J. Appl. Math. & Informatics Vol. **32***(2014), No. 5 - 6, pp. 817 - 825*
http://dx.doi.org/10.14317/jami.2014.817

STABILITY OF A CUBIC FUNCTIONAL EQUATION IN 2-NORMED SPACES

CHANG IL KIM AND KAP HUN JUNG*

Abstract. In this paper, we prove the generalized Hyers-Ulam stability of the following cubic funtional equation

$$f(2x+y)+f(2x-y)=2f(x+2y)-4f(x+y)+18f(x)-12f(y)$$

by the direct method in 2-normed spaces.

AMS Mathematics Subject Classification : 39B52, 39B72, 39B82.
Key words and phrases : Hyers-Ulam stability, 2-Banach spaces, cubic funtional equation.

1. Introduction and preliminaries

Gähler [4, 5] has introduced the concept of 2-normed spaces and Gähler and White [16] introduced the concept of 2-Banach spaces. Lewandowska published a series of papers on 2-normed sets and generalized 2-normed spaces [10, 11]. Recently, Park [12] investigated approximate additive mappings, approximate Jensen mappings and approximate quadratic mappings in 2-Banach spaces.

We list some definitions related to 2-normed spaces.

Definition 1.1. Let X be a linear space over $\mathbb{R}$ with dim $X > 1$ and let $\|\cdot,\cdot\| : X \times X \to \mathbb{R}$ be a function satisfying the following properties :
(1) $\|x, y\| = 0$ if and only if x and y are linearly dependent,
(2) $\|x, y\| = \|y, x\|$,
(3) $\|ax, y\| = |a|\|x, y\|$, and
(4) $\|x, y + z\| \le \|x, y\| + \|x, z\|$
for all $x, y, z \in X$ and $a \in \mathbb{R}$. Then the function $\|\cdot,\cdot\|$ is called *a 2-norm on X* and $(X, \|\cdot,\cdot\|)$ is called a *2-normed space*.

Let $(X, \|\cdot,\cdot\|)$ be a 2-normed space. Suppose that $x \in X$ and $\|x, y\| = 0$ for all $y \in X$. Suppose that $x \neq 0$. Since dim $X > 1$, choose $y \in X$ such that $\{x, y\}$

Received November 10, 2013. Revised January 28, 2014. Accepted February 27, 2014.
*Corresponding author.

is linearly independent and so by (1) in Definition1.1, we have $\|x, y\| \neq 0$, which is a contradiction. Hence we have the following lemma.

Lemma 1.2. *Let $(X, \|\cdot,\cdot\|)$ be a 2-normed space. If $x \in X$ and $\|x, y\| = 0$ for all $y \in X$, then $x = 0$.*

Definition 1.3. A sequence $\{x_n\}$ in a 2-normed space $(X, \|\cdot,\cdot\|)$ is called *a 2-Cauchy sequence* if

$$\lim_{n,m\to\infty} \|x_n - x_m,\ x\| = 0$$

for all $x \in X$.

Definition 1.4. A sequence $\{x_n\}$ in a 2-normed space $(X, \|\cdot,\cdot\|)$ is called *2-convergent* if

$$\lim_{n\to\infty} \|x_n - x, y\| = 0$$

for all $y \in X$ and for some $x \in X$. In case, $\{x_n\}$ said to be *converge to* x and denoted by $x_n \to x$ *as* $n \to \infty$ or $\lim_{n\to\infty} x_n = x$.

A 2-normed space $(X, \|\cdot,\cdot\|)$ is called a *2-Banach space* if every 2-Cauchy sequence in X is 2-convergent. Now, we state the following results as lemma [12].

Lemma 1.5. *Let $(X, \|\cdot,\cdot\|)$ be a 2-normed space. Then we have the following :*
(1) $|\|x, z\| - \|y, z\|| \leq \|x - y, z\|$ *for all* $x, y, z \in X$,
(2) *if* $\|x, z\| = 0$ *for all* $z \in X$, *then* $x = 0$, *and*
(3) *for any 2-convergent sequence* $\{x_n\}$ *in* X,

$$\lim_{n\to\infty} \|x_n, z\| = \|\lim_{n\to\infty} x_n, z\|$$

for all $z \in X$.

