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NONCONFORMING SPECTRAL ELEMENT METHOD FOR ELASTICITY INTERFACE PROBLEMS

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ABSTRACT. An exponentially accurate nonconforming spectral element me--thod for elasticity systems with discontinuities in the coefficients and the flux across the interface is proposed in this paper. The method is least-squares spectral element method. The jump in the flux across the interface is incorporated (in appropriate Sobolev norm) in the functional to be minimized. The interface is resolved exactly using blending elements. The solution is obtained by the preconditioned conjugate gradient method. The numerical solution for different examples with discontinuous coefficients and non-homogeneous jump in the flux across the interface are presented to show the efficiency of the proposed method.

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1. Introduction

The elliptic interface problems arises in many engineering problems, for example, in heat conduction or elasticity problems where the domain of definition is composed of different materials. In this paper we study the nonconforming spectral element method for elasticity interface problems. These problems have wide applications in continuum mechanics, multi-phase elasticity problems, etc.

There exists several methods in the literature to solve elliptic interface problems (see [26]). There are two types of finite element methods for elliptic interface problems: fitted and unfitted finite element methods. Fitted finite element method is a common approach where the mesh is fitted to the interface, so the interface conditions are satisfied in the weak formulation. The interface is approximated by the sides of isoparametric elements in the discretization. The accuracy of the method depends on the approximation quality of the interface. In this case, the method converges with optimal rates in h. If the mesh is not

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fitted to the interface, suboptimal convergence behavior will occur or may be the method does not converge at all. To avoid this difficulty, in [2] Babuska have formulated an equivalent minimization problem with all boundary and jump conditions incorporated in the cost functions.

Unfitted finite element methods are based on a mesh which is independent of interface. In [3] unfitted finite element method based on penalized problem as in [2] has been proposed. With an appropriate choice of penalty term the approximation converges to the solution at optimal rate (in h) in H^1 norm. Only lower order finite elements have been studied in the literature (for more details see [27]). In [27], a conforming higher order finite element method has been analyzed for elliptic interface problems. One can also look for different formats of finite element methods for interface problems in [24, 33, 35, 41]. Many nonconforming approaches are available in the literature like discontinuous Galerkin methods, Mortar finite element methods, etc (see [26]). In [8] a priori and a posteriori error estimates have been derived for discontinuous Galerkin method. Quasi optimal a priori estimates for interface problems even with lower smoothness conditions on the solution were derived.

This problem is also studied in the framework of least-squares finite element method [4, 5, 6, 11]. In these formulations, the given differential equation is converted into first order partial differential system and a suitable least-squares formulation is applied. Optimal convergence rates in h have been shown. The first order system least-squares method (FOSLS) for linear elasticity problems has been proposed in [9, 10]. Least-squares spectral element method has been proposed in [19, 20]. In [31, 32] iterative substructuring methods for spectral element discretizations of elliptic systems have been proposed. The method provides an efficient preconditioner with an optimal condition number. The extended finite element method (XFEM) or generalized finite element method (GFEM) is a useful method for approximating the solutions with singularities and solutions of interface problems. This method extends the FEM approach by enriching the solution space. The approximation consists of standard finite element approximation and the enrichment through the partition of unity concepts [17, 18].

Immersed interface method has been widely studied for elliptic interface problems [28]. Finite difference based explicit jump immersed interface method for elasticity systems was described in [25]. An immersed finite element method for elasticity equations with interfaces has been studied in [29, 39, 40]. In this method the mesh is independent of the interface and basis functions are chosen such that they satisfies the interface conditions. Optimal convergence rates in h have been derived. In [30], linear and bilinear immersed finite elements for planar elasticity interface problems have been discussed. $2D$ linear, bilinear immersed finite elements which satisfy the interface jump conditions were used. Optimal convergence rates in h were shown in L^2 and semi H^1 norms. Details and complete citation list on immersed finite element methods can be found in

[38]. In [12] an adaptive immersed interface finite element method for elasticity interface problem was presented.

In [22] Nitsche's method has been described. In [23] a finite element solution of elliptic interface problem using an approach due to Nitsche has been proposed. The method allows for discontinuities, internal to the elements, in the approximation across the interface and it was shown to be second order accurate (in h) in L^2 norm. In this method the interface conditions are satisfied weakly by means of variant of Nitsche's method. A hp Nitsche's method for interface problems with nonconforming unstructured finite element meshes have been proposed and error estimates with optimal bound in h and suboptimal bound in p by degree $p^{1/2}$ were obtained in [13].

