

ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF MATRIX LYAPUNOV INTEGRO DIFFERENTIAL EQUATIONS

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ABSTRACT. The asymptotic behavior of solutions of Lyapunov type matrix Volterra integro differential equation, in which the coefficient matrices are not stable, is studied by the method of reduction.

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1. Introduction

Integro differential equations are emerging into the main stream of research because of their wide applications in mathematical biology. Many mathematical models of biology can be represented through integro differential equations [5]. Recently the researcher [1],[2],[3],[4] & [11], studied different methods for solving the integro differential equations. The problem of stability of integro differential equations is studied by several methods like admissibility of integral operators, defining Lyapunov like functions and so on. Burton [4] studied the stability of the Volterra integro differential equation

$$X'(t) = A(t)X(t) + \int_0^t C(t,s)X(s)ds + f(t)$$

where X and f are n -vectors, A and C are $n \times n$ continuous matrices, by constructing a suitable Lyapunov function under various conditions on C and f . Burton also studied the stability of the Volterra integro differential equation in which A is a constant $n \times n$ matrix whose characteristic roots have negative real parts and the uniform stability of Volterra equations [5] and [6]. Grossman

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and Miller [7] studied the asymptotic behavior of solutions of Volterra integro differential system of the form

$$X'(t) = A(t)X(t) + \int_0^t B(t,s)X(s)ds + f(t) + g(X(t)), \quad X(0) = X_o,$$

as a perturbation of the linear system

$$y'(t) = A(t)y(t) + \int_0^t B(t,s)y(s)ds + f(t), \quad y(0) = x_o,$$

where A and B are $n \times n$ continuous matrices.

Murty, Srinivas and Narasimham [12] studied the asymptotic behaviour of solutions of matrix integro differential equation

$$X'(t) = X(t)B(t) + \int_0^t X(s)K(t,s)ds + F(t)$$

where $X(t)$, $B(t)$, $K(t,s)$ and $F(t)$ are $n \times n$ continuous matrices and $B(t)$ is not necessarily stable. Asymptotic behavior of solutions of Volterra equations also studied by Levin [10].

In many of the control engineering problems, we often come across the following important matrix of Lyapunov integro differential equation.

$$T'(t) = A(t)T(t) + T(t)B(t) + \int_0^t (K_1(t,s)T(s) + T(s)K_2(t,s))ds + F(t), \quad (1)$$

$$T(o) = T_o,$$

where $A(t)$, $B(t)$, $K_1(t,s)$, $K_2(t,s)$ are $(n \times n)$ matrices defined on $0 \leq t < \infty$ and $0 \leq s \leq t < \infty$, and $F(t)$ is an $(n \times n)$ matrix whose elements are continuous on $0 \leq t < \infty$.

This paper investigates the asymptotic behavior of the solutions of the matrix integro differential equation (1) of Volterra type in which $A(t)$ and $B(t)$ are not necessarily stable, by the method of reduction. This paper is organized as follows. In section 2, we obtain the solution of (1) in terms of resolvent functions which establishes variation of parameters formula. In section 3, we derive an equivalent equation of (1) which involves an arbitrary functions and by a proper choice of these functions we find new coefficient matrices $A_1(t)$ and $B_1(t)$ (corresponding to $A(t)$ and $B(t)$) to be stable. In section 4, we present our main results on asymptotic stability.

2. Variation of parameters formula

Theorem 2.1. *The solution of the matrix linear integro differential equation*

$$T'(t) = A(t)T(t) + \int_0^t K_1(t,s)T(s)ds + F(t), T(0) = T_o \quad (2)$$

where $A(t), K_1(t, s)$ are $(n \times n)$ continuous matrices for $t \in R^+$ and $(t, s) \in R^+ \times R^+, K_1(t, s) = 0$ for $s > t > 0$ and $F \in C[R, R^{n \times n}]$ is given by

$$T(t) = R_1(t, 0)T_0 + \int_0^t R_1(t, s)F(s)ds,$$

where $R_1(t, s)$ is the unique solution of

$$\frac{\partial R_1(t, s)}{\partial s} + R_1(t, s)A(s) + \int_s^t R_1(t, \sigma)K_1(\sigma, s)d\sigma = 0 \tag{3}$$

with $R_1(t, t) = I$ and given by

$$R_1(t, s) = \begin{cases} I + \int_s^t R_1(t, \sigma)\Psi_1(\sigma, s)d\sigma & \text{if } 0 \leq s \leq \infty \\ 0 & \text{if } 0 \leq t < s \end{cases}$$

where

$$\Psi_1(t, s) = \begin{cases} A(t) + \int_s^t K_1(t, \sigma)d\sigma & \text{if } 0 \leq s < \infty \\ 0 & \text{if } 0 \leq t < s \end{cases}$$

