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CAYLEY INTUITIONISTIC FUZZY GRAPHS

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Abstract. In this paper, we introduce the notion of Cayley intuitionistic fuzzy graphs and investigate some of their properties. We present some interesting properties of intuitionistic fuzzy graphs in terms of algebraic structures. We discuss connectedness in Cayley intuitionistic fuzzy graphs. We also describe different types of α-connectedness in Cayley intuitionistic fuzzy graphs.

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1. Introduction

Graph theory has numerous applications to problems in different areas including computer science, electrical engineering, system analysis, operations research, economics, networking routing, transportation, and optimization. Point-to-point interconnection networks for parallel and distributed systems are usually modeled by *directed graphs* (or digraphs). A digraph is a graph whose edges have direction and are called *arcs* (edges). Arrows on the arcs are used to encode the directional information: an arc from vertex (node)x to vertex y indicates that one may move from x to y but not from y to x. The Cayley graph was first considered for finite groups by Cayley in 1878. Max Dehn in his unpublished lectures on group theory from 1909-10 reintroduced Cayley graphs under the name Gruppenbild (group diagram), which led to the geometric group theory of today. His most important application was the solution of the word problem for the fundamental group of surfaces with genus, which is equivalent to the topological problem of deciding which closed curves on the surface contract to a point [6, 7].

Fuzzy set is a newly emerging mathematical frame work to exemplify the phenomenon of uncertainty in real life tribulations. It was introduced by Zadeh

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in 1965, and the concepts were pioneered by various independent researches. Kaufmann's initial definition of a fuzzy graph [13] was based on Zadeh's fuzzy relations [24]. Rosenfeld [17] introduced the fuzzy analogue of several basic graph-theoretic concepts. Mordeson and Peng [16] defined the concept of complement of fuzzy graph and studied some operations on fuzzy graphs. Atanassov [8] introduced the concept of intuitionistic fuzzy relations and intuitionistic fuzzy graphs and further were studied in [4, 12, 19, 20]. In 1983, Atanassov [9] introduced the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets [23]. Atanassov added a new component (which determines the degree of nonmembership) in the definition of fuzzy set. The fuzzy sets give the degree of membership of an element in a given set (and the nonmembership degree equals one minus the degree of membership), while intuitionistic fuzzy sets give both a degree of membership and a degree of nonmembership which are more-or-less independent from each other, the only requirement is that the sum of these two degrees is not greater than 1. Akram *et al.*[1-3] introduced many new concepts, including intuitionistic fuzzy hypergraphs, strong intuitionistic fuzzy graphs, intuitionistic fuzzy cycles and intuitionistic fuzzy trees. Wu [22] discussed fuzzy digraphs. Shahzamanian *et al.*[21] introduced the notion of roughness in Cayley graphs and investigated several properties. Namboothiri *et al.*[18] studied Cayley fuzzy graphs. In this paper, we introduce the notion of Cayley intuitionistic fuzzy graphs and investigate some of their properties. We present some interesting properties of intuitionistic fuzzy graphs in terms of algebraic structures. We discuss connectedness in Cayley intuitionistic fuzzy graphs. We also describe different types of α- connectedness in Cayley intuitionistic fuzzy graphs.

We have used standard definitions and terminologies in this paper. For other notations, terminologies and applications not mentioned in the paper, the readers are referred to [5-7, 11, 21].

2. Preliminaries

In this section, we review some elementary concepts whose understanding is necessary fully benefit from this paper.

A digraph is a pair $G^* = (V, E)$, where V is a finite set and $E \subseteq V \times V$. In this paper, we will write $xy \in E$ to mean $x \to y \in E$, and if $e = xy \in E$, we say x and y are *adjacent* such that x is a starting node and y is an ending node.

The study of vertex-transitive graphs has a long and rich history in discrete mathematics. Prominent examples of vertex-transitive graphs are Cayley graphs which are important in both theory as well as applications, e.g., Cayley graphs are good models for interconnection networks.

Definition 2.1. Let G be a finite group and let S be a minimal generating set of G. A Cayley graph (G, S) has elements of G as its vertices, the edge-set is given by $\{(g, gs) : g \in G, s \in S\}$. Two vertices g_1 and g_2 are adjacent if $g_2 = g_1.s$, where $s \in S$. Note that a generating set S is minimal if S generates G but no proper subset of S does.

Theorem 2.2. *All Cayley graphs are vertex transitive.*

Definition 2.3. Let $(V, *)$ be a group and A be any subset of V. Then the Cayley graph induced by $(V, *, A)$ is the graph $G = (V, R)$, where $R = \{(x, y) : x^{-1}y \in A\}$.

Definition 2.4 ([23, 24])**.** A *fuzzy subset* μ on a set X is a map $\mu : X \to [0, 1]$. A *fuzzy binary relation* on X is a fuzzy subset μ on $X \times X$.

Definition 2.5 ([22])**.** Let V be a finite set, $A =< V, \mu_A >$ a fuzzy set of V and $B =< V \times V, \mu_B >$ a fuzzy relation on V, then the ordered pair (A, B) is called a *fuzzy digraph.*

Definition 2.6 ([18])**.** Let $(V, *)$ be a group and let μ be a fuzzy subset of V. Then the fuzzy relation R on $V \times V$ defined by

$$R(x, y) = \{(\mu(x^{-1} * y) \text{ for all x, y} \in V\}$$

induces a fuzzy graph $G = (V, R)$, called the *Cayley fuzzy graph* induced by the $(V, *, \mu)$.

