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THIRD ORDER THREE POINT FUZZY BOUNDARY VALUE PROBLEM UNDER GENERALIZED DIFFERENTIABILITY

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Abstract. In this article, we investigate third order three-point fuzzy boundary value problem to using a generalized differentiability concept. We present the new concept of solution of third order three-point fuzzy boundary value problem. Some illustrative examples are provided.

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1. Introduction

Fuzzy differential equations is a natural way to model dynamical systems under possibility uncertainty. In [14], Puri and Ralescu introduced the concept of H-derivative of a fuzzy number valued function. Bede [4] proved that the fuzzy two-point boundary value problem is not equivalent to the integral equation expressed by Green's function under Hukuhara differentiability [16] (generalization of the H-derivative) in the fuzzy differential equation and using fuzzy Auman-type integral in the integral equation. Satio [15] gave a new representation of fuzzy numbers with bounded supports and proved that fuzzy number means a bounded continuous curve in the two-dimensional metric space. Under this new structure and certain conditions, Prakash et.al [13] proved a third order three-point boundary value problem of fuzzy differential equation is equivalent to a corresponding fuzzy integral equation. Bede [5] defined the generalized differentiability of fuzzy number valued functions. Two point boundary value problem under generalized differentiability is considered in [9]. In [12] the existence and uniqueness of solution for a first-order linear fuzzy differential equation with impulses subject to periodic boundary conditions are obtained. Recently an algorithm for the solution of second order fuzzy initial value problems with fuzzy

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coefficients, fuzzy initial values and fuzzy forcing functions is given in [2]. Analytical and numerical solution of fuzzy initial value problems under generalized differentiability are considered in [1, 3]. However it should be emphasized that most of the works in this direction are mainly concerned with fuzzy initial value problem, periodic boundary value problem and two point boundary value problem there has been no attempts made to study third order three-point fuzzy boundary value problem under generalized differentiability.

2. Preliminaries

Let us denote by $\mathbb{R}_F$ the class of fuzzy subsets $u : \mathbb{R} \to [0,1]$, satisfying the following properties:

(1) u is normal, that is, there exist $x_0 \in \mathbb{R}$ with $u(x_0) = 1$.
(2) u is convex fuzzy set, that is,

$$u(\lambda x + (1-\lambda)y) \quad \geq \quad \min\{u(x), u(y)\},\ \forall x, y \in \mathbb{R},\ \forall \lambda \in [0,1].$$

(3) u is upper semi-continuous on $\mathbb{R}$.
(4) $\overline{\{x \in \mathbb{R} | u(x) > 0\}}$ is compact, where $\bar{A}$ denotes the closure of A.

Then $\mathbb{R}_F$ is called the space of fuzzy numbers. For $0 < r \leq 1$, set $[u]^r = \{s \in \mathbb{R} | u(s) \geq r\}$ and $[u]^0 = cl\{s \in \mathbb{R} | u(s) > 0\}$. Then the r- level set $[u]^r$ is a non-empty compact interval for all $0 \leq r \leq 1$. The following Theorem gives the parametric form of a fuzzy number.

Theorem 2.1 ([7, 8]). *The necessary and sufficient conditions for $(\underline{u}(r), \overline{u}(r))$ to define the parametric form of a fuzzy number are as follows:*

(1) $\underline{u}(r)$ *is a bounded monotonic increasing (non-decreasing) left-continuous function* $\forall r \in (0,1]$ *and right-continuous for* $r = 0$.
(2) $\overline{u}(r)$ *is a bounded monotonic decreasing (non-increasing) left-continuous function* $\forall r \in (0,1]$ *and right-continuous for* $r = 0$.
(3) $\underline{u}(r) \leq \overline{u}(r), \quad 0 \leq r \leq 1$.

We refer to $\underline{u}$ and $\overline{u}$ as the lower and upper branches on u, respectively. For $u \in \mathbb{R}_F$, we define the length of u as: $len(u) = \overline{u} - \underline{u}$. A crisp number α is simply represented by $\overline{u}(r) = \underline{u}(r) = \alpha \quad (0 \leq r \leq 1)$ is called singleton. For $u, v \in \mathbb{R}_F$ and $\alpha \in \mathbb{R}$, the sum $u+v$ and the scalar multiplication αu are defined by $u + v = (\underline{(u+v)}(r), \overline{(u+v)}(r)) = (\underline{u}(r) + \underline{v}(r), \overline{u}(r) + \overline{v}(r))$,

$$\alpha u = \begin{cases} (\alpha \underline{u}(r), \alpha \overline{u}(r)), & \alpha \geq 0, \\ (\alpha \overline{u}(r), \alpha \underline{u}(r)), & \alpha < 0. \end{cases}$$

For $u, v \in \mathbb{R}_F$, we say $u = v$ if and only if $\underline{u}(r) = \underline{v}(r)$ and $\overline{u}(r) = \overline{v}(r)$.
The metric structure is given by the Hausdorff distance $D : \mathbb{R}_F \times \mathbb{R}_F \to \mathbb{R}_+ \cup \{0\}$, by

$$D(u,v) = \sup_{r \in [0,1]} max\{| \underline{u}(r) - \underline{v}(r) |, | \overline{u}(r) - \overline{v}(r) |\}.$$

Definition 2.2. Let $x, y \in \mathbb{R}_F$. If there exists $z \in \mathbb{R}_F$ such that $x = y + z$ then z is called the H-difference of x, y and it is denoted $x \ominus y$.
In this paper the sign "$\ominus$" stands always for H-difference and $x \ominus y \neq x + (-1)y$ in general. Usually we denote x+(-1)y by x-y, while $x \ominus y$ stands for the H-difference. In the sequel, we fix $I = [a, c]$, for $a, c \in \mathbb{R}$.