In 1940, S.M.Ulam [15] proposed the following stability problem :

"Let G_1 be a group and G_2 a metric group with the metric d. Given a constant $\delta > 0$, does there exists a constant $c > 0$ such that if a mapping $f : G_1 \to G_2$ satisfies $d(f(xy), f(x)f(y)) < 0$ for all $x, y \in G_1$, then there exists a unique homomorphism $h : G_1 \to G_2$ with $d(f(x), h(x)) < \delta$ for all $x \in G_1$?"

In the next year, D. H. Hyers [7] gave a partial solution of Ulam's problem for the case of approximate additive mappings. Subsequently, his result was generalized by T. Aoki [1] for additive mappings and by TH. M. Rassias [14] for linear mappings, to consider the stability problem with unbounded Cauchy differences. During the last decades, the stability problems of funtional equations have been extensively investigated by a number of mathematicians.

Rassias [13] introduced the cubic functional equation

$$f(2x + y) - 3f(x + y) + 3f(x) - f(x - y) = 6f(y) \tag{1}$$

and Jun and Kim [8] introduced the following cubic funtional equation

$$f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x) \tag{2}$$

In this paper, we inverstigate the following cubic funtional equation

$$f(2x+y)+f(2x-y)=2f(x+2y)-4f(x+y)+18f(x)-12f(y) \quad (3)$$

which is a linear combination of (1) and (2) and proved the generalized Hyers-Ulam stability of (3) in 2-normed spaces.

2. Stability of (3) from a normed space to a 2-Banach space

Thoughout this section, $(X,\|\cdot\|)$ or simply X is a real normed space and $(Y,\|\cdot,\cdot\|)$ or simply Y is a 2-Banach space. We start the following theorem.

Theorem 2.1. *A mapping $f : X \to Y$ satisfies* (3) *if and only if f is cubic.*

Proof. Suppose that f satisfies (3). Letting $x=y=0$ in (3), we have $f(0)=0$ and letting $y=0$ in (3), we have

$$f(2x)=8f(x) \quad (4)$$

for all $x\in X$. Letting $x=0$ in (3), by (4), we have $f(y)=-f(-y)$ for all $y\in X$ and so f is odd. Letting $y=-y$ in (3), we have

$$f(2x-y)+f(2x+y)-2f(x-2y)+4f(x-y)-18f(x)-12f(y)=0 \quad (5)$$

for all $x,y\in X$ and by (3) and (5), we have

$$\begin{aligned} &f(2x+y)+f(2x-y)-f(x+2y)-f(x-2y)+2f(x+y)\\ &+2f(x-y)-18f(x)=0 \end{aligned} \quad (6)$$

for all $x,y\in X$. Hence by (3) and (6), we have

$$f(x+2y)-f(x-2y)-2f(x+y)+2f(x-y)-12f(y)=0 \quad (7)$$

for all $x,y\in X$. Interching x and y in (7), since f is odd, f satisfies (2) and hence f is cubic. □

For any function $f : X\to Y$, we define the difference operator $D_f : X\times X\to Y$ by

$$D_f(x,y)=f(2x+y)+f(2x-y)-2f(x+2y)+4f(x+y)-18f(x)+12f(y).$$

Now we prove the generalized Hyers-Ulam stability of (3).

Theorem 2.2. *Let $\varepsilon\ge 0$, p and q be positive real numbers with $p+q<3$ and $r>0$. Suppose that $f : X\to Y$ is a function such that*

$$\|D_f(x,y),z\| \ \le\ \varepsilon\ (\|x\|^p\|y\|^q+\|x\|^p+\|y\|^q)\|z\|^r \quad (8)$$

for all $x,y\in X$ and $z\in Y$. Then there exists a unique cubic function $C : X\to Y$ satisfying (3) *and*

$$\|f(x)-C(x),z\| \ \le\ \frac{\varepsilon\|x\|^p\|z\|^r}{2(8-2^p)} \quad (9)$$

for all $x\in X$ and $z\in Y$.