Definition 2.7 ([8])**.** An intuitionistic fuzzy set (IFS, for short) on a universe X is an object of the form

$$A = \{< x, \mu_A(x), \nu_A(x) > | x \in X\},$$

where $\mu_A(x) \in [0, 1]$ is called degree of membership of x in A and $\nu_A(x) \in [0, 1]$ is called degree of nonmembership of x in A, and μ_A, ν_A satisfies the following condition for all $x \in X$, $\mu_A(x) + \nu_A(x) \leq 1$.

Definition 2.8. Let X be intuitionistic fuzzy set. For any subset A and for $\alpha \in [0, 1]$, $\{x|\mu_A(x) \geq \alpha, \nu_A(x) \leq \alpha\}$ is called α - cut of A. It is denoted by A_α.

Definition 2.9. Let X be intuitionistic fuzzy set. For any subset A and for $\alpha \in [0, 1]$, $\{x|\mu_A(x) > \alpha, \nu_A(x) < \alpha\}$ is called strong α - cut of A. It is denoted by A^+_α

Definition 2.10. Let X be intuitionistic fuzzy set. For any subset A of X, the support of A is the set$\{x \in X|\mu_A(x) \geq 0, \nu_A(x) > 0\}$. It is denoted by *supp*(A). It can also be denoted as *supp*$(A) = A^+_0$.

Definition 2.11. An intuitionistic fuzzy relation $R = (\mu_R(x, y), \nu_R(x, y))$ in a universe $X \times X$ ($R(X \to X)$, for short) is an intuitionistic fuzzy set of the form

$$R = \{< (x, y), \mu_R(x, y), \nu_R(x, y) > |(x, y) \in X \times X\},$$

where $\mu_R : X \times X \to [0, 1]$ and $\nu_R : X \times X \to [0, 1]$. The intuitionistic fuzzy relation R satisfies $\mu_R(x, y) + \nu_R(x, y) \leq 1$ for all x, $y \in X$.

Definition 2.12. Let R be an intuitionistic fuzzy relation on universe X. Then R is called *an intuitionistic fuzzy equivalence relation* on X if it satisfies the following conditions:

(a) R is intuitionistic fuzzy reflexive, i.e., $R(x, x) = (1, 0)$ for each $x \in X$,
(b) R is intuitionistic fuzzy symmetric, i.e., $R(x, y) = R(y, x)$ for any x, $y \in X$,
(c) R is intuitionistic fuzzy transitive, i.e., $R(x, z) \geq \bigvee_y (R(x, y) \bigwedge R(y, z))$.

Definition 2.13. Let R be an intuitionistic fuzzy relation on universe X. Then R is called *an intuitionistic fuzzy partial order relation* on X if it satisfies the following conditions:

(a) R is intuitionistic fuzzy reflexive, i.e., $R(x, x) = (1, 0)$, for each $x \in X$,
(b) R is intuitionistic fuzzy antisymmetric, i.e., for all $x, y \in X$ $R(x, y) \neq R(y, x)$,
(c) R is intuitionistic fuzzy transitive, i.e., $R(x, z) \geq \bigvee_y (R(x, y) \bigwedge R(y, z))$.

Definition 2.14. Let R be an intuitionistic fuzzy relation on universe X. Then R is called *an intuitionistic fuzzy linear order relation* on X if it satisfies the following conditions:

(a) R is intuitionistic fuzzy partial relation,
(b) $(R \vee R^{-1})(x, y) > 0$ for all x, $y \in X$.

3. Cayley Intuitionistic Fuzzy Graphs

In this section, we introduce Cayley intuitionistic fuzzy graphs and prove that all Cayley intuitionistic fuzzy graphs are regular.

Definition 3.1. An *intuitionistic fuzzy digraph* of a digraph G^* is a pair $G = (A, B)$, where $A =< V, \mu_A, \nu_A >$ is an intuitionistic fuzzy set in V and $B =< V \times V, \mu_B, \nu_B >$ is an intuitionistic fuzzy relation on V such that

$$\mu_B(xy) \leq \min(\mu_A(x), \mu_A(y)) \text{ and } \nu_B(xy) \leq \max(\nu_A(x), \nu_A(y)),$$

$0 \leq \mu_B(xy)+\nu_B(xy) \leq 1$ for all x, $y \in V$. We note that B may not be symmetric relation.

Definition 3.2. Let G be an intuitionistic fuzzy digraph. The in-degree of a vertex x in G is defined by $ind(x) = (ind_\mu(x), ind_\nu(x))$, where $ind_\mu(x) = \sum_{y \neq x} \mu_A(xy)$ and $ind_\nu(x) = \sum_{y \neq x} \nu_A(xy)$. Similarly, the out-degree of a vertex x in G is defined by $outd(x) = (outd_\mu(x), outd_\nu(x))$, where $outd_\mu(x) = \sum_{y \neq x} \mu_B(yx)$ and $outd_\nu(x) = \sum_{y \neq x} \nu_B(yx)$. An intuitionistic fuzzy digraph in which each vertex has same out-degree r is called *an out-regular digraph* with index of out-regularity r. In-regular digraphs are defined similarly.

Example 3.3. Consider an intuitionistic fuzzy digraph G of $G^* = (V, E)$, where $V = \{v_1, v_2, v_3, v_4\}$, $E = \{v_1v_2, v_2v_3, v_3v_4, v_4v_1\}$. By routine computations, it is easy to see from Fig. 1 that the intuitionistic fuzzy digraph is nether out-regular digraph nor in-regular digraph.