Remark 2.1. A function F is said to be a fuzzy number valued function if its range is a space of fuzzy numbers.

Definition 2.3. Let $F : I \to \mathbb{R}_F$ be a fuzzy number valued function. If there exists an element $F'(t_0) \in \mathbb{R}_F$ such that for all $h > 0$ sufficiently near to 0, $F(t_0 + h) \ominus F(t_0), F(t_0) \ominus F(t_0 - h)$ exist and the limits (in the metric D)

$$\lim_{h \to 0^+} \frac{F(t_0 + h) \ominus F(t_0)}{h} \text{ and } \lim_{h \to 0^+} \frac{F(t_0) \ominus F(t_0 - h)}{h}$$

exist and equal to $F'(t_0)$, then F said to be differentiable at $t_0 \in (a, c)$. If t_0 is the end points of I, then we consider the corresponding one-sided derivative. Here the limits are taken in the metric space $(\mathbb{R}_F, D)$.

In this paper we considered the following third order three-point fuzzy boundary value problem

$$y'''(t) = f(t, y(t), y'(t), y''(t)), \tag{1}$$

with boundary conditions

$$y(a) = y(b) = y(c) = \widetilde{0}, \tag{2}$$

where $\widetilde{0} = (0, 0) \in \mathbb{R}_F$ and $f : I \times \mathbb{R}_F \times \mathbb{R}_F \times \mathbb{R}_F \to \mathbb{R}_F$ is continuous fuzzy function.

3. Generalized fuzzy derivatives

The definition of the Hukuhara differentiability is a straightforward generalization of the Hukuhara differentiability of a set-valued function. Bede and Gel in [5] showed that if $F(t) = c.g(t)$ where c is a fuzzy number and $g : [a, b] \to R^+$ is a function with $g'(t) < 0$, then F is not Hukuhara differentiable. To avoid this difficulty, they introduced a more general definition of derivative for fuzzy function.

Definition 3.1. Let $F : I \to \mathbb{R}_F$ and fix $t_0 \in (a, c)$. If there exists an element $F'(t_0) \in \mathbb{R}_F$ such that for all $h > 0$ sufficiently near to 0, $F(t_0+h)\ominus F(t_0), F(t_0)\ominus F(t_0 - h)$ exist and the limits (in the metric D)

$$\lim_{h \to 0^+} \frac{F(t_0 + h) \ominus F(t_0)}{h} \text{ and } \lim_{h \to 0^+} \frac{F(t_0) \ominus F(t_0 - h)}{h}$$

exist and equal to $F'(t_0)$, then F said to be (1)-differentiable at t_0 and it is denoted by $D_1^1 F(t_0)$. If for all $h > 0$ sufficiently near to 0, $F(t_0) \ominus F(t_0 +$

$h)$, $F(t_0 - h) \ominus F(t_0)$ exist and the limits (in the metric D)

$$\lim_{h \to 0^+} \frac{F(t_0) \ominus F(t_0 + h)}{-h} \text{ and } \lim_{h \to 0^+} \frac{F(t_0 - h) \ominus F(t_0)}{-h} = F'(t_0)$$

exist and equal to $F'(t_0)$, then F is said to be (2)-differentiable and it is denoted by $D_2^1 F(t_0)$. If t_0 is the end points of I, then we consider the corresponding one-sided derivative.

Theorem 3.2 ([6, 10]). *Let $F : I \to \mathbb{R}_F$ and let $F(t) = (f(t,r), g(t,r))$ for each* $r \in [0, 1]$.

(1) *If F is (1)-differentiable then $f(t,r)$ and $g(t,r)$ are differentiable functions and $D_1^1 F(t) = (f'(t,r), g'(t,r))$.*
(2) *If F is (2)-differentiable then $f(t,r)$ and $g(t,r)$ are differentiable functions and $D_2^1 F(t) = (g'(t,r), f'(t,r))$.*

Definition 3.3. Let $F : I \to \mathbb{R}_F$ and let $n, m \in \{1, 2\}$. If $D_n^1 F$ exists on a neighborhood of t_0 as a fuzzy number valued function and it is (m)-differentiable at t_0 as a fuzzy number valued function, then F is said to be (n, m)-differentiable at $t_0 \in I$ and is denoted by $D_{n,m}^2 F(t_0)$.

Theorem 3.4 ([10]). *Let $F : I \to \mathbb{R}_F$, $D_1^1 F : I \to \mathbb{R}_F$ and $D_2^1 F : I \to \mathbb{R}_F$ and let $F(t) = (f(t,r), g(t,r))$.*

(1) *If $D_1^1 F$ is (1)-differentiable, then $f'(t,r)$ and $g'(t,r)$ are differentiable functions and $D_{1,1}^2 F(t) = (f''(t,r), g''(t,r))$.*
(2) *If $D_1^1 F$ is (2)-differentiable, then $f'(t,r)$ and $g'(t,r)$ are differentiable functions and $D_{1,2}^2 F(t) = (g''(t,r), f''(t,r))$.*
(3) *If $D_2^1 F$ is (1)-differentiable, then $f'(t,r)$ and $g'(t,r)$ are differentiable functions and $D_{2,1}^2 F(t) = (g''(t,r), f''(t,r))$.*
(4) *If $D_2^1 F$ is (2)-differentiable, then $f'(t,r)$ and $g'(t,r)$ are differentiable functions and $D_{2,2}^2 F(t) = (f''(t,r), g''(t,r))$.*

Remark 3.1. For each of these four derivatives, we have again two possibilities.
$D_1^1(D_1^1(D_1^1 F(t)))$, $D_2^1(D_1^1(D_1^1 F(t)))$,
$D_1^1(D_2^1(D_1^1 F(t)))$, $D_2^1(D_2^1(D_1^1 F(t)))$,
$D_1^1(D_1^1(D_2^1 F(t)))$, $D_2^1(D_1^1(D_2^1 F(t)))$ and
$D_1^1(D_2^1(D_2^1 F(t)))$, $D_2^1(D_2^1(D_2^1 F(t)))$.