Proof. Letting $x = y = 0$ in (8), we have $\|2f(0), z\| = 0$ for all $z \in Y$ and by the definition of 2-norm, we have $f(0) = 0$. Putting $y = 0$ in (8), we have

$$\|2f(2x) - 16f(x), z\| \ \leq \varepsilon \|x\|^p \|z\|^r \tag{10}$$

for all $x \in X$ and $z \in Y$ and so

$$\left\| \frac{f(2x)}{8} - f(x), z \right\| \ \leq \ \frac{\varepsilon}{16} \ \|x\|^p \|z\|^r \tag{11}$$

for all $x \in X$ and $z \in Y$. Replacing x by $2x$ in (11), we get

$$\left\| \frac{f(4x)}{8} - f(2x), z \right\| \ \leq \ \frac{2^p \varepsilon}{16} \ \|x\|^p \|z\|^r \tag{12}$$

for all $x \in X$ and $z \in Y$. By (11) and (12), we get

$$\begin{aligned} \left\| \frac{f(4x)}{8^2} - f(2x), z \right\| \ &\leq \ \left\| \frac{f(4x)}{8^2} - \frac{f(2x)}{8}, z \right\| + \left\| \frac{f(2x)}{8} - f(x), z \right\| \\ &= \ \frac{\varepsilon}{16} \left[1 + \frac{2^p}{8} \right] \|x\|^p \|z\|^r \end{aligned}$$

for all $x \in X$ and $z \in Y$. By induction on n, we can show that

$$\left\| \frac{f(2^n x)}{8^n} - f(x), z \right\| \ \leq \ \frac{\varepsilon}{16} \ \frac{1 - 2^{(p-3)n}}{1 - 2^{p-3}} \ \|x\|^p \|z\|^r \tag{13}$$

for all $x \in X$ and $z \in Y$. For $m, n \in \mathbb{N}$ with $n < m$ and $x \in X$, by (13), we have

$$\begin{aligned} \left\| \frac{f(2^m x)}{8^m} - \frac{f(2^n x)}{8^n}, z \right\| \ &= \ \frac{1}{8^n} \left\| \frac{f(2^{m-n} 2^n x)}{8^{m-n}} - f(2^n x) \ , \ z \right\| \\ &\leq \ \frac{\varepsilon}{16} \ \frac{2^{(p-3)n}(1 - 2^{(p-3)(m-n)})}{1 - 2^{p-3}} \ \|x\|^p \|z\|^r. \end{aligned} \tag{14}$$

Since $p < 3$, $\{\frac{f(2^n x)}{8^n}\}$ is a 2- Cauchy sequence in Y for all $x \in X$. Since Y is a 2-Banach space, the sequence $\{\frac{f(2^n x)}{8^n}\}$ is a 2-convergent in Y for all $x \in X$ and so we can define a mapping $C : X \to Y$ as

$$C(x) = \lim_{n \to \infty} \ \frac{f(2^n x)}{8^n}$$

for all $x \in X$. By (14), we have

$$\lim_{n \to \infty} \left\| \frac{f(2^n x)}{8^n} - f(x), z \right\| \ \leq \ \frac{\varepsilon}{16} \ \|x\|^p \|z\|^r \frac{1}{1 - 2^{p-3}}$$

for all $x \in X$ and $z \in Y$ and by Lemma 1.5 , we have

$$\|C(x) - f(x), z\| \ \leq \ \frac{\varepsilon \|x\|^p \ \|z\|^r}{2(8 - 2^p)}$$

for all $x \in X$ and $z \in Y$. Next we will show that C satisfies (3). By (8), we have

$$\begin{aligned} \|D_C(x, y), z\| &= \lim_{n \to \infty} \frac{1}{8^n} \ \|D_f(2^n x, 2^n y), z\| \\ &\leq \lim_{n \to \infty} \varepsilon \left[2^{(p+q-3)n} \|x\|^p \|y\|^q + 2^{(p-3)n} \|x\|^p + 2^{(q-3)n} \|y\|^q \right] \|z\|^r = 0 \end{aligned}$$

for all $z \in Y$, because $p < 3$, $q < 3$, $p+q < 3$ and so $D_C(x,y) = 0$ for all $x, y \in X$. By Theoem 2.1, C is cubic.