Definition 3.4. Let $(G, *)$ be a group and let S be a nonempty finite subset of G. Then the Cayley intuitionistic fuzzy graph $G = (V, R)$ is an intuitionistic

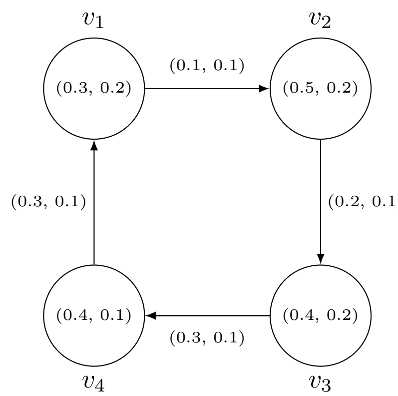

FIGURE 1. Intuitionistic fuzzy digraph

fuzzy graph with the vertex set $V = G$ and let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy subset of V. The intuitionistic fuzzy relation $R(x, y)$ on V is defined by $R(x, y) = \{(\mu_A(x^{-1}y), \mu_A(x^{-1}y))|x, y \in G \text{ and } x^{-1}y \in S\}$.

Example 3.5. Consider the group Z_3 and take $V = Z_3 = \{0, 1, 2\}$. Define $\mu_A : V \to [0, 1]$ and $\nu_A : V \to [0, 1]$ by $\mu_A(0) = \mu_A(1) = \mu_A(2) = 0.5$, $\nu_A(0) = \nu_A(1) = \nu_A(2) = 0.4$. Then the Cayley intuitionistic fuzzy graph $G = (V, R)$ induced by $(Z_3, +, A)$ is given by the following Table 1 and Figure 2.

TABLE 1. $R(a, b)$ for Cayley intuitionistic fuzzy graph

a	b	(-a)+ b	R(a,b)
0	0	0	(0.4, 0.4)
0	1	1	(0.3, 0.3)
0	2	2	(0.3, 0.3)
1	0	2	(0.3, 0.3)
1	1	0	(0.4, 0.4)
1	2	1	(0.3, 0.3)
2	0	1	(0.3, 0.3)
2	1	2	(0.3, 0.3)
2	2	0	(0.4, 0.4)

By routine computations, it is easy to see from Fig. 2 that $G = (V, R)$ is a Cayley intuitionistic fuzzy graph, and it is regular.

We notice that Cayley intuitionistic fuzzy graphs are actually intuitionistic fuzzy digraphs. Furthermore, the relation R in the above definition describes the strength of each directed edge. Let G denote an intuitionistic fuzzy graph $G = (V, R)$ induced by the triple $(V, *, A)$.

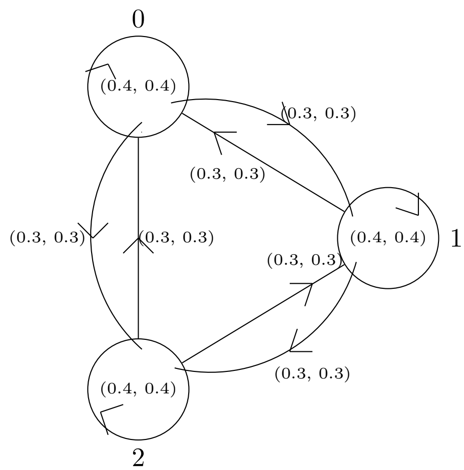

FIGURE 2. Cayley Intuitionistic fuzzy graph

Definition 3.6 ([15])**.** Let $(S, *)$ be a semigroup. Let $A' = (\mu'_A, \nu'_A)$ be an intuitionistic fuzzy subset of S. Then A' is said to be *an intuitionistic fuzzy subsemigroup* of S if for all $x, y \in S$, $\mu_{A'}(xy) \geq \min(\mu_{A'}(x), \mu_{A'}(y))$ and $\nu_{A'}(xy) \leq \max(\mu_{A'}(x), \nu_{A'}(y))$.

Theorem 3.7. *The Cayley intuitionistic fuzzy graph G is vertex transitive.*

Proof. Let $a,\ b \in V$. Define $\psi : V \to V$ by $\psi(x) = ba^{-1}x\ \forall x \in V$. Clearly, ψ is a bijective map. For each $x,\ y \in V$,

$$
\begin{aligned}
R(\psi(x), \psi(y)) &= (R_\mu(\psi(x), \psi(y)), R_\nu(\psi(x), \psi(y))). \\
\text{Now } R_\mu(\psi(x), \psi(y)) &= R_\mu(ba^{-1}x, ba^{-1}y) \\
&= \mu_A((ba^{-1}x)^{-1}(ba^{-1}x)) \\
&= \mu_A(x^{-1}y) \\
&= R_\mu(x, y). \\
R_\nu(\psi(x), \psi(y)) &= R_\nu(ba^{-1}x, ba^{-1}y) \\
&= \nu_A((ba^{-1}x)^{-1}(ba^{-1}x)) \\
&= \nu_A(x^{-1}y) \\
&= R_\nu(x, y).
\end{aligned}
$$

Therefore, $R(\psi(x), \psi(y)) = R(x, y)$. Hence ψ is an automorphism on G. Also $\psi(a) = b$. Hence G is vertex transitive. □

Theorem 3.8. *Every vertex transitive intuitionistic fuzzy graph is regular.*

Proof. Let $G = (V, R)$ be any vertex transitive intuitionistic fuzzy graph. Let $u,v \in V$. Then there is an automorphism f on G such that $f(u) = v$. Note that

$$\begin{aligned} ind(u) &= \sum_{x \in V} R(x,u) = \sum_{x \in V} (R_\mu(x,u), R_\nu(x,u)) \\ &= \sum_{x \in V} (R_\mu(f(x), f(u)), R_\nu(f(x), f(u)) \\ &= \sum_{x \in V} (R_\mu(f(x), v), R_\nu(f(x), v)) \\ &= \sum_{x \in V} (R_\mu(y,v), R_\nu(y,v) \\ &= ind(v), \end{aligned}$$

$$\begin{aligned} outd(u) &= \sum_{x \in V} R(x,u) = \sum_{x \in V} (R_\mu(u,x), R_\nu(u,x)) \\ &= \sum_{x \in V} (R_\mu(f(u), f(x)), R_\nu(f(u), f(x)) \\ &= \sum_{x \in V} (R_\mu(v, f(x)), R_\nu(v, f(x)) \\ &= \sum_{x \in V} (R_\mu(v,y), R_\nu(v,y) \\ &= outd(v). \end{aligned}$$

Hence, G is regular. □

From Theorem 3.6 and Theorem 3.7, we have.