Definition 3.5. Let $F : I \to \mathbb{R}_F$ and let $n, m, l \in \{1, 2\}$. If $D_n^1 F$ and $D_{n,m}^2 F$ exist on a neighborhood of t_0 as fuzzy number valued functions and $D_{n,m}^2 F$ is (l)-differentiable at t_0 as a fuzzy number valued function, then F is said to be (n, m, l)-differentiable at $t_0 \in I$ and it is denoted by $D_{n,m,l}^3 F(t_0)$.

Theorem 3.6 ([10]). *Let $F : I \to \mathbb{R}_F$ $D_n^1 F : I \to \mathbb{R}_F$, $D_{n,m}^2 F : I \to \mathbb{R}_F$ for $n, m \in \{1, 2\}$and let $F(t) = (f(t,r), g(t,r))$.*

(1) *If $D_{1,1}^2 F$ is (1)-differentiable, then $f''(t,r)$ and $g''(t,r)$ are differentiable functions and $D_{1,1,1}^3 F(t) = (f'''(t,r), g'''(t,r))$.*

(2) *If $D^2_{1,1}F$ is (2)-differentiable, then $f''(t,r)$ and $g''(t,r)$ are differentiable functions and $D^3_{1,1,2}F(t) = (g'''(t,r), f'''(t,r))$.*
(3) *If $D^2_{1,2}F$ is (1)-differentiable, then $f''(t,r)$ and $g''(t,r)$ are differentiable functions and $D^3_{1,2,1}F(t) = (g'''(t,r), f'''(t,r))$.*
(4) *If $D^2_{1,2}F$ is (2)-differentiable, then $f''(t,r)$ and $g''(t,r)$ are differentiable functions and $D^3_{1,2,2}F(t) = (f'''(t,r), g'''(t,r))$.*
(5) *If $D^2_{2,1}F$ is (1)-differentiable, then $f''(t,r)$ and $g''(t,r)$ are differentiable functions and $D^3_{2,1,1}F(t) = (g'''(t,r), f'''(t,r))$.*
(6) *If $D^2_{2,1}F$ is (2)-differentiable, then $f''(t,r)$ and $g''(t,r)$ are differentiable functions and $D^3_{2,1,2}F(t) = (f'''(t,r), g'''(t,r))$.*
(7) *If $D^2_{2,2}F$ is (1)-differentiable, then $f''(t,r)$ and $g''(t,r)$ are differentiable functions and $D^3_{2,2,1}F(t) = (f'''(t,r), g'''(t,r))$.*
(8) *If $D^2_{2,2}F$ is (2)-differentiable, then $f''(t,r)$ and $g''(t,r)$ are differentiable functions and $D^3_{2,2,2}F(t) = (g'''(t,r), f'''(t,r))$.*

4. Three-point fuzzy boundary value problem

In this section, we consider fuzzy boundary value problem (1)-(2) with generalized differentiability and introduce a new class of solutions.

Definition 4.1. Let $y : I \to \mathbb{R}_F$ and let $n, m, l \in \{1, 2\}$. If $D^1_n y, D^2_{n,m}y$ and $D^3_{n,m,l}y$ exist on I as fuzzy number valued functions, $D^3_{n,m,l}y(t) = f(t, y(t), D^1_n y(t), D^2_{n,m}y(t))$ for all $t \in I$ and $y(a) = y(b) = y(c) = \widetilde{0}$, then y is said to be a (n, m, l) solution for the fuzzy boundary value problem (1)-(2) on I,

Definition 4.2. Let $n, m, l \in \{1, 2\}$ and let I_1 be an interval such that $I_1 \subset I$. If $y,\ D^1_n y, D^2_{n,m}y$ and $D^3_{n,m,l}y$ exist on I_1 as fuzzy number valued functions and $D^3_{n,m,l}y(t) = f(t, y(t), D^1_n y(t), D^2_{n,m}y(t))$ for all $t \in I_1$, then y is said to be a (n, m, l) solution for the fuzzy differential equation (1) on I_1.

Definition 4.3. Let $n_i, m_i, l_i \in \{1, 2\}$ and $i \in \{1, 2, 3, 4\}$. If there exists a fuzzy number valued function $y : I \to \mathbb{R}_F$ such that

$$y(t) = \begin{cases} y_1(t) & \text{if } t \in (a, b) \\ y_2(t) & \text{if } t \in (b, c) \\ y_1(t) = y_2(t) & \text{if } t \in \{a, b, c\} \end{cases}$$

where $y_1 : [a, b] \cup \{c\} \to \mathbb{R}_F$ and $y_2 : [b, c] \cup \{a\} \to \mathbb{R}_F$ are the fuzzy number valued functions with $y_1(c) = y_2(a) = \widetilde{0}$ and if there exist $t_1 \in (a, b)$ and $t_2 \in (b, c)$ such that $y_1(a) = y_1(b) = y_2(b) = y_2(c) = \widetilde{0}$, y_1 is a (n_1, m_1, l_1)-solution and a (n_2, m_2, l_2)-solution of the equation (1) on (a, t_1) and on (t_1, b) respectively and y_2 is a (n_3, m_3, l_3)-solution and a (n_4, m_4, l_4) solution of the equation (1) on (b, t_2) and on (t_2, c) respectively. Then we say that y is a generalized solution of the fuzzy boundary value problem (1)-(2).

