{y})$ is the positive equilibrium of Eq. (3.8), and the linearised system of Eq. (3.8) about $(\overline{x}, \overline{y})$ is the system

$$z_{n+1} = B z_n, \;\text{ where }\; B = \begin{pmatrix} \frac{-\overline{y}}{\overline{x}^2} & \frac{1}{\overline{x}} \\ \frac{-1}{\overline{x}^2} & \frac{-\alpha}{\overline{y}^2} \end{pmatrix}, \quad z_n = \begin{pmatrix} u_n \\ v_n \end{pmatrix}.$$

The characteristic equation of B is

$$\lambda^2 + \lambda \left(\frac{\alpha}{\overline{y}^2} + \frac{\overline{y}}{\overline{x}^2} \right) + \frac{\alpha}{\overline{x}^2 \overline{y}} + \frac{1}{\overline{x}^3} = 0. \tag{3.9}$$

Using Eq. (3.6), from Eq. (3.7), since $\overline{x} > \alpha, \; \overline{y} > \beta$ we have

$$\frac{\alpha}{\overline{y}^2} + \frac{\overline{y}}{\overline{x}^2} + \frac{\alpha}{\overline{x}^2 \overline{y}} + \frac{1}{\overline{x}^3} < \frac{\alpha}{\beta^2} + \frac{1}{\alpha\beta} + \frac{1}{\alpha^3} + 1 - \frac{\alpha}{\overline{x}} < 1$$

and we obtain

$$\frac{\alpha}{\beta^2} + \frac{1}{\alpha\beta} + \frac{1}{\alpha^3} < \frac{\alpha}{\overline{x}} < 1. \tag{3.10}$$

Then, from (3.10) and Theorem 1.3.7 of Kocic and Ladas in [4], all the roots of Eq. (3.9) are modulus less than 1. Therefore, from Proposition 3.1, system (3.8) is asymptotically stable. The proof is complete. □

Theorem 3.3. *Consider Eq.* (1.1) *when the case* $\alpha \neq \beta$. *Assume that* $\alpha > 1, \; \beta > 1$. *Then every positive solution of Eq.* (1.1) *converges to a two-periodic solution of Eq.* (1.1).

Proof. Since $\alpha > 1, \beta > 1$, we know by Theorem 2.1 that every positive solution of Eq. (1.1) is bounded, it follows that there are finite

$$s = \liminf_{n\to\infty} x_{2n+1} \text{ and } S = \limsup_{n\to\infty} x_{2n+1},$$
$$l = \liminf_{n\to\infty} x_{2n} \text{ and } L = \limsup_{n\to\infty} x_{2n}$$

exist. Then it is easy to see from Eq. (1.2) and (1.3) that

$$s \geq \alpha + \frac{l}{S} \;\text{ and }\; S \leq \alpha + \frac{L}{s}$$

and

$$l \geq \beta + \frac{s}{L} \text{ and } L \leq \beta + \frac{S}{l}.$$

Thus, we have

$$sS \geq \alpha S + l \text{ and } Ss \leq \alpha s + L$$

and

$$Ll \geq \beta L + s \text{ and } Ll \leq \beta l + S.$$

This implies that

$$\alpha S + l \leq Ss \leq \alpha s + L$$

and

$$\beta L + l \leq Ll \leq \beta l + S.$$

Then, we get

$$\alpha(S - s) \leq (L - l) \tag{3.11}$$

and

$$\beta(L - l) \leq (S - s). \tag{3.12}$$

Now, we shall prove that $s = S$ and $l = L$. It is clear that if $l = L$, then by (3.11) it must be $s = S$. Similarly, if $s = S$, then by (3.12) it must be $l = L$.

Hence we assume that $s < S$ and $l < L$. From (3.11) and (3.12) we have

$$\alpha(S - s) + \beta(L - l) \leq (S - s) + (L - l),$$

then we obtain a contradiction. So, we get $s = S$ and $l = L$

Moreover, it is obvious that since $\alpha \neq \beta$, then from Eq. (1.2) and Eq. (1.3)

$$\lim_{n\to\infty} x_{2n+1} \neq \lim_{n\to\infty} x_{2n}.$$

Then it is clear that every positive solution of Eq. (1.1) converges to a two-periodic solution of Eq. (1.1). The proof is complete. □

Finally, using Proposition 3.1, Theorems 3.1 and 3.2, we have the following Theorem.

Theorem 3.4. *Consider Eq.* (1.1) *when the case* $\alpha \neq \beta$. *Assume that* $\alpha > 1$, $\beta > 1$ *and that* (3.6) *holds. Then two-period solutions of Eq.* (1.1) *are globally asymptotically stable.*

References

1. R. Devault, G. Ladas and S.W. Schultz, *On the recursive sequence* $x_{n+1} = (A/x_n) + (1/x_{n-2})$, Proc. Amer. Math. Soc. **126**(11), 3257–3261 (1998).
2. E.A. Grove and G. Ladas, *Periodicities in nonlinear difference equations*, Chapman and Hall/Crc, 2005.
3. V. Kocić and G. Ladas, *Global behavior of nonlinear difference equations of higher order with applications*, Kluwer Academic Publishers, Dordrecht, 1993.
4. M.R.S. Kulenović, G. Ladas and C.B. Overdeep, *On the dynamics of* $x_{n+1} = p_n + (x_{n-1}/x_n)$, J. Difference Equ. Appl., **9**(11), 1053-1056 (2003).
5. M.R.S. Kulenović, G. Ladas and C.B. Overdeep, *On the dynamics of* $x_{n+1} = p_n + (x_{n-1}/x_n)$ *with a period-two coefficient*, J. Difference Equ. Appl., **10**(10), 905-914 (2004).
6. Ö. Öcalan, *Dynamics of the difference equation* $x_{n+1} = p_n + (x_{n-k})/(x_n)$ *with a Period-two Coefficient*, Appl. Math. Comput., **228**, 31–37 (2014).
7. V.G. Papanicolaou, *On the asymptotic stability of a class of linear difference equations*, Mathematics Magazine, **69**, 34-43 (1996).
8. S. Stević, *On the recursive sequence* $x_{n+1} = \alpha_n + (x_{n-1}/x_n)$, Int. J. Math. Sci., **2**(2), 237-243 (2003).

Özkan Öcalan received M. Sc. from Afyon Kocatepe University and Ph. D. at Ankara University. Since 1995 he has been working at Afyon Kocatepe University. His research interests include oscillation and stability of difference and differential equations.

Afyon Kocatepe University, Faculty of Science and Arts, Department of Mathematics, ANS Campus, 03200, Afyonkarahisar, Turkey.

e-mail: ozkan@aku.edu.tr