Theorem 3.9. *Cayley intuitionistic fuzzy graphs are in-regular and out-regular.*

Theorem 3.10. *If for all $u,v \in V$, $ind_\mu(u) = \sum_{v \in V} \mu_B(v) = outd_\mu(u)$ and $ind_\nu(u) = \sum_{v \in V} \nu_B(v) = outd_\nu(u)$. Then Cayley intuitionistic fuzzy graphs are regular.*

Theorem 3.11. *Let $G = (V, R)$ be an intuitionistic fuzzy graph. Then intuitionistic fuzzy relation R is reflexive if and only if $R(1,1) = (1,0)$, that is, $\mu_A(1) = 1$ and $\nu_A(1) = 0$.*

Proof. R is reflexive if and only if $R(x,x) = (1,0)$ for all $x \in V$. Now

$$\begin{aligned} R(x,x) &= (\mu_A(x^{-1}x), \nu_A(x^{-1}x)) \\ &= (\mu_A(1), \nu_A(1)) \text{ for all x} \in V. \end{aligned}$$

Hence R is reflexive if and only if $\mu_A(1) = 1$ and $\nu_A(0) = 0$. □

Theorem 3.12. *Let $G = (V, R)$ be an intuitionistic fuzzy graph. Then intuitionistic fuzzy relation R is symmetric if and only if $(\mu_A(x), \nu_A(x)) = (\mu_A(x^{-1}), \nu_A(x^{-1}))$ for all $x \in V$.*

Proof. Suppose that R is symmetric. Then for any $x \in V$,

$$\begin{aligned}(\mu_A(x), \nu_A(x)) &= (\mu_A(x^{-1}x^2), \nu_A(x^{-1}x^2)) \\ &= R(x, x^2) = R(x^2, x), \text{since R is symmetric} \\ &= (\mu_A((x^2)^{-1}x), \nu_A(x^2)^{-1}x) \\ &= (\mu_A((x^{-2}x), \nu_A(x^{-2}x) \\ &= (\mu_A((x^{-1}), \nu_A(x^{-1})\end{aligned}$$

Conversely, suppose that $(\mu_A(x), \nu_A(x)) = (\mu_A(x^{-1}), \nu_A(x^{-1}))$ for all $x \in V$. Then for all $x, y \in V$,

$$\begin{aligned}R(x, y) &= (\mu_A(x^{-1}y), \nu_A(x^{-1}y)) \\ &= (\mu_A(y^{-1}x), \nu_A(y^{-1}x)) \\ &= R(y, x).\end{aligned}$$

Hence R is symmetric. □

Theorem 3.13. *An intuitionistic fuzzy relation R is anti symmetric if and only if $\{x : (\mu_A(x), \nu_A(x)) = (\mu_A(x^{-1}), \nu_A(x^{-1}))\} = \{(1, 1)\}$.*

Proof. Suppose that R is anti symmetric and let $x \in V$. Then

$$\begin{aligned}(\mu_R(x), \nu_R(x)) = (\mu_A(x^{-1}), \nu_A(x^{-1}))\text{which implies } R(1, x) &= R(x, 1) \\ \text{Hence } x = 1, [\text{since R is anti symmetric.}]\end{aligned}$$

Conversely, suppose that $\{x : (\mu_A(x), \nu_A(x)) = (\mu_A(x^{-1}), \nu_A(x^{-1}))\} = \{(1, 0)\}$. Then for any $x, y \in V$, $R(x, y) = R(y, x) \Leftrightarrow (\mu_A(x^{-1}y), \nu_A(y^{-1}x))$. This implies that $(\mu_A(x^{-1}y), \nu_A(y^{-1}x)) = (\mu_A((x^{-1}y)^{-1}), \nu_A((x^{-1}y)^{-1}))$. That is $x^{-1}y = 1$. Equivalently, $x = y$. Hence R is anti symmetric. □

Theorem 3.14. *An intuitionistic fuzzy relation R is transitive if and only if (μ_A, ν_A) is an intuitionistic fuzzy subsemigroup of $(G, *)$.*

Proof. Suppose that R is transitive and let $x, y \in V$. Then $R^2 \leq R$. Now for any $x \in V$, we have $R(1, x) = (\mu_A(x), \nu_A(x))$. This implies that $\vee\{R(1, z) \wedge R(z, xy)|z \in V\} = R^2(1, xy) \leq R(1, xy)$. That is $\vee\{\mu_R(z) \wedge \mu_R(z^{-1}xy)|z \in V\} \leq \mu_R(xy)$ and $\wedge\{\nu_R(z) \vee \nu_R(z^{-1}xy)|z \in V\} \geq \nu_R(xy)$. Hence $\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(xy) \leq \mu_A(x) \vee \nu_A(y)$. Hence (μ_A, ν_A) is an intuitionistic fuzzy subsemigroup of $(S, *)$.
Conversely, suppose that $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy subsemigroup of $(G, *)$. That is, for all $x, y \in V \mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(xy) \leq$