By Theorem 3.2, Theorem 3.4 and Theorem 3.6, we can translate the fuzzy boundary value problem (1)-(2) to a system of ordinary boundary value problems hereafter, called corresponding (n,m,l)-system for problem (1)-(2). Therefore, possible system of ordinary boundary value problems for the problem (1)-(2) are as follows:

(1,1,1)-system:

$$\underline{y}'''(t,r) = \underline{f}(t, y(t,r), D_1^1 y(t,r), D_{1,1}^2 y(t,r)),$$
$$\overline{y}'''(t,r) = \overline{f}(t, y(t,r), D_1^1 y(t,r), D_{1,1}^2 y(t,r)),$$

$$\left.\begin{array}{ll} \underline{y}(a,r) = 0, & \overline{y}(a,r) = 0, \\ \underline{y}(b,r) = 0, & \overline{y}(b,r) = 0, \\ \underline{y}(c,r) = 0, & \overline{y}(c,r) = 0. \end{array}\right\} \tag{3}$$

(1,1,2)-system:

$$\overline{y}'''(t,r) = \underline{f}(t, y(t,r), D_1^1 y(t,r), D_{1,1}^2 y(t,r)),$$
$$\underline{y}'''(t,r) = \overline{f}(t, y(t,r), D_1^1 y(t,r), D_{1,1}^2 y(t,r)),$$

with the boundary condition as in (3).

(1,2,1)-system:

$$\overline{y}'''(t,r) = \underline{f}(t, y(t,r), D_1^1 y(t,r), D_{1,2}^2 y(t,r)),$$
$$\underline{y}'''(t,r) = \overline{f}(t, y(t,r), D_1^1 y(t,r), D_{1,2}^2 y(t,r)),$$

with the boundary condition as in (3).

(1,2,2)-system:

$$\underline{y}'''(t,r) = \underline{f}(t, y(t,r), D_1^1 y(t,r), D_{1,2}^2 y(t,r)),$$
$$\overline{y}'''(t,r) = \overline{f}(t, y(t,r), D_1^1 y(t,r), D_{1,2}^2 y(t,r)),$$

with the boundary condition as in (3).

(2,1,1)-system:

$$\overline{y}'''(t,r) = \underline{f}(t, y(t,r), D_2^1 y(t,r), D_{2,1}^2 y(t,r)),$$
$$\underline{y}'''(t,r) = \overline{f}(t, y(t,r), D_2^1 y(t,r), D_{2,1}^2 y(t,r)),$$

with the boundary condition as in (3).

(2,1,2)-system:

$$\underline{y}'''(t,r) = \underline{f}(t, y(t,r), D_2^1 y(t,r), D_{2,1}^2 y(t,r)),$$
$$\overline{y}'''(t,r) = \overline{f}(t, y(t,r), D_2^1 y(t,r), D_{2,1}^2 y(t,r)),$$

with the boundary condition as in (3).

(2,2,1)-system:

$$\underline{y}'''(t,r) = \underline{f}(t, y(t,r), D_2^2 y(t,r), D_{2,2}^2 y(t,r)),$$
$$\overline{y}'''(t,r) = \overline{f}(t, y(t,r), D_2^1 y(t,r), D_{2,2}^2 y(t,r)),$$

with the boundary condition as in (3).

(2,2,2)-system:

$$\overline{y}'''(t,r) = \underline{f}(t, y(t,r), D_2^1 y(t,r), D_{2,2}^2 y(t,r)),$$
$$\underline{y}'''(t,r) = \overline{f}(t, y(t,r), D_2^1 y(t,r), D_{2,2}^2 y(t,r)),$$

with the boundary condition as in (3).
Our strategy of solving (1)-(2) is based on the selection of derivative type in the fuzzy boundary value problem. We first choose the type of solution and translate problem (1)-(2) to the corresponding system of boundary value problems. Then, we solve the obtained boundary value problems system. Finally we find such a domain in which the solution and its derivatives have valid level sets according to the type of differentiability and using the Representation theorem [11] we can construct the solution of the fuzzy boundary problem (1)-(2).

Remark 4.1. If y is the (n,m,l)-solution of (1) on $I_1 \subseteq I$ for $m,n,l \in \{1,2\}$, then y is (n,m,l)-differentiable on I_1 and $y(t)$ is not (n,m,l)-differentiable in $t_0 \in (I \backslash I_1)$.

5. Examples

Example 5.1. Consider the following third order three-point fuzzy boundary value problem:

$$y'''(t) = (r, 2-r), \tag{4}$$

$$y(0) = y(1) = y(2) = \widetilde{0}. \tag{5}$$

If y is a (1,1,1)-solution of (4)-(5), then

$y'(t) = (\underline{y}'(t,r), \overline{y}'(t,r)),\ y''(t) = (\underline{y}''(t,r), \overline{y}''(t,r)),\ y'''(t) = (\underline{y}'''(t,r), \overline{y}'''(t,r)),$

$y(0) = (\underline{y}(0,r), \overline{y}(0,r)) = \widetilde{0},\ y(1) = (\underline{y}(1,r), \overline{y}(1,r)) = \widetilde{0}$ and $y(2) = (\underline{y}(2,r),$ $\overline{y}(2,r)) = \widetilde{0}$ and satisfies the (1,1,1)-system associated with (4). Similarly for other system. On the other hand, by direct calculation, the corresponding solution of the (1,1,1), (1,2,2), (2,1,2), and (2,2,1) systems has necessarily the following expression

$$y(t) = \left(\frac{r}{6}(t^3 - 3t^2 + 2t), \frac{2-r}{6}(t^3 - 3t^2 + 2t)\right). \tag{6}$$