Proof. Clearly $R_1(t, s)$ defined as above exists and satisfies (3). Let $T(t)$ be the solution of (2) for $t \geq 0$. Set $P(s) = R_1(t, s)T(s)$ then,

$$\begin{aligned} P'(s) &= \frac{\partial R_1}{\partial s}T(s) + R_1(t, s)T'(s) \\ &= \frac{\partial R_1}{\partial s}T(s) + R_1(t, s)[A(s)T(s) + \int_0^s [K_1(s, u)T(u)du + F(s)]. \end{aligned}$$

Integrating between 0 to t gives

$$\begin{aligned} P(t) - P(0) &= \int_0^t \left[\frac{\partial R_1}{\partial s}T(s) + R_1(t, s)A(s)T(s) + R_1(t, s)F(s) \right] ds \\ &\quad + \int_0^t R_1(t, s) \left[\int_0^s K_1(s, u)T(u)du \right] ds. \end{aligned}$$

Using Fubini's theorem we get

$$\begin{aligned} T(t) - R_1(t, 0)T_0 &= \int_0^t \left[\frac{\partial R_1}{\partial s} + R_1(t, s)A(s) + \int_s^t R_1(t, u)K_1(u, s)du \right] T(s) ds \\ &\quad + \int_0^t R_1(t, s)F(s)ds. \end{aligned}$$

Therefore

$$T(t) = R_1(t, 0)T_0 + \int_0^t R_1(t, s)F(s)ds.$$

Conversely suppose that $T(t)$ is given as above. We will show that it satisfies (2). Consider

$$\int_0^t R_1(t, s)T'(s)ds = R_1(t, t)T(t) - R_1(t, 0)T_0 - \int_0^t \frac{\partial R_1}{\partial s}T(s)ds.$$

$$= \int_0^t R_1(t, s)F(s)ds - \int_0^t \frac{\partial R_1}{\partial s} T(s)ds$$

From (3) and using Fubini's theorem we get

$$\int_0^t R_1(t, s)[T'(s) - A(s)T(s) - \int_0^s K_1(s, u)T(u)du + F(s)]du = 0.$$

Since $R_1(t, s)$ is non zero continuous function for $t_0 \leq s \leq t < \infty$ we will get

$$T'(s) - A(s)T(s) - \int_0^s K_1(s, u)T(u)du + F(s) = 0.$$

i.e. $T(t)$ satisfies (2). □

Theorem 2.2. *The solution of the matrix linear integro differential equation*

$$T'(t) = T(t)B(t) + \int_0^t T(s)K_2(t, s)ds + F(t), \quad T(0) = T_0$$

with $K_2(t, s) = 0$ for $s > t > 0$ is given by

$$T(t) = T_0 R_2^*(t, 0) + \int_0^t F(s)R_2^*(t, s)ds$$

where $R_2(t, s)$ is the unique solution of

$$\frac{\partial R_2(t, s)}{\partial s} + R_2(t, s)B^*(s) + \int_s^t R_2(t, \sigma)K_2^*(\sigma, s)d\sigma = 0$$

with $R_2(t, t) = I$ and given by

$$R_2(t, s) = \begin{cases} I + \int_s^t R_2(t, \sigma)\Psi_2(\sigma, s)d\sigma & \text{if } 0 \leq s < \infty \\ 0 & \text{if } 0 \leq t < s \end{cases}$$

where

$$\Psi_2(t, s) = \begin{cases} B^*(t) + \int_s^t K_2(t, \sigma)d\sigma & \text{if } 0 \leq s \leq t < \infty \\ 0 & \text{if } 0 \leq t < s \end{cases}$$

Proof. Proof is the consequence of the theorem (2.1). □

With these two results as a tool one can obtain the solution of (1) in terms of $R_1(t, s)$ and $R_2(t, s)$ which establishes the variation of parameters formula for (1).