$\nu_A(x) \vee \nu_A(y)$. Then for any $x, y \in V$,

$$\begin{aligned} R^2(x,y) &= (R^2_\mu(x,y), R^2_\nu(x,y)) \\ R^2_\mu(x,y) &= \vee\{R_\mu(x,z) \wedge R_\mu(z,y) | z \in V\} \\ &= \vee\{\mu_A(x^{-1}z) \wedge \mu_A(z^{-1}y) | z \in V\} \\ &\leq \mu_A(x^{-1}y) \\ &= R_\mu(x,y). \\ R^2_\nu(x,y) &= \wedge\{R_\nu(x,z) \vee R_\nu(z,y) : z \in V\} \\ &= \wedge\{\nu_A(x^{-1}z) \vee \nu_A(z^{-1}y) : z \in V\} \\ &\geq \nu_A(x^{-1}y) \\ &= R_\nu(x,y). \end{aligned}$$

Hence $R^2_\mu(x,y) \leq R_\mu(x,y)$ and $R^2_\nu(x,y) \geq R_\nu(x,y)$. Hence R is transitive. □

We conclude that:

Theorem 3.15. *An intuitionistic fuzzy relation R is a partial order if and only if $A = (\mu_B, \nu_B)$ is an intuitionistic fuzzy subsemigroup of $(V, *)$ satisfying*

(i) $\mu_A(1) = 1$ *and* $\nu_A(1) = 0$,
(ii) $\{x : (\mu_A(x), \nu_A(x)) = (\mu_A(x^{-1}), \nu_A(x^{-1}))\} = \{(1, 0)\}$.

Theorem 3.16. *An intuitionistic fuzzy relation R is a linear order if and only if (μ_B, ν_B) is an intuitionistic fuzzy subsemigroup of $(V, *)$ satisfying*

(i) $\mu_A(1) = 1$ *and* $\nu_A(1) = 0$,
(ii) $\{x | (\mu_A(x), \nu_A(x)) = (\mu_A(x^{-1}), \nu_A(x^{-1}))\} = \{(1, 0)\}\}$,
(ii) $R^2 \leq R$, *that is,*

$$\{x, y \mid \mu_R(x,y) \geq \mu_{R\circ R}(x,y), \nu_R(x,y) \leq \nu_{R\circ R}(x,y)` \mid x, y \in V\},$$

(iv) $\{x \mid \mu_A(x) \vee \mu_A(x^{-1}) > 0, \nu_A(x) \wedge \nu_A(x^{-1}) > 0\}$.

Proof. Suppose R is a linear order. Then by Theorem 3.15, the conditions (i),(ii) and (iii) are satisfied. For any $x \in V$, $(R \vee R^{-1})(1, x) > 0$. This implies that $R(1, x) \vee R(x, 1) > 0$. Hence $\{x : \mu_A(x) \vee \mu_A(x^{-1}) > 0, \nu_A(x) \wedge \nu_A(x^{-1}) > 0\}$. Conversely, suppose that the conditions (i), (ii) and (iii) hold. Then by Theorem 3.15, R is partial order. Now for any $x, y \in V$, we have $(x^{-1}y)$, $(y^{-1}x) \in V$. Then by condition (iv), $\{x : \mu_A(x) \vee \mu_A(x^{-1}) > 0, \nu_A(x) \wedge \nu_A(x^{-1}) > 0\}$. That is $R(1, x) \vee R(x, 1) > 0$. Hence $(R \vee R^{-1})(x, y) > 0$. Therefore R is linear order. □

Theorem 3.17. *An intuitionistic fuzzy relation R is a equivalence relation if and only if (μ_A, ν_A) is an intuitionistic fuzzy sub semigroup of $(G, *)$ satisfying*

(i) $\mu_A(1) = 1$ *and* $\nu_A(1) = 0$,
(ii) $(\mu_A(x), \nu_A(x)) = (\mu_A(x^{-1}), \nu_A(x^{-1}))$ *for all* $x \in V$.

Proof. Proof follows from Theorem 3.15 and Theorem 3.16. □

4. Cayley graphs induced by Cayley intuitionistic fuzzy graphs

Definition 4.1. Let $(V,*)$ be a group, let A be an intuitionistic fuzzy set of V and $G=(V,R)$ be the Cayley intuitionistic fuzzy graph induced by $(V,*,A)$. For any $\alpha\in[0,1]$, let A_α be $\alpha-$ cut of A and A_α^+ be the strong $\alpha-$ cut of A. We define S_{A_α} and $S_{A_\alpha^+}$ as $S_{A_\alpha}=\{(x,y)\in V\times V : x^{-1}y\in A_\alpha\}$,
$S_{A_\alpha^+}=\{(x,y)\in V\times V : x^{-1}y\in A_\alpha^+\}$. Then it is clear that for any $\alpha\in[0,1]$, the Cayley intuitionistic fuzzy graph induced by $(V,*,A)$ induces the Cayley graphs (V,S_{A_α}) and $(V,S_{A_\alpha^+})$.

Note that for any $\alpha\in[0,1]$, $S_{A_\alpha}=R_\alpha$ and $S_{A_\alpha^+}=R_\alpha^+$. Thus for any $\alpha\in[0,1]$, the Cayley intuitionistic fuzzy graph (V,R) induced by Cayley intuitionistic graphs (V,R_α) and (V,R_α^+).

Remark 4.1. Let $G=(V,R)$ be any intuitionistic fuzzy graph, then G is connected(weakly connected, semi-connected, locally connected or quasi connected) if and only if the induce fuzzy graph (V,R_0^+) is connected(weakly connected, semi-connected, locally connected or quasi connected).

We now observe the following definition and lemma to study different types of connectedness of G.

Definition 4.2. Let $(L,*)$ be a semigroup and let $A=(\mu_A,\nu_A)$ be an intuitionistic fuzzy subset of L. Then the subsemigroup generated by A is the meet of all intuitionistic fuzzy subsemigroups of L which contains A. It is denoted by $< A >$.