In this paper, we propose a least-squares spectral element method for elasticity interface problems based on the method proposed in [26]. In the least-squares formulation of the method, a solution is sought which minimizes the sum of the squares of a squared norms of the residuals in the partial differential equation and the sum of the residuals in the boundary conditions in fractional Sobolev norms and the sum of the jumps in the displacement and the flux across the interface in appropriate fractional Sobolev norms and enforce the continuity along the inter element boundaries by adding a term which measures the sum of the squares of the jump in the function and its derivatives in fractional Sobolev norms. The proposed numerical formulation is based on the regularity estimate for the interface problems stated in [7] and the stability estimate proved in [26].

This method is nonconforming (in terms of approximation). This formulation is different from the standard techniques in LSFEM used to convert the second order elliptic equations into first order system. The interface is resolved completely using blending elements [21]. Higher order spectral elements are used to approximate the solution. The spectral elements are the sum of tensor products of the polynomials of degree W in each variable. The solution is obtained using preconditioned conjugate gradient method (PCGM) without storing the stiffness matrix and load vector. Even though we do not store the matrix, the added advantage of the proposed method is the resulting stiffness matrix is symmetric and positive definite. The integrals involved in the residual computations are obtained efficiently and inexpensively [36] (a brief description is given in the Appendix).

The rest of the paper is organized as follows: In Section 2 the elasticity interface problem is defined. The discretization of the domain is given in Section 3 and the numerical scheme is described. Finally in Section 4 numerical results are presented for various examples.

2. Elasticity interface problem

In this section we state the elasticity interface problem on a domain $\Omega \subseteq \mathbb{R}^2$. First we define the function spaces which we need in the latter sections.

Denoting $H^k(\Omega)$, the usual Sobolev space of integer order k with the norm

$\|.\|_{k,\Omega}$ as given below,

$$\|u(x,y)\|_{k,\Omega}^2 = \int_\Omega \sum_{\alpha_1+\alpha_2 \le k} \left|\partial_x^{\alpha_1}\partial_y^{\alpha_2} u(x,y)\right|^2 dx\,dy.$$

Further, let

$$\|u\|_{s,J}^2 = \int_J u^2(x)dx + \int_J\int_J \frac{|u(x)-u(x')|^2}{|x-x'|^{1+2s}} dx\,dx',$$

denote the fractional Sobolev norm of order s, where $0 < s < 1$. Here J denotes an interval contained in $\mathbb{R}$.

We denote vectors and vector spaces by bold characters. For example, $\mathbf{u} = (u_1, u_2)^T$, $\mathbf{H}^k(\Omega) = H^k(\Omega) \times H^k(\Omega)$, etc. The norms are given by $\|\mathbf{u}\|_{k,\Omega}^2 = \|u_1\|_{k,\Omega}^2 + \|u_2\|_{k,\Omega}^2$ for $\mathbf{u} \in \mathbf{H}^k(\Omega)$, $\|\mathbf{u}\|_{s,J}^2 = \|u_1\|_{s,J}^2 + \|u_2\|_{s,J}^2$, etc.

2.1. Linear elasticity system. Let $\mathbf{x} = (x,y)$ be a point in space, $\mathbf{u} = (u_1(x,y), u_2(x,y))^T$ be the displacement vector and $\epsilon = (\epsilon_{ij})$ be the strain tensor. If u_1, u_2 are the two planar displacement components, then the strain-displacement relation is given by

$$\epsilon_{11} = \frac{\partial u_1}{\partial x}\,,\ \epsilon_{22} = \frac{\partial u_2}{\partial y}\,,\ \epsilon_{12} = \epsilon_{21} = \frac{1}{2}\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right).$$

The relation between stresses and strains (from the Hooke's law) is given by,

$$\sigma_{ij} = \lambda(\bigtriangledown.\mathbf{u})\delta_{ij} + 2\mu\varepsilon_{ij}(\mathbf{u}),\ i,j = 1,2, \tag{1}$$

where λ and μ are the Lame coefficients, and

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}, \qquad \bigtriangledown.\mathbf{u} = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y}.$$

Let $\sigma = (\sigma_{ij})$ be the stress tensor, $\mathbf{f}(\mathbf{x}) = (f_1, f_2)^T$ be the applied body forces, then the stress tensor satisfies the following partial differential equations,

$$-\bigtriangledown .\sigma = \mathbf{f}, \qquad \text{i.e,} \qquad \begin{cases} -\frac{\partial \sigma_{11}}{\partial x} - \frac{\partial \sigma_{12}}{\partial y} = f_1 \\ -\frac{\partial \sigma_{21}}{\partial x} - \frac{\partial \sigma_{22}}{\partial y} = f_2. \end{cases} \tag{2}$$

