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ON POLAR TAXICAB GEOMETRY IN A PLANE

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Abstract. Most distance functions, including taxicab distance, are defined on Cartesian plane, and recent studies on distance functions have been mainly focused on Cartesian plane. However, most streets in cities include not only straight lines but also curves. Therefore, there is a significant need for a distance function to be defined on a curvilinear coordinate system. In this paper, we define a new function named polar taxicab distance, using polar coordinates. We prove that this function satisfies the conditions of distance function. We also investigate the geometric properties and classifications of circles in the plane with polar taxicab distance.

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1. Introduction

In Euclidean geometry, the distance between two points is called Euclidean distance and is defined as the length of a line segment connected by two endpoints in a straight line. Euclidean geometry has been widely used because it is easy to understand intuitively, and appropriate for applying various theories.

However, there is a limitation in applying the Euclidean distance function to measure the distance between two places in real life since there are many obstacles, such as structures and roads on a route. Accordingly, the idea of how a taxi travels in modern cities was developed into a practical distance notion, called the taxicab distance[4]. Taxicab distance measures the shortest distance between two points when only movements along axis-directions are permitted.

Nevertheless, taking account of the fact that not all routes in real life are composed of right angles, taxicab distance was generalized into the alpha distance[3, 5], which includes taxicab distance and Chinese checker distance[2, 5] as special cases. Also, a distance function called generalized absolute-value metric[1] was

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introduced to generalize taxicab distance, Chinese checker distance, and alpha distance.

All distance functions mentioned above are defined on the Cartesian plane, and recent studies on distance functions have been mainly focused on the Cartesian plane. However, most streets in cities include not only straight lines but also curves. Therefore, there is a significant need for a distance function to be defined on a curvilinear coordinate system. Hence, we introduce a new distance function, namely, the polar taxicab distance, using the polar coordinate system.

In this study, we prove that the polar taxicab distance satisfies the conditions of a distance function and that the polar taxicab distance between any two points is preserved by a reflection across the line through the origin and a rotation around the origin. Further, we study the geometric properties and classifications of circles in the plane with the polar taxicab distance function.

2. Polar Taxicab Distance

Let $P(r, \theta)$ be a point in the polar coordinate plane with $r \geq 0$ and $0 \leq \theta < 2\pi$. Now, we define the function $d_{PT}(A, B)$ as

$$d_{PT}(A,B) = \begin{cases} \min\{r_1, r_2\} \times |\theta_2 - \theta_1| + |r_2 - r_1| & (0 \leq |\theta_2 - \theta_1| \leq 2) \\ r_1 + r_2 & (2 < |\theta_2 - \theta_1| \leq \pi), \end{cases} \tag{1}$$

where $A(r_1, \theta_1)$ and $B(r_2, \theta_2)$ are points in a plane expressed by polar coordinates. A few useful lemmas are introduced to prove that d_{PT} is a distance function.

Lemma 2.1. *For two points A and B in the polar coordinate plane, $d_{PT}(A, B)$ is preserved by the reflection on the line that passes through the origin.*

Proof. Let $A(r_1, \theta_1)$ and $B(r_2, \theta_2)$ be the points in the polar coordinate plane, and let A' and B' be the reflections of A and B on the line $\theta = \phi$. Then, the polar coordinates of A' and B' are given by $(r_1, 2\phi - \theta_1)$ and $(r_2, 2\phi - \theta_2)$, respectively. Henceforth, it is sufficient to consider the following two cases.

Case 1. $0 \leq |\theta_1 - \theta_2| \leq 2$

Since $0 \leq |(2\phi - \theta_1) - (2\phi - \theta_2)| \leq 2$, we obtain that $d_{PT}(A, B) = |r_1 - r_2| + \min\{r_1, r_2\}|\theta_1 - \theta_2|$ and $d_{PT}(A', B') = |r_1 - r_2| + \min\{r_1, r_2\}|(2\phi - \theta_1) - (2\phi - \theta_2)|$. Hence, we see that $d_{PT}(A, B) = d_{PT}(A', B')$.

Case 2. $2 < |\theta_1 - \theta_2| \leq \pi$

Since $2 < |(2\phi - \theta_1) - (2\phi - \theta_2)| \leq \pi$, we obtain that $d_{PT}(A, B) = r_1 + r_2$ and $d_{PT}(A', B') = r_1 + r_2$. Hence, we get $d_{PT}(A, B) = d_{PT}(A', B')$.