Example 4.3. Consider $L=Z_3$ and $A=(\mu_A,\nu_A)$ as in Example 3.5. Then $< A >$ is given by $<\mu_A>(0)=1$, $<\nu_A>(0)=0$, and $<\mu_A>(y)=0.5$, $<\nu_A>(y)=0.5$ for $y=1,2$.

Lemma 4.4. *Let $(L,*)$ be a semigroup and $A=(\mu_A,\nu_A)$ be an intuitionistic fuzzy subset of L. Then intuitionistic fuzzy subset $< A >$ is precisely given by* $<\mu_A>(x)=\vee\{\mu_A(x_1)\wedge\mu_A(x_2)\wedge\ \ldots\ \wedge\mu_A(x_n) : x=x_1x_2\ \ldots\ x_n \text{ with } \mu_A(x_i)>0\ for\ i=1,2,\ \ldots\ ,n\}$, $<\nu_A>(x)=\wedge\{\nu_A(x_1)\vee\nu_A(x_2)\vee\ \ldots\ \vee\nu_A(x_n) : x=x_1x_2\ \ldots\ x_n \text{ with } \nu_A(x_i)>0\ for\ i=1,2,\ \ldots\ ,n\}$ *for any $x\in L$.*

Proof. Let $A'=(\mu_A',\nu_A')$ be an intuitionistic fuzzy subset of L defined by $\mu_A'(x)=\vee\{\mu_A(x_1)\wedge\mu_A(x_2)\wedge\ \ldots\ \wedge\mu_A(x_n) : x=x_1x_2\ \ldots\ x_n\ with\ \mu_A(x_i)>0\ for\ i=1,2,\ \ldots\ ,n\}$, $\nu_A'(x)=\wedge\{\nu_A(x_1)\vee\nu_A(x_2)\vee\ \ldots\ \vee\nu_A(x_n) : x=x_1x_2\ \ldots\ x_n \text{ with } \nu_A(x_i)>0\ for\ i=1,2,\ \ldots\ ,n\}$, for any $x\in L$. Let $x,y\in L$. If $\mu_A(x)=0\ or\ \mu_A(y)=0$, then $\mu_A(x)\wedge\mu_A(y)=0$ and $\nu_A(x)=0\ or\ \nu_A(y)=0$, then $\nu_A(x)\vee\nu_A(y)=0$. Therefore, $\mu_A'(xy)\geq\mu_A(x)\wedge\mu_A(y)$ and $\nu_A'(xy)\leq\nu_A(x)\vee\nu_A(y)$. Again , if $\mu_A(x)\neq 0,\nu_A(x)\neq 0$, then by definition of $\mu_A'(x)$ and $\nu_A'(x)$, we have $\mu_A'(xy)\geq\mu_A(x)\wedge\mu_A(y)$ and $\nu_A'(xy)\leq\nu_A(x)\vee\nu_A(y)$. Hence (μ_A',ν_A') is an intuitionistic fuzzy subsemigroup of L containing (μ_A,ν_A). Now let L' be any intuitionistic fuzzy subsemigroup of L' containing (μ_A,ν_A). Then for any $x\in L$ with $x=x_1x_2\ \ldots\ x_n\ with\ \mu_A(x_i)>0,\ \nu_A(x_i)>0,$

for $i = 1, 2, \dots, n$, we have $\mu_{L'}(x_i) \geq \mu_{L'}(x_1) \wedge \mu_{L'}(x_2) \wedge \dots \wedge \mu_{L'}(x_n) \geq \mu_A(x_1) \wedge \mu_A(x_2) \wedge \dots \wedge \mu_A(x_n)$ and $\nu_{L'}(x_i) \leq \nu_{L'}(x_1) \wedge \nu_{L'}(x_2) \wedge \dots \wedge \nu_{L'}(x_n) \leq \nu_A(x_1) \wedge \nu_A(x_2) \wedge \dots \wedge \nu_A(x_n)$. Thus $\mu_{L'}(x) \geq \vee\{\mu_A(x_1) \wedge \mu_A(x_2) \wedge \dots \wedge \mu_A(x_n) : x = x_1 x_2 \dots x_n \ with\ \mu_A(x_i) > 0 \ for\ i = 1, 2, \dots, n\}$, $\nu_{L'}(x) \leq \wedge\{\nu_A(x_1) \vee \nu_A(x_2) \vee \dots \vee \nu_A(x_n) : x = x_1 x_2 \dots x_n \ with\ \nu_A(x_i) > 0 \ for\ i = 1, 2, \dots, n\}$, for any $x \in L$. Hence $\mu_{L'}(x) \geq \mu'_A(x), \nu_{L'}(x) \leq \nu'_A(x)$, for all $x \in L$. Thus $\mu'_A(x) \leq \mu_{L'}(x)$, $\nu'_A(x) \geq \mu_A(x)$. Thus $A' = (\mu'_A, \nu'_A)$ is the meet of all intuitionistic fuzzy subsemigroup containing (μ_A, ν_A). □

Theorem 4.5. *Let $(L, *)$ be a semigroup and $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy subset of L. Then for any $\alpha \in [0, 1]$, $(<\mu_\alpha>, <\nu_\alpha>) = (<\mu>_\alpha, <\nu>_\alpha)$ and $(<\mu^+_\alpha>, <\nu^+_\alpha>) = (<\mu>^+_\alpha, <\nu>^+_\alpha)$, where $(<\mu_\alpha>, <\nu_\alpha>)$ denotes the subsemigroup generated by (μ_α, ν_α) and $<(\mu, \nu)>$ denotes intuitionistic fuzzy subsemigroup generated by (μ, ν).*

Proof.