By the Representation Theorem [11] and Theorem 2.1, we see $(\overline{y}(t,r), \underline{y}(t,r))$ represents a valued fuzzy number when $t^3 - 3t^2 + 2t \geq 0$. Hence (6) represents fuzzy number for $t \in [0,1]$ or $t = 2$. Now we find the range of (1,1,1),(1,2,2),(2,1,2) and (2,2,1)-solutions of the differential equation (4) separately.
(1,1,1)-solution
The (1)-derivative of (6) in that case is given by:

$$y'(t) = \left(\frac{r}{6}(3t^2 - 6t + 2), \frac{2-r}{6}(3t^2 - 6t + 2)\right)$$

and it is a fuzzy number when $t \in [0, \frac{1}{3}(3-\sqrt{3}]$ or $t=2$. Then it is again (1)-differentiable

$$y''(t) = (r(t-1), (2-r)(t-1))$$

and it is a fuzzy number when $t=2$. By the Definition 3.5, $D^3_{1,1,1}y(t)$ does not exist. Hence y in (6) is not a (1,1,1)-solution of the fuzzy differential equation (4).
$(1,2,2)$-solution
$y'(t) = (\frac{r}{6}(3t^2-6t+2), \frac{2-r}{6}(3t^2-6t+2))$ is a fuzzy number when $t \in [0, \frac{1}{3}(3-\sqrt{3}]$ or $t=2$. $y''(t) = ((2-r)(t-1), r(t-1))$ and $y'''(t) = (r, 2-r)$ are fuzzy numbers when $t \in [0, \frac{1}{3}(3-\sqrt{3})]$. Hence $y(t)$, $D^1_1y(t)$, $D^2_{1,2}y(t)$ and $D^3_{1,2,2}y(t)$ are valid fuzzy numbers for $t \in [0, \frac{1}{3}(3-\sqrt{3}]$ and y in (6) is a (1,2,2)-solution of the fuzzy differential equation (4) on $[0, \frac{1}{3}(3-\sqrt{3}]$.
$(2,1,2)$-solution
$y'(t) = (\frac{2-r}{6}(3t^2-6t+2), \frac{r}{6}(3t^2-6t+2))$, $y''(t) = ((2-r)(t-1), r(t-1))$ and $y'''(t) = (r, 2-r)$ are fuzzy numbers when $t \in [\frac{1}{3}(3-\sqrt{3}), 1]$. Hence $y(t)$, $D^1_2y(t)$, $D^2_{2,1}y(t)$ and $D^3_{2,1,2}y(t)$ are valid fuzzy numbers for $t \in [\frac{1}{3}(3-\sqrt{3}), 1]$ and y in (6) is a (2,1,2)-solution of the fuzzy differential equation (4) on $[\frac{1}{3}(3-\sqrt{3}), 1]$.
$(2,2,1)$-solution
$y'(t) = (\frac{2-r}{6}(3t^2-6t+2), \frac{r}{6}(3t^2-6t+2))$ is a fuzzy numbers when $t \in [\frac{1}{3}(3-\sqrt{3}), 1]$. $y''(t) = (r(t-1), (2-r)(t-1))$ is a fuzzy number when $t=1$. By the Definition 3.5, $D^3_{2,2,1}y(t)$ does not exist. Hence y in (6) is not a (2,2,1)-solution of the fuzzy differential equation (4).
The solution of the remaining four systems (1,1,2), (1,2,1), (2,1,1), and (2,2,2) has the following form

$$y(t) = \left(\frac{2-r}{6}(t^3-3t^2+2t), \frac{r}{6}(t^3-3t^2+2t)\right). \tag{7}$$

By the Representation Theorem [11] and Theorem 2.1, we see $(\overline{y}(t,r), \underline{y}(t,r))$ represents a valued fuzzy number $t^3-3t^2+2t \leq 0$. Hence (7) represents fuzzy real number for $t=0$ or $t \in [1,2]$. Now we find the range of (1,1,2), (1,2,1), (2,1,1), and (2,2,2)-solutions of the differential equation (4) separately.
$(1,1,2)$-solution
$y'(t) = (\frac{2-r}{6}(3t^2-6t+2), \frac{r}{6}(3t^2-6t+2))$ is a fuzzy number when $t \in [1, \frac{1}{3}(3+\sqrt{3})]$. $y''(t) = ((2-r)(t-1), r(t-1))$ is a fuzzy number when $t=1$. By the Definition 3.5, $D^3_{1,1,2}y(t)$ does not exist. Hence y in (7) is not a (1,1,2)-solution of the fuzzy differential equation (4).
$(1,2,1)$-solution
$y'(t) = (\frac{2-r}{6}(3t^2-6t+2), \frac{r}{6}(3t^2-6t+2))$, $y''(t) = (r(t-1), (2-r)(t-1))$ and $y'''(t) = (r, 2-r)$ are fuzzy numbers when $t \in [1, \frac{1}{3}(3+\sqrt{3})]$. Hence $y(t)$, $D^1_1y(t)$, $D^2_{1,2}y(t)$ and $D^3_{1,2,1}y(t)$ are valid fuzzy numbers for $t \in [1, \frac{1}{3}(3+$

$\sqrt{3})]$ and y in (7) is a (1,2,1)-solution of the fuzzy differential equation (4) on $t \in [1, \frac{1}{3}(3+\sqrt{3})]$.

$(2,1,1)$-solution

$y'(t) = (\frac{r}{6}(3t^2 - 6t + 2), \frac{2-r}{6}(3t^2 - 6t + 2))$ is a fuzzy number when $t = 0$ or $t \in [\frac{1}{3}(3+\sqrt{3}), 2]$. $y''(t) = (r(t-1), (2-r)(t-1))$ and $y'''(t) = (r, 2-r)$ are fuzzy numbers when $t \in [\frac{1}{3}(3+\sqrt{3}), 2]$. Hence $y(t)$, $D^1_2 y(t)$, $D^2_{2,1}y(t)$ and $D^3_{2,1,1}y(t)$ are valid fuzzy numbers for $t \in [\frac{1}{3}(3+\sqrt{3}), 2]$ and y in (7) is a (2,1,1)-solution of the fuzzy differential equation (4) on $t \in [\frac{1}{3}(3+\sqrt{3}), 2]$.