To show that C is unique, suppose there exists another cubic function $C' : X \to Y$ which satisfies (3) and (9). Since C and C' are cubic, $C(x) = \frac{C(2^n x)}{8^n}$ and $C'(x) = \frac{C'(2^n x)}{8^n}$ for all $x \in X$. It follows that

$$\begin{aligned} \|C'(x) - C(x), z\| &= \frac{1}{8^n}\ \|C'(2^n x) - C(2^n x), z\| \\ &\leq \frac{1}{8^n}\left[\|C'(2^n x) - f(2^n x), z\| + \|f(2^n x) - C(2^n x), z\|\right] \\ &\leq \frac{2^{(p-3)n}\ \varepsilon\|x\|^p\|z\|^r}{8-2^p} \quad \longrightarrow 0 \ \ as \ \ n \longrightarrow \infty. \end{aligned}$$

So $\| C'(x) - C(x), z \| = 0$ for all $z \in Y$ and hence $C'(x) = C(x)$ for all $x \in X$. □

Related with Theorem 2.2, we can also the following theorem.

Theorem 2.3. *Let $\varepsilon \geq 0$, p and q be positive real numbers with $p, q > 3$ and $r > 0$. Suppose that $f : X \to Y$ is a function satisfying* (8)*. Then there exists a unique cubic function $C : X \to Y$ satisfying* (3) *and*

$$\|f(x) - C(x), z\| \ \leq \ \frac{\varepsilon\|x\|^p\|z\|^r}{2(2^p - 8)} \tag{15}$$

for all $x \in X$ and $z \in Y$.

Proof. Letting $x = y = 0$ in (8), we have $\|2f(0), z\| = 0$ for all $z \in Y$ and so we have $f(0) = 0$. Putting $y = 0$ and replacing x by $\frac{x}{2}$ in (8), we get

$$\left\|2f(x) - 16f\Big(\frac{x}{2}\Big), z\right\| \ \leq \ 2^{-p}\varepsilon\|x\|^p\|z\|^r$$

for all $x \in X$ and $z \in Y$ and so

$$\left\|8f\Big(\frac{x}{2}\Big) - f(x), z\right\| \ \leq \ 2^{-p-1}\varepsilon\|x\|^p\|z\|^r \tag{16}$$

for all $x \in X$ and $z \in Y$. Replacing x by $\frac{x}{2}$ in (16), we get

$$\left\|8f\Big(\frac{x}{4}\Big) - f\Big(\frac{x}{2}\Big), z\right\| \ \leq \ 2^{-2p-1}\varepsilon\|x\|^p\|z\|^r \tag{17}$$

for all $x \in X$ and $z \in Y$. By (16) and (17), we get

$$\begin{aligned} \left\|8^2 f\Big(\frac{x}{4}\Big) - f(x), z\right\| &\leq \left\|8^2 f\Big(\frac{x}{4}\Big) - 8f\Big(\frac{x}{2}\Big), z\right\| + \left\|8f\Big(\frac{x}{2}\Big) - f(x), z\right\| \\ &= \frac{\varepsilon}{2}\left[2^{-p} + 8\cdot 2^{-2p}\right]\|x\|^p\ \|z\|^r \end{aligned}$$

for all $x \in X$ and $z \in Y$. By induction on n, we can show that

$$\left\|8^n f\Big(\frac{x}{2^n}\Big) - f(x), z\right\| \ \leq \ \frac{\varepsilon}{2}\ \|x\|^p\ \|z\|^r\ \frac{2^{-p}(1 - 2^{(3-p)n})}{1 - 2^{3-p}} \tag{18}$$

for all $x \in X$ and $z \in Y$. For $m, n \in \mathbb{N}$ with $n < m$ and $x \in X$, by (18), we have