Therefore, we have $d_{PT}(A, B) = d_{PT}(A', B')$, which implies that the reflection on the line which passes through the origin preserves $d_{PT}(A, B)$. □

Lemma 2.2. *For two points A and B in the polar coordinate plane, $d_{PT}(A,B)$ is preserved by the rotation around the origin.*

Proof. Let $A(r_1, \theta_1)$ and $B(r_2, \theta_2)$ be the points in the polar coordinate plane and let A' and B' be the rotations by ϕ around the origin. Then, the polar coordinates of A' and B' are given by $(r_1, \theta_1 + \phi)$ and $(r_2, \theta_2 + \phi)$, respectively. Henceforth, it is sufficient to consider the following two cases.

Case 1. $0 \le |\theta_1 - \theta_2| \le 2$

Since $0 \le |(\theta_1 + \phi) - (\theta_2 + \phi)| \le 2$, we obtain that

$$d_{PT}(A,B) = |r_1 - r_2| + \min\{r_1, r_2\}|\theta_1 - \theta_2|$$

and

$$d_{PT}(A',B') = |r_1 - r_2| + \min\{r_1, r_2\}|(\theta_1 + \phi) - (\theta_2 + \phi)|.$$

Hence, we see that $d_{PT}(A,B) = d_{PT}(A',B')$.

Case 2. $2 < |\theta_1 - \theta_2| \le \pi$

Since $2 < |(\theta_1 + \phi) - (\theta_2 + \phi)| \le \pi$, we obtain that $d_{PT}(A,B) = r_1 + r_2$ and $d_{PT}(A',B') = r_1 + r_2$. Hence, we get $d_{PT}(A,B) = d_{PT}(A',B')$.

Therefore, we have $d_{PT}(A,B) = d_{PT}(A',B')$, which implies that the rotation around the origin preserves $d_{PT}(A,B)$. □

Theorem 2.3. *The function d_{PT} defined as* (1) *determines a distance function for $\mathbb{R}^2$.*

Proof. Let O be the origin. For any three points on the polar coordinate plane, without loss of generality, we can label these three points to satisfy the following conditions:

(i) Three points are assigned as A, B, and C counterclockwise.

(ii) Let $\angle AOB = \phi_1$, $\angle BOC = \phi_2$, and $\angle COA = \phi_3$. Then $0 \le \phi_1$, ϕ_2, $\phi_3 \le \pi$, and $\phi_3 \ge \max\{\phi_1, \phi_2\}$.

Now, let the coordinates of A, B, and C be $A(r_1, \theta_1)$, $B(r_2, \theta_2)$, and $C(r_3, \theta_3)$, respectively.

Since r_1, $r_2 \ge 0$, $|\theta_2 - \theta_1| \ge 0$, and $|r_2 - r_1| \ge 0$, we have $d_{PT}(A,B) \ge 0$. Also, $d_{PT}(A,B)$ equals 0 if and only if either $r_1 + r_2 = 0$ or $|r_2 - r_1| = |\theta_2 - \theta_1| = 0$. Therefore, $d_{PT}(A,B) = 0$ if and only if $A = B$. Thus, d_{PT} is positive definite.

Clearly, $d_{PT}(A,B) = d_{PT}(B,A)$.

Finally, for any two points X and Y with $\angle XOY = \phi$ $(0 \le \phi \le \pi)$, we define a function f as

$$f(X,Y) = \begin{cases} 0 & (0 \le \phi \le 2) \\ 1 & (2 < \phi \le \pi). \end{cases}$$

Then $f(A,B) + f(B,C) + f(C,A)$ is equal to one of 0, 1, 2, and 3. Hence, we investigate the cases according to the values of $f(A,B) + f(B,C) + f(C,A)$.

Case 1. $f(A,B)+f(B,C)+f(C,A)=0$

Let A' and C' be the reflections of A and C on the line OB. Then, $d_{PT}(A,B)=d_{PT}(A',B)$, $d_{PT}(B,C)=d_{PT}(B,C')$, and $d_{PT}(C,A)=d_{PT}(C',A')$ by Lemma 2.1. If $r_1<r_3$, then $d_{PT}(O,A')>d_{PT}(O,C')$. Renaming C', B, and A' as A, B, and C, respectively, would lead $r_1>r_3$. Therefore, we can assume $r_1\geq r_3$ without loss of generality.