Theorem 2.3. *The solution of (1) is given by*

$$T(t) = R_1(t, 0)T_0 R_2^*(t, 0) + \int_0^t R_1(t, s)F(s)R_2^*(t, s)ds \quad (4)$$

where $R_1(t, s)$ and $R_2(t, s)$ are stated as in previous theorems.

Proof. We refer [9]. □

3. Equivalence equation

In this section we derive an equation, equivalent to (1) by defining proper choice of arbitrary functions.

Theorem 3.1. *Let $\varphi_1(t, s)$ and $\varphi_2(t, s)$ are $n \times n$ matrix functions which are continuously differentiable on $0 \leq s \leq t < \infty$ and commute with $T(t)$. Then the equation (1) with $T(0) = T_0$ is equivalent to*

$$Y'(t) = A_1 Y(t) + Y(t) B_1(t) + \int_0^t L_1(t, s) Y(s) + Y(s) L_2(t, s) ds + H(t), \quad (5)$$

with $Y(0) = T_0$ where $A_1(t) = A(t) - \varphi_1(t, t)$, $B_1(t) = B(t) - \varphi_2(t, t)$.

$$L_1(t, s) = K_1(t, s) + \frac{\partial \varphi_1}{\partial s} + \varphi_1(t, s) A(s) + A(s) \varphi_2(t, s) + \int_s^t (\varphi_1(t, u) K_1(u, s) + K_1(u, s) \varphi_2(t, u)) du,$$

$$L_2(t, s) = K_2(t, s) + \frac{\partial \varphi_2}{\partial s} + \varphi_2(t, s) B(s) + B(s) \varphi_2(t, s) + \int_s^t (\varphi_1(t, u) K_2(u, s) + K_2(u, s) \varphi_2(t, u)) du,$$

$$H(t) = F(t) + \varphi_1(t, 0) T_0 + T_0 \varphi_2(t, 0) + \int_0^t (\varphi_1(t, s) F(s) + F(s) \varphi_2(t, s)) ds.$$

Proof. Let $T(t)$ be any solution of (1) with $T(0) = T_0$ then,

$$T'(t) = A(t) T(t) + T(t) B(t) + \int_0^t (K_1(t, s) T(s) + T(s) K_2(t, s)) ds + F(t).$$

Now consider

$$\begin{aligned} & L_1(t, s) T(s) + T(s) L_2(t, s) \\ = & K_1(t, s) T(s) + T(s) K_2(t, s) + \varphi_1(t, s) A(s) T(s) + T(s) B(s) \varphi_2(t, s) \\ & + \frac{\partial \varphi_1}{\partial s} T(s) + \frac{\partial \varphi_2}{\partial s} T(s) + A(s) \varphi_2(t, s) T(s) + T(s) \varphi_2(t, s) B(s) + \int_s^t (\varphi_1(t, u) K_1(u, s) \\ & + K_1(u, s) \varphi_2(t, u)) du T(s) + T(s) \int_s^t (\varphi_1(t, u) K_2(u, s) + K_2(u, s) \varphi_2(t, u)) du. \end{aligned}$$

Integrating on both sides from 0 to t and using Fubini's theorem, we get

$$\begin{aligned} & \int_0^t (L_1(t, s) T(s) + T(s) L_2(t, s)) ds \\ = & \int_0^t (K_1(t, s) T(s) + T(s) K_2(t, s)) ds + \int_0^t (\varphi_1(t, s) A(s) T(s) + T(s) B(s) \varphi_2(t, s)) ds \\ & + \int_0^t \left(\frac{\partial \varphi_1}{\partial s} T(s) + T(s) \frac{\partial \varphi_2}{\partial s} \right) ds + \int_0^t (A(s) \varphi_1(t, s) T(s) + T(s) \varphi_2(t, s) B(s)) ds \end{aligned}$$

$$\begin{aligned}
& + \int_0^t \int_s^t (\varphi_1(t, u)K_1(u, s) + K_1(u, s)\varphi_2(t, u))du T(s)ds \\
& + \int_0^t T(s) \int_s^t [\varphi_1(t, u)K_2(u, s) + K_2(u, s)\varphi_2(t, u)]du ds.
\end{aligned}$$