$$\begin{aligned} x \in (<\mu>_\alpha, <\nu>_\alpha) &\Leftrightarrow \text{there exists } x_1, x_2, \dots, x_n \ in\ (\mu_\alpha, \nu_\alpha) \text{ such that} \\ &\quad x = x_1 x_2 \dots x_n \\ &\Leftrightarrow \text{ there exists } x_1, x_2, \dots, x_n \text{ in L such that } \mu(x_i) \geq \alpha, \\ &\quad \nu(x_i) \leq \alpha, \text{for all } i = 1, 2, \dots, n \text{ and } x = x_1 x_2 \dots x_n \\ &\Leftrightarrow <\mu>(x) \geq \alpha \text{ and } <\nu>(x) \leq \alpha \\ &\Leftrightarrow x \in <\mu>_\alpha \text{ and } x \in <\nu>_\alpha . \end{aligned}$$

Therefore, $(<\mu_\alpha>, <\nu_\alpha>) = (<\mu>_\alpha, <\nu>_\alpha)$. Similarly, we have $(<\mu^+_\alpha>, <\nu^+_\alpha>) = (<\mu>^+_\alpha, <\nu>^+_\alpha)$. □

Remark 4.2. Let $(L, *)$ be a semigroup and $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy subset of L. Then by Theorem 4.5, we have $< supp(A) >= supp < A >$.

5. Connectedness in Cayley intuitionistic fuzzy graphs

In this section, first we state the the basic Theorems which are used to prove the forthcoming Theorems.
Let G denotes the Cayley intuitionistic fuzzy graphs $G = (V, R)$ induced by $(V, *, A)$ and $G' = (V', R')$ be the crisp Cayley graph induced by $(V', *, A)$. Then we conclude the following results.

Theorem 5.1. *Let A be any subset of V' and $G' = (V', R')$ be the Cayley graph induced by $(V', *, A)$. Then G' is connected if and only if $< A > \supseteq V - v_1$.*

Theorem 5.2. *Let A be any subset of a set V' and $G' = (V', R')$ be the Cayley graph induced by the triplet $(V', *, A)$. Then G' is weakly connected if and only if $< A \cup A^{-1} > \supseteq V - v_1$, where $A^{-1} = \{x^{-1} : x \in A\}$.*

Theorem 5.3. *Let A be any subset of a set V' and $G' = (V', R')$ be the Cayley graph induced by the triplet $(V', *, A)$. Then G' is semi-connected if and only if $< A > \cup < A^{-1} > \supseteq V - v_1$, where $A^{-1} = \{x^{-1} : x \in A\}$.*

Theorem 5.4. *Let $G' = (V', R')$ be the Cayley graph induced by the triplet $(V', *, A)$. Then G' is locally connected if and only if $< A >=< A^{-1} >$, where $A^{-1} = \{x^{-1} : x \in A\}$.*

Theorem 5.5. *Let $G' = (V', R')$ be the Cayley graph induced by the triplet $(V', *, A)$, where V' is finite. Then G' is quasi connected if and only if it is connected.*

Definition 5.6. Let $(L, *)$ be a group and A be an intuitionistic fuzzy subset of L. Then we define A^{-1} as intuitionistic fuzzy subset of L given by $A^{-1}(x) = A(x^{-1})$ for all $x \in L$.

Theorem 5.7. *G is connected if and only if $supp < A > \supseteq V - v_1$.*

Proof. By Remark 4.1,4.2 and by Theorem 5.1,

$$\begin{aligned} G \text{ is connected} &\Leftrightarrow (V, R_0^+)\text{is connected} \\ &\Leftrightarrow < A_0^+ > \supseteq V - v_1 \\ &\Leftrightarrow < supp(A) > \supseteq V - v_1 \\ &\Leftrightarrow supp < A > \supseteq V - v_1. \end{aligned}$$

□

Theorem 5.8. *G is weakly connected if and only if $supp\ (< A \cup A^{-1} >) \supseteq V - v_1$.*

Proof. By Remark 4.1,4.2 and by Theorem 5.2,

$$\begin{aligned} G \text{ is weakly connected} &\Leftrightarrow (V, R_0^+)\text{is weakly connected} \\ &\Leftrightarrow < A_0^+ \cup (A_0^+)^{-1} > \supseteq V - v_1 \\ &\Leftrightarrow < supp(A) \cup supp(A)^{-1} > \supseteq V - v_1 \\ &\Leftrightarrow supp < A \cup (A)^{-1} > \supseteq V - v_1 \\ &\Leftrightarrow supp < A \cup A^{-1} > \supseteq V - v_1. \end{aligned}$$

□

Theorem 5.9. *G is semi- connected if and only if $supp\ (< A > \cup < A^{-1} >) \supseteq V - v_1$.*

Proof. By Remark 4.1,4.2 and by Theorem 5.3,

$$\begin{aligned} G \text{ is semi- connected} &\Leftrightarrow (V, R_0^+)\text{is semi connected} \\ &\Leftrightarrow < A_0^+ > \cup < (A_0^+)^{-1} > \supseteq V - v_1 \\ &\Leftrightarrow < supp(A) > \cup < supp(A)^{-1} > \supseteq V - v_1 \\ &\Leftrightarrow supp < A > \cup < (A)^{-1} > \supseteq V - v_1 \\ &\Leftrightarrow supp(< A > \cup < A^{-1} >) \supseteq V - v_1. \end{aligned}$$

□

Theorem 5.10. *Let G is locally connected if and only if supp$(< A >) =$ supp$(< A^{-1} >)$.*

Proof. By Remark 4.1,4.2 and by Theorem 5.4,

$$\begin{aligned} G \text{ is locally connected} &\Leftrightarrow (V, R_0^+) \text{ is locally connected} \\ &\Leftrightarrow < A_0^+ > = < (A_0^+)^{-1} > \\ &\Leftrightarrow < supp(A) > = < supp(A)^{-1} > \\ &\Leftrightarrow supp < A > = supp < A^{-1} > \end{aligned}$$