$(2,2,2)$-solution

$y'(t) = (\frac{r}{6}(3t^2 - 6t + 2), \frac{2-r}{6}(3t^2 - 6t + 2))$ is a fuzzy number when $t = 0$ or $t \in [\frac{1}{3}(3+\sqrt{3}), 2]$. $y''(t) = ((2-r)(t-1), r(t-1))$ is a fuzzy number when $t = 0$. By the Definition 3.5, $D^3_{2,2,2}y(t)$ does not exist. Hence y in (7) is not a (2,2,2)-solution of the fuzzy differential equation (4).

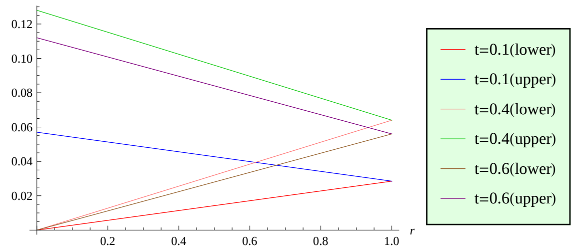

FIGURE 1. $\underline{y}_1(t,r)$ and $\overline{y}_1(t,r))$ for different t.

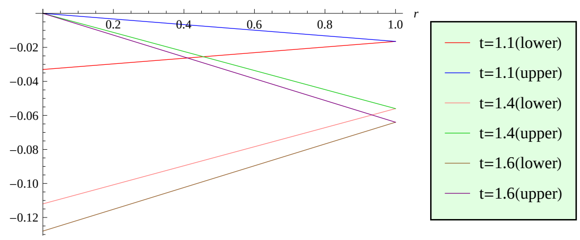

FIGURE 2. $\underline{y}_2(t,r)$ and $\overline{y}_2(t,r)$ for different $t \in [1,2]$.

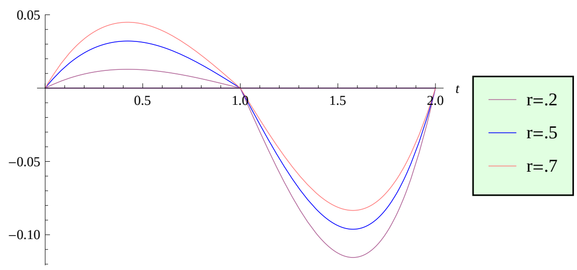

FIGURE 3. Lower branch of generalized solution different r.

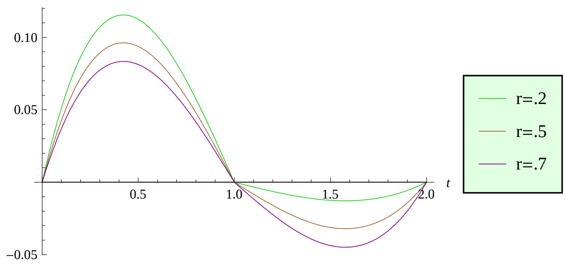

FIGURE 4. Upper branch of generalized solution different r.

There exists a fuzzy number valued function $y : [0, 2] \to \mathbb{R}_F$ such that

$$y(t) = \begin{cases} y_1(t) & \text{if } t \in (0, 1) \\ y_2(t) & \text{if } t \in (1, 2) \\ y_1(t) = y_2(t) & \text{if } t \in \{0, 1, 2\} \end{cases} \tag{8}$$

where $y_1(t) = \left(\frac{r}{6}(t^3 - 3t^2 + 2t), \frac{2-r}{6}(t^3 - 3t^2 + 2t)\right)$ for all $t \in [0, 1] \cup \{2\}$ and $y_2(t) = \left(\frac{2-r}{6}(t^3 - 3t^2 + 2t), \frac{r}{6}(t^3 - 3t^2 + 2t)\right)$ for all $t \in [1, 2] \cup \{0\}$ are fuzzy number valued functions and there exist $\frac{1}{3}(3 - \sqrt{3})$ and $\frac{1}{3}(3 + \sqrt{3})$ such that y_1 is a $(1, 2, 2)$ solution and $(2, 1, 2)$ solution of the equation (4) on $[0, \frac{1}{3}[3 - \sqrt{3})]$ and on $[\frac{1}{3}(3 - \sqrt{3}), 1]$ respectively, y_2 is a $(1, 2, 1)$ solution and $(2, 1, 1)$ solution of the equation (4) on $[1, \frac{1}{3}(3 + \sqrt{3})]$ and on $[\frac{1}{3}(3 + \sqrt{3}), 2]$ respectively and y_1 y_2

satisfy the boundary conditions (5). Therefore y in (8) is a generalized solution of the fuzzy boundary value problem (4)-(5). y_1 and y_2 are shown in Figure 1 and Figure 2 respectively for different values of t. From these figures we see that y_1 and y_2 are fuzzy number valued functions. In Figure 3 and Figure 4, lower and upper branch of the generalized solution y are shown respectively for different values of r.