Now using

$$\begin{aligned}
\int_0^t \frac{\partial \varphi_1}{\partial s} T(s) ds &= \varphi_1(t, t)T(t) - \varphi_1(t, o)T_o - \int_0^t \varphi_1(t, s)T'(s) ds, \\
\int_0^t T(s) \frac{\partial \varphi_2}{\partial s} &= T(t)\varphi_2(t, t) - T_o\varphi_2(t, o) - \int_0^t T'(s)\varphi_2(t, s) ds
\end{aligned}$$

and substituting $T'(s)$ and using Fubini's Theorem we will get

$$\begin{aligned}
& \int_0^t (L_1(t, s)T(s) + T(s)L_2(t, s)) ds \\
& = T'(t) - A(t)T(t) - T(t)B(t) - F(t) + \varphi_1(t, t)T(t) - \varphi_1(t, o)T_o + T(t)\varphi_2(t, t) \\
& \quad - T_o\varphi_2(t, o) - \int_0^t \varphi_1(t, s)F(s) ds \\
& \quad - \int_0^t \varphi_1(t, s) \left[A(s)T(s) + T(s)B(s) + \int_0^s (K_1(s, u)T(u) + T(u)K_2(s, u)) du \right] ds \\
& \quad - \int_0^t \left[A(s)T(s) + T(s)B(s) + \int_0^s (K_1(s, u)T(u) + T(u)K_2(s, u)) du \right] \varphi_2(t, s) ds \\
& \quad - \int_0^t F(s)\varphi_2(t, s) ds + \int_0^t (\varphi_1(t, s)A(s)T(s) + T(s)B(s)\varphi_2(t, s)) ds \\
& \quad + \int_0^t (A(s)\varphi_2(t, s)T(s) + T(s)\varphi_1(t, s)B(s)) ds \\
& \quad + \int_0^t \int_s^t (\varphi_1(t, u)K_1(u, s) + K_1(u, s)\varphi_2(t, u))du T(s) ds \\
& \quad + \int_0^t T(s) \int_s^t (\varphi_1(t, u)K_2(u, s) + K_2(u, s)\varphi_2(t, u))du ds.
\end{aligned}$$

i.e

$$\begin{aligned}
& \int_0^t (L_1(t, s)T(s) + T(s)L_2(t, s)) ds \\
& = T'(t) - [A(t) - \varphi_1(t, t)]T(t) - T(t)[B(t) - \varphi_2(t, t)] \\
& \quad - F(t) - \varphi_1(t, o)T_o - T_o\varphi_2(t, o) - \int_0^t (\varphi_1(t, s)F(s) + F(s)\varphi_2(t, s)) ds
\end{aligned}$$

Therefore

$$T'(t) = A_1(t)T(t) + T(t)B_1(t) + \int_0^t (L_1(t, s)T(s) + T(s)L_2(t, s)) ds + H(t).$$

Hence $T(t)$ is also a solution of (5). To prove the converse, let $Y(t)$ be any solution of (3.1) existing on $[0, \infty)$. Define

$$Z(t) = Y'(t) - A(t)Y(t) - Y(t)B(t) - \int_0^t (K_1(t, s)Y(s) + Y(s)K_2(t, s))ds - F(t).$$

Substitute $Y'(t)$ from (3.1) we get

$$\begin{aligned} Z(t) = & -(\varphi_1(t, t)Y(t) - \varphi_1(t, o)T_o) - (Y(t)\varphi_2(t, t) - T_o\varphi_2(t, o)) \\ & + \int_0^t (L_1(t, s)Y(s) + Y(s)L_2(t, s))ds + \int_0^t (\varphi_1(t, s)F(s) + F(s)\varphi_2(t, s))ds \\ & - \int_0^t (K_1(t, s)Y(s) + Y(s)K_2(t, s))ds. \end{aligned}$$

Substituting $L_1(t, s)$ and $L_2(t, s)$ we get

$$\begin{aligned} Z(t) = & -(\varphi_1(t, t)Y(t) - \varphi_1(t, o)T_o) - (Y(t)\varphi_2(t, t) - T_o\varphi_2(t, o)) \\ & + \int_0^t (\varphi_1(t, s)F(s) + F(s)\varphi_2(t, s))ds + \int_0^t \left[\frac{\partial \varphi_1}{\partial s} Y(s) + Y(s) \frac{\partial \varphi_2}{\partial s} \right] ds \\ & + \int_0^t (\varphi_1(t, s)A(s)Y(s) + A(s)\varphi_2(t, s)Y(s) + Y(s)\varphi_2(t, s)B(s) + Y(s)B(s)\varphi_2(t, s))ds \\ & + \int_0^t \left[\int_s^t (\varphi_1(t, u)K_1(u, s) + K_1(u, s)\varphi_2(t, u))du \right] Y(s)ds \\ & + \int_0^t Y(s) \left[\int_s^t (\varphi_1(t, u)K_2(u, s) + K_2(u, s)\varphi_2(t, u))du \right] ds. \end{aligned}$$