Then, $d_{PT}(A,B)=\min\{r_1,r_2\}\phi_1+|r_1-r_2|$, $d_{PT}(B,C)=\min\{r_2,r_3\}\phi_2+|r_2-r_3|$, and $d_{PT}(C,A)=\min\{r_3,r_1\}(\phi_1+\phi_2)+|r_3-r_1|$. Hence, we can see that

$$d_{PT}(A,B)+d_{PT}(B,C)-d_{PT}(C,A)=\begin{cases}2(r_2-r_1)+(r_1-r_3)\phi_1 & (r_2\geq r_1\geq r_3)\\ (r_2-r_3)\phi_1 & (r_1\geq r_2\geq r_3)\\ (r_3-r_2)(2-\phi_1-\phi_2) & (r_1\geq r_3\geq r_2),\end{cases} \tag{2}$$

$$d_{PT}(B,C)+d_{PT}(C,A)-d_{PT}(A,B)=\begin{cases}2(r_2-r_3)+r_1(2-\phi_1)+r_3(\phi_1+2\phi_2) & (r_2\geq r_1\geq r_3)\\ (r_2-r_3)(2-\phi_1)+2r_3\phi_2 & (r_1\geq r_2\geq r_3)\\ (r_3-r_2)\phi_1+(r_2+r_3)\phi_2 & (r_1\geq r_3\geq r_2),\end{cases} \tag{3}$$

and

$$d_{PT}(C,A)+d_{PT}(A,B)-d_{PT}(B,C)=\begin{cases}(r_1+r_3)\phi_1 & (r_2\geq r_1\geq r_3)\\ 2(r_1-r_2)+(r_1+r_2)\phi_1 & (r_1\geq r_2\geq r_3)\\ 2(r_1-r_3)+(r_2+r_3)\phi_1+(r_3-r_2)\phi_2 & (r_1\geq r_3\geq r_2).\end{cases} \tag{4}$$

From our hypotheses, we can easily obtain that each expression of (2), (3) and (4) is nonnegative. Therefore, for Case 1, the triangle inequality holds.

Case 2. $f(A,B)+f(B,C)+f(C,A)=1$

If $r_1<r_3$, let A' and C' be the reflections of A and C on the line OB. Since $d_{PT}(A,B)=d_{PT}(A',B)$, $d_{PT}(B,C)=d(B,C')$, and $d_{PT}(C,A)=d_{PT}(C',A')$, by Lemma 2.1, considering A, B, and C as C', B, and A', respectively, generates the same situation. Thus, we can assume $r_1\geq r_3$ without loss of generality. Then, $0\leq\phi_1$, $\phi_2\leq 2$, $2<\phi_1+\phi_2\leq\pi$ since $f(A,B)+f(B,C)+f(C,A)=1$.

Then, $d_{PT}(A,B)=\min\{r_1,r_2\}\phi_1+|r_1-r_2|$, $d_{PT}(B,C)=\min\{r_2,r_3\}\phi_2+|r_2-r_3|$, and $d_{PT}(C,A)=r_1+r_3$. In order to prove that the triangle inequality holds, we should check the signs of the following three equations.

Firstly,

$$d_{PT}(A,B)+d_{PT}(B,C)-d_{PT}(C,A)=\begin{cases}2(r_2-r_3)+r_1(\phi_1+\phi_2-2) & (r_2\geq r_1\geq r_3) & (5)\\ r_2\phi_1+r_3\phi_2-2r_3 & (r_1\geq r_2\geq r_3) & (6)\\ r_2(\phi_1+\phi_2-2) & (r_1\geq r_3\geq r_2). & (7)\end{cases}$$

From our hypotheses, it is trivial that each expression of (5) and (7) is nonnegative. Note that $r_2\phi_1 + r_3\phi_2 - 2r_3 \geq r_3(\phi_1 + \phi_2 - 2) \geq 0$ since $2 < \phi_1 + \phi_2 \leq \pi$. Thus, the expression (6) is also nonnegative.

Next,

$$d_{PT}(B,C) + d_{PT}(C,A) - d_{PT}(A,B) = \begin{cases} r_1(2-\phi_1) + r_3\phi_2 & (r_2 \geq r_1 \geq r_3) \quad (8) \\ r_2(2-\phi_1) + r_3\phi_2 & (r_1 \geq r_2 \geq r_3) \quad (9) \\ 2r_3 + r_2(\phi_2 - \phi_1) & (r_1 \geq r_3 \geq r_2)\ . \quad (10) \end{cases}$$

Using our hypotheses, each expression of (8) and (9) is clearly nonnegative. Since $0 \leq \phi_1 \leq 2$, we have $2r_3 + r_2(\phi_2 - \phi_1) \geq r_2(2 - \phi_1 + \phi_2) \geq 0$ so that the expression (10) is also nonnegative.