Example 5.2. Consider the following third order three-point boundary value problem:

$$y'''(t) = (r-1, 1-r), \tag{9}$$

$$y(0) = y(0.5) = y(1) = \widetilde{0} \tag{10}$$

By direct calculation, the corresponding solution of the (1,1,1), (1,2,2), (2,1,2), and (2,2,1) systems has necessarily the following expression

$$y(t) = \left(\frac{r-1}{12}(2t^3 - 3t^2 + t), \frac{1-r}{12}(2t^3 - 3t^2 + t)\right). \tag{11}$$

By the Representation Theorem [11] and Theorem 2.1, we see $(\overline{y}(t,r), \underline{y}(t,r))$ represents a valued fuzzy number $2t^3 - 3t^2 + t \geq 0$. Hence (11) represents fuzzy number for $t \in [0, \frac{1}{2}]$ or $t = 1$. Now we find the range of (1,1,1),(1,2,2),(2,1,2) and (2,2,1)-solutions of the fuzzy differential equation (9) separately.
(1,1,1)-solution
The (1)-derivative of (9) in that case is given by:

$$y'(t) = \left(\frac{r-1}{12}(6t^2 - 6t + 1), \frac{1-r}{12}(6t^2 - 6t + 1)\right)$$

and it a fuzzy number when $t \in [0, \frac{1}{6}(3 - \sqrt{3}]$ or $t = 1$. Then it is again (1)-differentiable

$$y''(t) = \left(\frac{r-1}{12}(12t - 6), \frac{1-r}{12}(12t - 6)\right)$$

and it is a fuzzy number when $t = 1$. By the Definition 3.5, $D^3_{1,1,1}y(t)$ does not exist. Hence y in (11) is not a (1,1,1)-solution of the fuzzy differential equation (9).
$(1, 2, 2)$-solution
$y'(t) = (\frac{r-1}{12}(6t^2 - 6t + 1), \frac{1-r}{12}(6t^2 - 6t + 1))$ is a fuzzy number when $t \in (0, \frac{1}{6}(3 - \sqrt{3}])$ or $t = 1$. $y''(t) = (\frac{1-r}{12}(12t - 6), \frac{r-1}{12}(12t - 6))$ and $y'''(t) = (r - 1, 1 - r)$ are fuzzy numbers when $t \in [0, \frac{1}{6}(3 - \sqrt{3})]$. Hence $y(t)$, $D^1_1 y(t)$, $D^2_{1,2}y(t)$ and $D^3_{1,2,2}y(t)$ are valid fuzzy numbers for $t \in [0, \frac{1}{6}(3 - \sqrt{3}]$ and y (11) is (1,2,2)-solution of the fuzzy differential equation (9) on $[0, \frac{1}{6}(3 - \sqrt{3})]$.
$(2, 1, 2)$-solution
$y'(t) = (\frac{1-r}{12}(6t^2 - 6t + 1), \frac{r-1}{12}(6t^2 - 6t + 1))$, $y''(t) = (\frac{1-r}{12}(12t - 6), \frac{r-1}{12}(12t - 6))$ and $y'''(t) = (r - 1, 1 - r)$ are fuzzy numbers when $t \in [\frac{1}{6}(3 - \sqrt{3}), \frac{1}{2}]$. Hence $y(t)$, $D^1_2 y(t)$, $D^2_{2,1}y(t)$ and $D^3_{2,1,2}y(t)$ are valid fuzzy numbers for $t \in$

$[\frac{1}{6}(3-\sqrt{3}), \frac{1}{2}]$ and y (11) is (2,1,2)-solution of the fuzzy differential equation (9) on $t \in [\frac{1}{6}(3-\sqrt{3}), \frac{1}{2}]$.

$(2,2,1)$-solution

$y'(t) = (\frac{1-r}{12}(6t^2-6t+1), \frac{r-1}{12}(6t^2-6t+1))$ is a fuzzy number when $t \in [\frac{1}{6}(3-\sqrt{3}), \frac{1}{2}]$. $y''(t) = (\frac{r-1}{12}(12t-6), \frac{1-r}{12}(12t-6))$ is a fuzzy number when $t = \frac{1}{2}$. By the Definition 3.5, $D^3_{2,2,1}y(t)$ does not exist. Hence y in (11) is not a (2,2,1)-solution of the fuzzy differential equation (11).

The solution of the remaining four systems (1,1,2), (1,2,1), (2,1,1), and (2,2,2) has the following form

$$y(t) = (\frac{1-r}{12}(2t^3-3t^2+t), \frac{r-1}{12}(2t^3-3t^2+t)). \tag{12}$$

By the Representation Theorem [11] and Theorem 2.1, we see $(\overline{y}(t,r), \underline{y}(t,r))$ represents a valued fuzzy number $2t^3-3t^2+t \leq 0$. Hence (12) represents fuzzy real number for $t = 0$ or $t \in [\frac{1}{2}, 1]$. Now we find the range of (1,1,2), (1,2,1), (2,1,1), and (2,2,2)-solutions of the fuzzy boundary value problem separately.

$(1,1,2)$-solution

$y'(t) = (\frac{1-r}{12}(6t^2-6t+1), \frac{r-1}{12}(6t^2-6t+1))$ is a fuzzy number when $t \in [\frac{1}{2}, \frac{1}{6}(3+\sqrt{3})]$. $y''(t) = (\frac{1-r}{12}(12t-6), \frac{r-1}{12}(12t-6))$is a fuzzy number when $t = \frac{1}{2}$. By the Definition 3.5, $D^3_{1,1,2}y(t)$ does not exist. Hence y in (12) is not a (1,1,2)-solution of the fuzzy differential equation (9).

$(1,2,1)$-solution

$y'(t) = (\frac{1-r}{12}(6t^2-6t+1), \frac{r-1}{12}(6t^2-6t+1))$ $y''(t) = (\frac{r-1}{12}(12t-6), \frac{1-r}{12}(12t-6))$ and $y'''(t) = (r-1, 1-r)$ are fuzzy numbers when $t \in [\frac{1}{2}, \frac{1}{6}(3+\sqrt{3})]$. Hence $y(t)$, $D^1_1 y(t)$, $D^2_{1,2}y(t)$ and $D^3_{1,2,1}y(t)$ are valid fuzzy numbers for $t \in [\frac{1}{2}, \frac{1}{6}(3+\sqrt{3})]$ and y (12) is (1,2,1)-solution of the fuzzy differential equation (9) on $t \in [\frac{1}{2}, \frac{1}{6}(3+\sqrt{3})]$.