$$\begin{aligned} \left\|8^m f\Big(\frac{x}{2^m}\Big) - 8^n f\Big(\frac{x}{2^n}\Big), z\right\| &= 8^n \left\|8^{m-n} f\Big(\frac{x}{2^{m-n}}\ \frac{x}{2^n}\Big) - f\Big(\frac{x}{2^n}\Big),\ z\right\| \\ &\leq \frac{\varepsilon}{2}\ \|x\|^p\ \|z\|^r\ \frac{2^{(3-p)n-p}\Big(1 - 2^{(3-p)(m-n)}\Big)}{1 - 2^{3-p}} \end{aligned}$$

and since $p > 3$, $\{8^n f(\frac{x}{2^n})\}$ is a 2- Cauchy sequence in Y for all $x \in X$. Since Y is a 2-Banach space, the sequence $\{8^n f(\frac{x}{2^n})\}$ is a 2-convergent in Y for all $x \in X$. Define $C : X \to Y$ as

$$C(x) = \lim_{n\to\infty}\ 8^n f\Big(\frac{x}{2^n}\Big)$$

for all $x \in X$. By (18), we have

$$\lim_{n\to\infty} \left\|8^n f\Big(\frac{x}{2^n}\Big) - f(x), z\right\| \ \leq\ \frac{\varepsilon}{2}\ \|x\|^p\ \|z\|^r \frac{2^{-p}}{1 - 2^{3-p}}$$

for all $x \in X$ and $z \in Y$ and by Lemma 1.5, we have

$$\|C(x) - f(x), z\| \ \leq\ \frac{\varepsilon\|x\|^p\ \|z\|^r}{2(2^p - 8)}$$

for all $x \in X$ and $z \in Y$. Next we will show that C satisfies (3).

$$\begin{aligned} \|D_C(x,y), z\| &= \lim_{n\to\infty} 8^n \left\|D_f\Big(\frac{x}{2^n}, \frac{y}{2^n}\Big), z\right\| \\ &\leq \lim_{n\to\infty} \varepsilon\Big[2^{(3-p-q)n}\|x\|^p\|y\|^q + 2^{(3-p)n}\|x\|^p + 2^{(3-q)n}\|y\|^q\Big]\|z\|^r = 0 \end{aligned}$$

for all $z \in Y$, because $p, q > 3$ and so $D_C(x,y) = 0$ for all $x, y \in X$. By Theoem 2.1, C is cubic.

To show that C is unique, suppose there exists another cubic function $C' : X \to Y$ which satisfies (3) and (15). Since C and C' are cubic, $C(x) = 8^n C(\frac{x}{2^n})$ and $C'(x) = 8^n C'(\frac{x}{2^n})$ for all $x \in X$. It follows that

$$\begin{aligned} \|C'(x) - C(x), z\| &= 8^n \left\|C'\Big(\frac{x}{2^n}\Big) - C\Big(\frac{x}{2^n}\Big), z\right\| \\ &\leq 8^n \Big[\ \left\|C'\Big(\frac{x}{2^n}\Big) - f\Big(\frac{x}{2^n}\Big), z\right\| + \left\|f\Big(\frac{x}{2^n}\Big) - C\Big(\frac{x}{2^n}\Big), z\right\|\ \Big] \\ &\leq\ \frac{2^{(3-p)n}\varepsilon\|x\|^p\|z\|^r}{2^p - 8} \quad \longrightarrow 0 \ \ as\ \ n \longrightarrow \infty. \end{aligned}$$

So $\| C'(x) - C(x), z \| = 0$ for all $z \in Y$ and hence $C'(x) = C(x)$ for all $x \in X$. □

3. Stability of (3) from a 2- normed space to a 2-Banach space

In this section, we study similar problems of (3). Let $(X, \|\cdot,\cdot\|)$ be a 2-normed space and $(Y, \|\cdot,\cdot\|)$ a 2- Banach space.