□

Theorem 5.11. *A finite Cayley intuitionistic fuzzy graph G is quasi connected if and only if it is connected.*

Proof. By Remark 4.1,4.2 and by Theorem 5.5,

$$\begin{aligned} G \text{ is quasi-connected} &\Leftrightarrow (V, R_0^+) \text{ is quasi-connected} \\ &\Leftrightarrow (V, R_0^+) \text{ is connected} \\ &\Leftrightarrow \text{ G is connected.} \end{aligned}$$

□

6. Different types of α-connectedness in Cayley intuitionistic fuzzy graphs

Definition 6.1. The $\mu-$*strength* of a path $P = v_1, v_2, \ldots, v_n$ is defined as $\min(\mu_2(v_i, v_j))$ for all i and j and is denoted by S_μ. The $\nu-$*strength* of a path $P = v_1, v_2, \ldots, v_n$ is defined as $\max(\nu_2(v_i, v_j))$ for all i and j and is denoted by S_ν. The strength of $P = \{\mu - strength, \mu - strength\}$

Definition 6.2. Let G be an intuitionistic fuzzy digraph. Then G is said to be:

(i) $\alpha-$*connected* if for every pair of vertices $x, y \in G$, there is a path P from x to y such that strength $(P) \geq \alpha$,
(ii) *weakly* $\alpha-$*connected* if an intuitionistic fuzzy graph $(V, R \vee R^{-1})$ is $\alpha-$connected,
(iii) *semi* $\alpha-$*connected* if for every $x, y \in V$, there is a path of strength greater than or equal to α from x to y or from y to x in G,
(iv) *locally* $\alpha-$*connected* if for every pair of vertices x and y, there is a path P of strength greater than or equal to α from x to y whenever there is a path P' of strength greater than or equal to α from y to x,
(v) *quasi* $\alpha-$*connected* if for every pair $x, y \in V$, there is some $z \in V$ such that there is directed path from z to x of strength greater than or equal to α and there is a directed path from z to y of strength greater than or equal to α.

Remark 6.1. Let $G = (V, R)$ be any intuitionistic fuzzy graph, then G is α-connected (weakly α-connected, semi α-connected, locally α-connected or quasi α-connected) if and only if the induce intuitionistic fuzzy graph (V, R_0^+) is connected(weakly connected, semi-connected, locally connected or quasi connected).

Let G denotes the Cayley intuitionistic fuzzy graphs $G = (V, R)$ induced by $(V, *, A)$. Also for any $\alpha \in [0, 1]$, then we have the following results.

Theorem 6.3. *G is $\alpha-$connected if and only if $< A >_\alpha \supseteq V - v_1$.*

Proof. By Remark 6.1 and by Theorems 4.5, 5.7,

$$\begin{aligned} G \text{ is connected} &\Leftrightarrow (V, R_\alpha)\text{is connected} \\ &\Leftrightarrow < A_\alpha > \supseteq V - v_1 \\ &\Leftrightarrow < A >_\alpha \supseteq V - v_1. \end{aligned}$$

□

Theorem 6.4. *G is weakly $\alpha-$connected if and only if $< A \cup A^{-1} >_\alpha \supseteq V - v_1$.*

Proof. By Remark 6.1 and by Theorems 4.5, 5.8,

$$\begin{aligned} G \text{ is weakly connected} &\Leftrightarrow (V, R_\alpha)\text{is weakly connected} \\ &\Leftrightarrow < A_\alpha \cup (A_\alpha)^{-1} > \supseteq V - v_1 \\ &\Leftrightarrow < (A \cup A^{-1})_\alpha > \supseteq V - v_1 \\ &\Leftrightarrow < A \cup (A)^{-1} >_\alpha \supseteq V - v_1. \end{aligned}$$

□

Theorem 6.5. *G is semi $\alpha-$connected if and only if $(< A >_\alpha \cup < A^{-1} >_\alpha) \supseteq V - v_1$.*

Proof. By Remark 6.1 and by Theorems 4.5, 5.9,

$$\begin{aligned} G \text{ is semi } \alpha\text{−connected} &\Leftrightarrow (V, R_0^+) \text{ is semi } \alpha\text{−connected} \\ &\Leftrightarrow < A_\alpha > \cup < A_\alpha^{-1} > \supseteq V - v_1 \\ &\Leftrightarrow < A >_\alpha \cup < A^{-1} >_\alpha \supseteq V - v_1. \end{aligned}$$

□

Theorem 6.6. *Let G is locally $\alpha-$connected if and only if $< A >_\alpha = < A_\alpha^{-1} >$.*

Proof. By Remark 6.1 and by Theorems 4.5, 5.10,

$$\begin{aligned} G \text{ is locally } \alpha\text{−connected} &\Leftrightarrow (V, R_{\alpha-})\text{is locally connected} \\ &\Leftrightarrow < A_\alpha > = < A_\alpha^{-1} > \\ &\Leftrightarrow < A >_\alpha = < A^{-1} >_\alpha . \end{aligned}$$

□

Theorem 6.7. *A finite Cayley intuitionistic fuzzy graph G is quasi $\alpha-$connected if and only if it is $\alpha-$connected.*

Proof. By Remark 6.1 and by Theorems 4.5, 5.11,

$$\begin{aligned} G \text{ is quasi } \alpha\text{−connected} &\Leftrightarrow (V, R_\alpha^+) \text{is quasi−connected} \\ &\Leftrightarrow (V, R_\alpha^+) \text{ is connected} \\ &\Leftrightarrow \text{G is } \alpha\text{−connected.} \end{aligned}$$

This completes the proof. □

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