Now using the fact that

$$\begin{aligned} (\varphi_1(t, t)Y(t) - \varphi_1(t, o)T_o) &= \int_0^t \left(\frac{\partial \varphi_1(t, s)}{\partial s} Y(s) + \varphi_1(t, s)Y'(s) \right) ds, \\ Y(t)\varphi_2(t, t) - T_o\varphi_2(t, o) &= \int_0^t \left(Y(s) \frac{\partial \varphi_2(t, s)}{\partial s} + Y'(s)\varphi_2(t, s) \right) ds \end{aligned}$$

and simplifying we get

$$Z(t) = - \int_0^t (\varphi_1(t, s)Z(s) + Z(s)\varphi_2(t, s))ds.$$

Since the solutions of the matrix Volterra integral equations are unique, then $Z(t) \equiv 0$. Therefore

$$Y'(t) = A(t)Y(t) + Y(t)B(t) + \int_0^t (K_1(t, s)Y(s) + Y(s)K_2(t, s))ds + F(t).$$

Hence $Y(t)$ is a solution of (1) and the proof is complete. □

Because (5) is equivalent to (1) the stability properties of (1) implies the stability properties of (5). If $A(t)$ and $B(t)$ are not stable in (1) we can find $A_1(t)$ and $B_1(t)$ (corresponding to $A(t)$ and $B(t)$) to be stable through the proper choice of ϕ_1 and ϕ_2 . If we are choosing ϕ_1 and ϕ_2 such that $L_1(t, s)$ and $L_2(t, s)$ are vanish then (5) reduced to differential equation equivalent to integro

differential equation (1). Now we will present our main theorems on asymptotic stability in next section.

4. Main Results

Lemma 4.1. *Let $A_1(t)$ and $B_1(t)$ are $(n \times n)$ continuous Matrices as defined previously and they commute with their integrals and let M and α are positive real numbers. Suppose the inequality*

$$|e^{\int_s^t (A_1(\tau) + B_1(\tau)) d\tau}| \leq M e^{-\alpha(t-s)} \quad 0 \leq s \leq t < \infty \quad (6)$$

holds then every solution of (5) with $Y(0) = T_0$ satisfies the inequality

$$\begin{aligned} |Y(t)| &\leq M|T_0|e^{-\alpha t} + M \int_0^t |H(s)|e^{-\alpha(t-s)} ds \\ &+ M \int_0^t \left[\int_s^t |L_1(w, u)|e^{-\alpha(t-w)} dw |Y(u)| \right. \\ &\quad \left. + |Y(u)| \int_s^t |L_2(w, u)|e^{-\alpha(t-w)} dw \right] du. \end{aligned}$$

Proof. Consider

$$\begin{aligned} Y'(t) &= A_1(t)Y(t) + Y(t)B_1(t) + \int_0^t (L_1(t, s)Y(s) + Y(s)L_2(t, s))ds + H(t) \\ \Rightarrow -A_1(t)Y(t) + Y'(t) - Y(t)B_1(t) &= \int_0^t (L_1(t, s)Y(s) + Y(s)L_2(t, s))ds + H(t). \end{aligned}$$

Pre-multiplying with $e^{-\int_0^t A_1(\tau) d\tau}$ and post multiplying with $e^{-\int_0^t B_1(\tau) d\tau}$ on both sides and rearranging we get

$$\begin{aligned} &\left(e^{-\int_0^t A_1(\tau) d\tau} Y(t) e^{-\int_0^t B_1(\tau) d\tau} \right)' \\ &= e^{-\int_0^t A_1(\tau) d\tau} \left[\int_0^t (L_1(t, s)Y(s) + Y(s)L_2(t, s))ds + H(t) \right] e^{-\int_0^t B_1(\tau) d\tau}. \end{aligned}$$

Integrating from 0 to t on both sides we get

$$\begin{aligned} &e^{-\int_0^t A_1(\tau) d\tau} Y(t) e^{-\int_0^t B_1(\tau) d\tau} \\ &= T_0 + \int_0^t e^{-\int_0^s A_1(\tau) d\tau} H(s) e^{-\int_0^s B_1(\tau) d\tau} ds \\ &\quad + \int_0^t e^{-\int_0^s A_1(\tau) d\tau} \int_0^s (L_1(s, u)Y(u) + Y(u)L_2(s, u)) du e^{-\int_0^s B_1(\tau) d\tau} ds \end{aligned}$$