Theorem 3.1. *Let $\varepsilon \geq 0$ and p and q be positive real numbers with $p + q < 3$. Suppose that $f : X \to Y$ is a function such that*

$$\|D_f(x,y), z\| \ \leq\ \varepsilon\ (\|x,z\|^p\|y,z\|^q + \|x,z\|^p + \|y,z\|^q) \tag{19}$$

for all $x, y \in X$ *and* $z \in Y$*. Then there exists a unique cubic function* $C : X \to X$ *satisfying* (3) *and*

$$\|f(x) - C(x), z\| \ \le \ \frac{\varepsilon \|x, z\|^p}{2(8 - 2^p)} \tag{20}$$

for all $x \in X$ *and* $z \in Y$*.*

Proof. Letting $x = y = 0$ in (19). We have $\|2f(0), z\| = 0$ for all $z \in Y$, so we have $f(0) = 0$. Putting $y = 0$ in (19), we have

$$\|2f(2x) - 16f(x), z\| \ \le \ \varepsilon \|x, z\|^p$$

for all $x \in X$ and $z \in Y$. Therefore

$$\left\| \frac{f(2x)}{8} - f(x), z \right\| \ \le \ \frac{\varepsilon}{16} \ \|x, z\|^p \tag{21}$$

for all $x \in X$ and $z \in Y$. Replacing x by $2x$ in (21), we get

$$\left\| \frac{f(4x)}{8} - f(2x), z \right\| \ \le \ \frac{2^p \varepsilon}{16} \ \|x, z\|^p$$

for all $x \in X$ and $z \in Y$. By induction on n, we can show that

$$\left\| \frac{f(2^n x)}{8^n} - f(x), z \right\| \ \le \ \frac{\varepsilon}{16} \ \frac{1 - 2^{(p-3)n}}{1 - 2^{p-3}} \ \|x, z\|^p \tag{22}$$

for all $x \in X$ and $z \in Y$. For $m, n \in \mathbb{N}$ with $n < m$ and $x \in X$, by (22), we get

$$\begin{aligned} \left\| \frac{f(2^m x)}{8^m} - \frac{f(2^n x)}{8^n}, z \right\| \ &= \ \left\| \frac{f(2^{m-n+n} x)}{8^{m-n+n}} - \frac{f(2^n x)}{8^n} \ , \ z \right\| \\ &= \frac{1}{8^n} \left\| \frac{f(2^{m-n} x)}{8^{m-n}} - f(2^n x) \ , \ z \right\| \\ &\le \ \frac{\varepsilon}{16} \|x, z\|^p \frac{2^{(p-3)n} \Big(1 - 2^{(p-3)(m-n)} \Big)}{1 - 2^{p-3}} \end{aligned}$$

for all $x \in X$ and $z \in Y$. Since $p < 3$, $\{\frac{f(2^n x)}{8^n}\}$ is a 2- Cauchy sequence in Y for all $x \in X$. Since Y is a 2-Banach space, the sequence $\{\frac{f(2^n x)}{8^n}\}$ is a 2-convergent in Y for all $x \in X$. Define $C : X \to Y$ as

$$C(x) \ = \lim_{n \to \infty} \frac{f(2^n x)}{8^n}$$

for all $x \in X$ and by Lemma 1.5 and (22), we have (20). Next we show that C satisfies (3). By (19), we have

$$\begin{aligned} \|D_C(x, y), z\| \ &= \ \lim_{n \to \infty} \frac{1}{8^n} \|D_f(2^n x, 2^n y), z\| \\ &\le \ \lim_{n \to \infty} \varepsilon \Big[2^{(p+q-3)n} \|x, z\|^p \|x, z\|^q + 2^{(p-3)n} \|x, z\|^p + 2^{(q-3)n} \|x, z\|^q \Big] \ = \ 0 \end{aligned}$$

for all $z \in Y$ and so $D_C(x, y) = 0$ for all $x, y \in X$. By Theoem 2.1, C is cubic.