$(2,1,1)$-solution

$y'(t) = (\frac{r-1}{12}(6t^2-6t+1), \frac{1-r}{12}(6t^2-6t+1))$ is a fuzzy number when $t = 0$ or $t \in [\frac{1}{6}(3+\sqrt{3}), 1]$. $y''(t) = (\frac{r-1}{12}(12t-6), \frac{1-r}{12}(12t-6))$ and $y'''(t) = (r-1, 1-r)$ are fuzzy numbers when $t \in [\frac{1}{6}(3+\sqrt{3}), 1]$. Hence $y(t)$, $D^1_2 y(t)$, $D^2_{2,1}y(t)$ and $D^3_{2,1,1}y(t)$ are valid fuzzy numbers for $t \in [\frac{1}{6}(3+\sqrt{3}), 1]$ and y (12) is (2,1,1)-solution of the fuzzy differential equation (9) on $t \in [\frac{1}{6}(3+\sqrt{3}), 1]$.

$(2,2,2)$-solution

$y'(t) = (\frac{r-1}{12}(6t^2-6t+1), \frac{1-r}{12}(6t^2-6t+1))$ is a fuzzy number when $t = 0$ or $t \in [\frac{1}{6}(3+\sqrt{3}), 1]$. $y''(t) = (\frac{1-r}{12}(12t-6), \frac{r-1}{12}(12t-6))$ is a fuzzy number when $t = 0$. By the Definition 3.5, $D^3_{2,2,1}y(t)$ does not exist. Hence y in (12) is not a (2,2,2)-solution of the fuzzy differential equation (9).

There exists a fuzzy number valued function $y : [0,1] \to \mathbb{R}_F$ such that

$$y(t) = \begin{cases} y_1(t) & \text{if } t \in (0, \frac{1}{2}) \\ y_2(t) & \text{if } t \in (\frac{1}{2}, 1) \\ y_1(t) = y_2(t) & \text{if } t \in \{0, \frac{1}{2}, 1\} \end{cases} \tag{13}$$

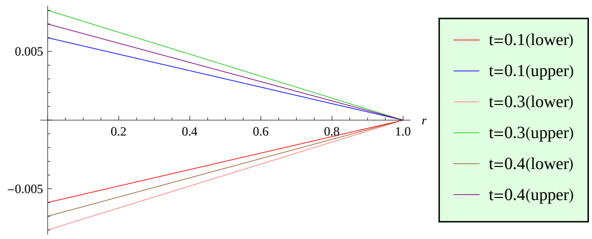

FIGURE 5. $\underline{y}_1(t, r)$ and $\overline{y}_1(t, r)$ for different t.

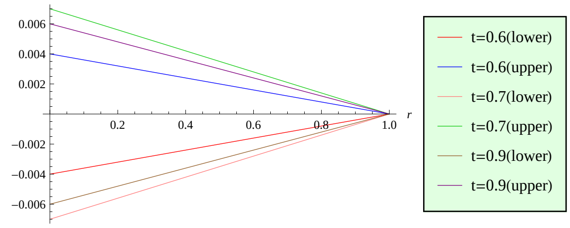

FIGURE 6. $\underline{y}_2(t, r)$ and $\overline{y}_2(t, r)$ for different t.

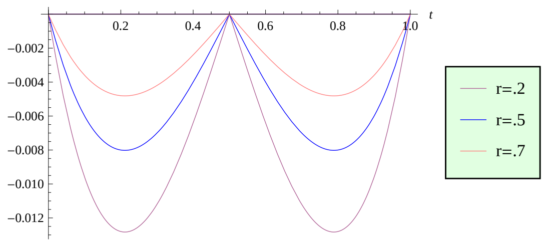

FIGURE 7. Lower branch of generalized solution for different r.

where $y_1(t) = \left(\frac{r-1}{12}(2t^3 - 3t^2 + t), \frac{1-r}{12}(2t^3 - 3t^2 + t)\right)$ for all $t \in [0, \frac{1}{2}] \cup \{1\}$ and $y_2(t) = \left(\frac{1-r}{12}(2t^3 - 3t^2 + t), \frac{r-1}{12}(2t^3 - 3t^2 + t)\right)$ for all $t \in [\frac{1}{2}, 1] \cup \{0\}$ are fuzzy number valued function and there exist $\frac{1}{6}(3-\sqrt{3})$ and $\frac{1}{6}(3+\sqrt{3})$ such that y_1 is a $(1, 2, 2)$-solution and $(2, 1, 2)$-solution of the equation (9) on $[0, \frac{1}{6}[3-\sqrt{3})]$ and on $[\frac{1}{6}(3-\sqrt{3}), \frac{1}{2}]$ respectively, y_2 is a $(1, 2, 1)$-solution and $(2, 1, 1)$-solution of the

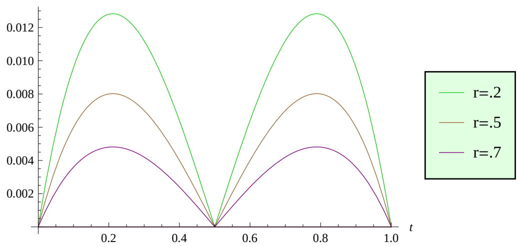

FIGURE 8. Upper branch of generalized solution for different r.

equation (9) on $[\frac{1}{2}, \frac{1}{6}(3+\sqrt{3})]$ and on $[\frac{1}{6}(3+\sqrt{3}), 1]$ respectively and y_1 y_2 satisfy the boundary conditions (10). Therefore y in (13) is a generalized solution of the fuzzy boundary value problem (9)-(10). y_1 and y_2 are shown in Figure 5 and Figure 6 respectively for different values of t. From these figures we see that y_1 and y_2 are fuzzy number valued functions. In Figure 7 and Figure 8, lower and upper branch of the generalized solution y are shown respectively for different values of r.

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