Therefore,

$$\begin{aligned} Y(t) &= e^{\int_0^t A_1(\tau) d\tau} T_0 e^{-\int_0^t B_1(\tau) d\tau} + \int_0^t e^{\int_0^s A_1(\tau) d\tau} H(s) e^{\int_0^s B_1(\tau) d\tau} ds \\ &\quad + \int_0^t e^{\int_0^s A_1(\tau) d\tau} \left[\int_0^s (L_1(s, u)Y(u) + Y(u)L_2(s, u)) du \right] e^{\int_0^s B_1(\tau) d\tau} ds. \end{aligned}$$

Taking norm on both sides and using the inequality (6) we get

$$|Y(t)| \leq M e^{-\alpha t} |T_0| + M \int_0^t |H(s)| e^{-\alpha(t-s)} ds + M \int_0^t \int_0^s (|L_1(s, u)| |Y(u)| + |Y(u)| |L_2(s, u)|) du e^{-\alpha(t-s)} ds.$$

Now using Fubini's Theorem we get

$$|Y(t)| \leq M e^{-\alpha t} |T_0| + M \int_0^t |H(s)| e^{-\alpha(t-s)} ds + M \int_0^t \left[\int_s^t (|L_1(w, u)| e^{-\alpha(t-w)} dw |Y(u)| + |Y(u)| \int_s^t |L_2(w, u)| e^{-\alpha(t-w)} dw) \right] du. \quad \square$$

Theorem 4.2. Let $\varphi_1(e, s)$ and $\varphi_2(t, s)$ are continuously differentiable matrix functions such that, for $0 \leq s \leq t < \infty$,

- (i) The hypothesis of Lemma (4.1) holds.
- (ii) $|\varphi_1(t, s) + \varphi_2(t, s)| \leq L_0 e^{-\gamma(t-s)}$.
- (iii) $\sup_{0 \leq s \leq t < \infty} \int_s^t (|L_1(w, u)| + |L_2(w, u)|) e^{-\alpha(w-s)} dw \leq \alpha_0$ where $L_0, \gamma (> \alpha), \alpha_0$ are positive real numbers.
- (iv) $F(t) \equiv 0$.

If $\alpha - M\alpha_0 > 0$, then every solution $T(t)$ of (1) tends to zero exponentially as $t \rightarrow \infty$.

Proof. In order to show every solution of (1) tends to zero exponentially, it is enough to show every solution of (5) tends to zero exponentially as $t \rightarrow \infty$. From the previous lemma (4.1) and condition (ii) and (iv) implies

$$|Y(t)| e^{\alpha t} \leq M |T_0| + M |T_0| L_0 \int_0^t |e^{-(\gamma-\alpha)s}| ds + M \int_t^t |Y(u)| \left[\int_s^t (|L_1(w, u)| + |L_2(w, u)|) e^{-\alpha(w)} dw \right] du$$

then,

$$|Y(t)| e^{\alpha t} \leq M |T_0| \left(1 + \frac{L_0}{\gamma - \alpha} \right) + \int_0^t |Y(u)| e^{\alpha s} M \alpha_0 du$$

Now applying Gronwall - Bellman inequality we get

$$|Y(t)| e^{\alpha t} \leq M |T_0| \left(1 + \frac{L_0}{\gamma - \alpha} \right) e^{M \alpha_0 t}.$$

Therefore

$$|Y(t)| \leq M |T_0| \left(1 + \frac{L_0}{\gamma - \alpha} \right) e^{-(\alpha - M \alpha_0)t}.$$

Since $\alpha - M\alpha_0 > 0$, the theorem follows. □

Remark 4.1. From the Theorem (4.2) the solution of (1) is exponentially asymptotically stable if $F(t) \equiv 0$.

Remark 4.2. If $F(t) \neq 0$ in the Theorem (4.2), then the solutions of (1) tends to zero as $t \rightarrow \infty$.

Remark 4.3. It is possible to select the matrices $\varphi_1(t, s)$ and $\varphi_2(t, s)$ satisfying the conditions (i) and (ii) of the Theorem (4.1).

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