To show that C is unique, suppose that there exists another cubic function $C' : X \to Y$ which satisfies (3) and (20). Since C and C' are cubic, $C(x) = \frac{C(2^n x)}{8^n}$ and $C'(x) = \frac{C'(2^n x)}{8^n}$ for all $x \in X$. Since $p < 3$,

$$\begin{aligned}\|C'(x) - C(x), z\| &= \frac{1}{8^n}\|C'(2^n x) - C(2^n x), z\| \\ &\leq \frac{1}{8^n}\left[\|C'(2^n x) - f(2^n x), z\| + \|f(2^n x) - C(2^n x), z\|\right] \\ &\leq \frac{\varepsilon\ 2^{(p-3)n}\|x, z\|^p}{8 - 2^p} \quad \longrightarrow 0 \ \ as \ \ n \longrightarrow \infty.\end{aligned}$$

So $\| C'(x) - C(x), z \| = 0$ for all $z \in Y$ and hence $C'(x) = C(x)$ for all $x \in X$. □

Similar to Theorem 3.1, we have the following theorem.

Theorem 3.2. *Let $(X, \|\cdot,\cdot\|)$ be a 2- Banach space. Let $\varepsilon \geq 0$, p and q be positive real numbers with $p, q > 3$. Suppose that $f : X \to X$ is a function satisfying* (19). *Then there exists a unique cubic function $C : X \to X$ satisfying* (3) *and*

$$\|f(x) - C(x), z\| \ \leq \ \frac{\varepsilon\|x, z\|^p}{2(2^p - 8)}$$

for all $x, z \in X$.

References

1. T.Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan **2** (1950), 64-66.
2. S.Czerwik, *Stability of Functional Equations of Ulam-Hyers-Rassias Type*, Hadronic Press, Palm Harbor, Florida, 2003.
3. G. L. Forti, *Hyers-Ulam stability of functional equations in several variables*, Aequationes Math., **50** (1995), 143-190.
4. S.Gähler, *2-metrische Räumen und ihr topologische structure*, Math. Nachr., **26** (1963), 115-148.
5. S.Gähler, *Linear 2-normierte Räumen*, Math. Nachr., **28** (1964), 1-43.
6. P. Gavruta, *A generalization of the Hyers-Ulam-Rassias of approximately additive mappings* , J. Math. Anal. Appl.**184** (1994), 431-436.
7. D. H. Hyers, *On the stability of the linear functional equations* , Proc. Natl. Acad. Sci. USA **27** (1941), 222-224.
8. K. W. Jun and H. M. Kim, *The generalized Hyers-Ulam-Rassias stability of a cubic functional equation* , J. Math. Anal. Appl., **274** (2002), 867-878.
9. S. M. Jung, *Hyers-Ulam stability of Fibonacci functional equation* , Bull. Iranian Math. Soc., **35** (2009), 217-227.
10. Z. Lewandowska, *Ggeneralized 2-normed spaces*, Stuspske Prace Matematyczno-Fizyczne 1 (2001), no.4, 33-40.
11. Z. Lewandowska, *On 2-normed sets*, Glasnik Mat. Ser. III, **42** (58) (2003), no.1, 99-110.
12. W. G. Park, *approximate additive mapping in 2-Banach spaces and related topics* , J. Math. Anal. Appl., **376** (2011), 193-202.
13. T.M Rassias, *Solution of the stability problem for cubic functional equation* , Glasnik Matematički., **36** (2001), 63-72.

14. J.M Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc., **72** (1978), 297-300.
15. S. M. Ulams *A Collection of Mathematical Problems*, Interscience Tracts in Pure and Applied Mathematics, No. 8, Interscience, New York, NY, USA, 1960.
16. A. White, *2-Banach spaces*, Math. Nachr., **42** (1969), 43-60.

Chang Il Kim received M.Sc. from Sogang Uninversity and Ph.D at Sogang Uninersity. Since 1993 he has been at Dankook University. His research interests include general topology and functional analysis.

Department of Mathematics Education, Dankook University, 152, Jukjeon, Suji, Yongin, Gyeonggi, 448-701, Korea.
e-mail: kci206@hanmail.net

Kap Hun Jung received M.Sc. and Ph.D. from Dankook University. He is now teaching at Seoul National University of Science and Technology as a lecturer. His research interests include functional analysis.

School of Liberal Arts, Seoul National University of Science and Technology, Seoul 139-743, Korea.
e-mail: jkh58@hanmail.net