Finally,

$$d_{PT}(C,A) + d_{PT}(A,B) - d_{PT}(B,C) = \begin{cases} r_3(2-\phi_2) + r_1\phi_1 & (r_2 \geq r_1 \geq r_3) \quad (11) \\ 2(r_1 - r_2) + r_3(2-\phi_2) + r_2\phi_1 & (r_1 \geq r_2 \geq r_3) \quad (12) \\ 2r_1 + r_2(\phi_1 - \phi_2) & (r_1 \geq r_3 \geq r_2). \quad (13) \end{cases}$$

From our hypotheses, we can easily obtain that each expression of (11) and (12) is nonnegative. Note that $0 \leq \phi_2 \leq 2$. Then we have $2r_1 + r_2(\phi_1 - \phi_2) \geq r_2(2 + \phi_1 - \phi_2) \geq 0$ so that the expression (13) is also nonnegative. Therefore, for Case 2, the triangle inequality holds.

Case 3. $f(A,B) + f(B,C) + f(C,A) = 2$

If $\phi_2 < \phi_1$, let A', C' be the reflections of A and C on the line OB. Since $d_{PT}(A,B) = d_{PT}(A',B)$, $d_{PT}(B,C) = d(B,C')$, and $d_{PT}(C,A) = d_{PT}(C',A')$, by Lemma 2.1, considering A, B, and C as C', B, and A', respectively, generates the same situation. Thus, we can assume $\phi_2 \geq \phi_1$ without loss of generality. Then, $0 \leq \phi_1 \leq 2$, $2 < \phi_2 \leq \pi$, and $2 < \phi_3 \leq \pi$ since $f(A,B) + f(B,C) + f(C,A) = 2$. Then, $d_{PT}(A,B) = \min\{r_1, r_2\}\phi_1 + |r_1 - r_2|$, $d_{PT}(B,C) = r_2 + r_3$, and $d_{PT}(C,A) = r_1 + r_3$. Hence, we can see that

$$d_{PT}(A,B) + d_{PT}(B,C) - d_{PT}(C,A) = \begin{cases} r_2\phi_1 & (r_1 \geq r_2) \\ 2(r_2 - r_1) + r_1\phi_1 & (r_1 < r_2), \end{cases} \quad (14)$$

$$d_{PT}(B,C) + d_{PT}(C,A) - d_{PT}(A,B) = \begin{cases} 2r_3 + r_2(2-\phi_1) & (r_1 \geq r_2) \\ 2r_3 + r_1(2-\phi_1) & (r_1 < r_2), \end{cases} \quad (15)$$

and

$$d_{PT}(C,A) + d_{PT}(A,B) - d_{PT}(B,C) = \begin{cases} 2(r_1 - r_2) + r_2\phi_1 & (r_1 \geq r_2) \\ r_1\phi_1 & (r_1 < r_2). \end{cases} \quad (16)$$

From our hypotheses, we can easily obtain that each expression of (14), (15), and (16) is nonnegative. Therefore, for Case 3, the triangle inequality holds.

Case 4. $f(A,B) + f(B,C) + f(C,A) = 3$

Then, $d_{PT}(A,B) = r_1 + r_2$, $d_{PT}(B,C) = r_2 + r_3$, and $d_{PT}(C,A) = r_1 + r_3$. Hence, we can see that

$$d_{PT}(A,B) + d_{PT}(B,C) - d_{PT}(C,A) = 2r_2 \geq 0,$$

$$d_{PT}(B,C) + d_{PT}(C,A) - d_{PT}(A,B) = 2r_3 \geq 0,$$

and

$$d_{PT}(C,A) + d_{PT}(A,B) - d_{PT}(B,C) = 2r_1 \geq 0.$$

Therefore, for Case 4, we easily obtain that the triangle inequality holds. Consequently, the proof of the theorem is completed. □

Due to Theorem 2.3, we define a new distance function.

Definition 2.4. A function d_{PT} defined as (1) is called a polar taxicab distance function.

3. Circles in Polar Taxicab Geometry

A polar taxicab circle in polar taxicab geometry is a set of points that has the same polar taxicab distance from a fixed point, as that in Euclidean geometry. The general shape of a polar taxicab circle changes as the center of the polar taxicab circle changes. In addition, its shape varies as the ratio of the radius and the distance from the center to the origin changes.

Theorem 3.1. *The locus of points whose polar taxicab distance from the origin is constant is a Euclidean circle.*

Proof. It is obvious that $d_{PT}(O,P) = |r|$ for any point $P(r,\theta)$ on the polar taxicab circle. For the points whose polar taxicab distance from O is R, Euclidean distance from O to P is also R. Therefore, the locus of P is a Euclidean circle. □

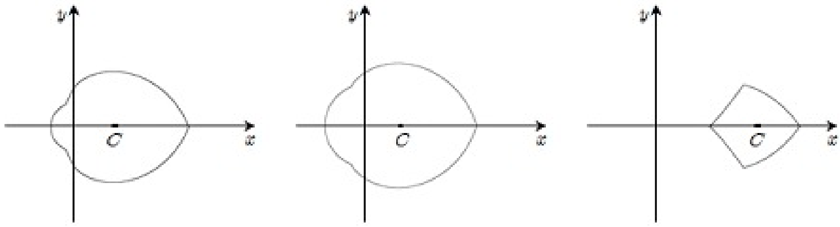
Theorem 3.2. *The locus of points whose polar taxicab distance from a point C is constant is one of three configurations shown in Figure 1(a), 1(b), and 1(c), where C is not the origin.*

Proof. Since rotation around the origin does not change the shape of figure by Lemma 2.2, we can put the point C on the axis by using rotation around the origin. Let the coordinate of C be $C(k,0)$, and let $P(r,\theta)$ be a point on the polar taxicab circle satisfying $d_{PT}(C,P) = R$.

Since the region where $0 \leq \theta \leq \pi$ and the region where $-\pi \leq \theta \leq 0$ are symmetric, it is sufficient to consider the region where $0 \leq \theta \leq \pi$. Thus, there are the following two cases to consider.

Case 1. $R \geq k$

For the region where $0 \leq \theta \leq 2$, if $r \leq k$, then $d_{PT}(C,P) = k - r + r\theta = R$, so $r(\theta - 1) = R - k$. If there is a point $P(r,\theta)$ such that $0 \leq \theta < 1$ and



(a) (b) (c)

FIGURE 1. types of polar taxicab circles.

$d_{PT}(C,P) = R$, then $d_{PT}(C,P) = k - r + r\theta = R$. However, $k - r + r\theta < k$ leads to a contradiction. Thus, if there exists P and $d_{PT}(C,P) = R$, then P lies on $1 \leq \theta \leq 2$. Also, $R - k \leq k(\theta - 1)$ holds since $r(\theta-1) \leq k(\theta-1)$ and $r(\theta - 1) = R - k$. Hence, if there exists θ which satisfies $\frac{R}{k} \leq \theta$, then $R \leq 2k$.

Therefore, there exists P only if $1 \leq \theta \leq 2$, $k \leq R \leq 2k$, and its locus is $r(\theta - 1) = R - k$.

If $r > k$, then $d_{PT}(C,P) = r - k + k\theta = R$, so $r + k\theta = k + R$. Also, $k + R - k\theta > k$, and $\theta < \frac{R}{k}$. Thus there always exists θ such that $\theta < \frac{R}{k}$, and its locus is $r + k\theta = k + R$.

For the region where $2 < \theta \leq \pi$, $d_{PT}(C,P) = r + k = R$, so $r = R - k$. Therefore, there always exists r which satisfies $r = R - k$ since $R - k > 0$ and its locus is $r = R - k$.

Case 2. $R < k$

If there is a point $P(r,\theta)$ such that $2 < \theta \leq \pi$ and $d_{PT}(C,P) = R$, then $d_{PT}(C,P) = r + k \geq k$. However, $R < k$ leads to a contradiction. Thus, if there exists P and $d_{PT}(C,P) = R$, then $\theta \leq 2$.

If $r \leq k$, then $d_{PT}(C,P) = k - r + r\theta = R$, so $r(1-\theta) = k - R$. $r(1-\theta) > 0$ since $R < k$ and there exists θ only if $\theta < 1$. Also, $k - R \leq k - k\theta$ holds since $r(1-\theta) \leq k(1-\theta)$. Therefore, there exists P only if $0 \leq \theta \leq \frac{R}{k}$ and its locus is $r(1-\theta) = k - R$.

If $r > k$, then $d_{PT}(C,P) = r - k + k\theta = R$, so $r + k\theta = R + k$. $k + R - k\theta > k$ since $r > k$, resulting in $\theta < \frac{R}{k}$. Therefore, there exists P only if $\theta < \frac{R}{k}$, and its locus is $r + k\theta = R + k$.

In Case 1, there are two types of polar taxicab circles where $1 \leq \frac{R}{k} < 2$ and $2 \leq \frac{R}{k}$. In Case 2, there is one type of circle where $0 < \frac{R}{k} < 1$ in polar taxicab geometry. Therefore, if point C is not the origin, there are three possible types of the locus of points, as illustrated in Figure 1(a), Figure 1(b), and Figure 1(c), whose polar taxicab distance from the point C is constant. □

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