From the above equations, we can re-write the above system as the system of plane elasticity equations of the following,

$$\begin{aligned} \mathcal{L}_1\mathbf{u} &= -\left\{(\lambda+2\mu)\frac{\partial^2 u_1}{\partial x^2} + (\lambda+\mu)\frac{\partial^2 u_2}{\partial x \partial y} + \mu\frac{\partial^2 u_1}{\partial y^2}\right\} = f_1, \\ \mathcal{L}_2\mathbf{u} &= -\left\{(\lambda+2\mu)\frac{\partial^2 u_2}{\partial y^2} + (\lambda+\mu)\frac{\partial^2 u_1}{\partial x \partial y} + \mu\frac{\partial^2 u_2}{\partial x^2}\right\} = f_2. \end{aligned} \tag{3}$$

The Lame coefficients λ and μ are given by

$$\mu = \frac{E}{2(1+\nu)},\ \lambda = \frac{\nu E}{(1-2\nu)(1+\nu)}(\text{plane strain}), \lambda = \frac{\nu E}{(1-\nu^2)}\ (\text{plane stress}) \tag{4}$$

where E is the Young's modulus and ν is the Poisson's ratio. Let us assume that the constants λ and μ have finite jumps across the interface; so does the flux $\sigma\mathbf{n}$. Now the elasticity interface problem is defined below.

2.2. The interface problem. Let Ω and $\Omega_1(\overline{\Omega}_1 \subset \Omega)$ be open bounded domains with boundaries $\partial\Omega = \Gamma$ $(\bar{\Omega} = \Omega \cup \partial\Omega)$ and Γ_0 respectively. Assume that the boundary Γ_0 is sufficiently smooth. Further, let $\Omega_2 = \Omega \setminus \overline{\Omega}_1$. Let $\mathbf{u}_1 = (u_1^1, u_2^1)^T = \mathbf{u}\mid_{\Omega_1}$ and $\mathbf{u}_2 = (u_1^2, u_2^2)^T = \mathbf{u}\mid_{\Omega_2}$. Now the elasticity interface problem can be written as follows:

$$\begin{aligned} \mathcal{L}\mathbf{u} &= \mathbf{f} \text{ in } \Omega_1 \cup \Omega_2 \\ [\mathbf{u}] &= 0 \quad \text{ on } \Gamma_0 \\ [\sigma\mathbf{n}] &= \mathbf{q}, \\ \mathbf{u} &= \mathbf{g} \text{ on } \Gamma \end{aligned} \tag{5}$$

where $\mathcal{L}\mathbf{u} = (\mathcal{L}_1\mathbf{u}, \mathcal{L}_2\mathbf{u})^T$, $\mathbf{f} = (f_1, f_2)^T$, $\mathbf{q} = (q_1, q_2)^T$, $\mathbf{g} = (g_1, g_2)^T$ are known vector functions. $\mathbf{n}$ is the unit outward normal to the interface Γ_0. The jump [.] is defined as the difference of the limiting values from the outside of the interface to the inside. The coefficients λ and μ are piecewise constant, *i.e.*,

$$\lambda = \begin{cases} \lambda_1 & \text{in } \Omega_1 \\ \lambda_2 & \text{in } \Omega_2 \end{cases} \text{ and } \mu = \begin{cases} \mu_1 & \text{in } \Omega_1 \\ \mu_2 & \text{in } \Omega_2 \end{cases}. \tag{6}$$

3. Discretization and Numerical Scheme

Considered the circular domain Ω_1 such that $\overline{\Omega}_1 \subset \Omega$, where Ω is square whose boundary is $\Gamma = \cup_{i=1}^4 \Gamma^i$ as shown in Figure 1, for brevity. Let $\Omega_2 = \Omega \setminus \overline{\Omega}_1$ and the interface is Γ_0 which is smooth as shown in Figure 1. The results presented are applicable to arbitrary smooth interfaces also.

Now the domain Ω_1 and Ω_2 are partitioned into finite number of quadrilateral subdomains (elements) $\Omega_1^1, \Omega_1^2, ..., \Omega_1^p$ and $\Omega_2^1, \Omega_2^2, ..., \Omega_2^q$ such that the subdomain divisions match on the interface. The interface is completely resolved using blending elements [21].

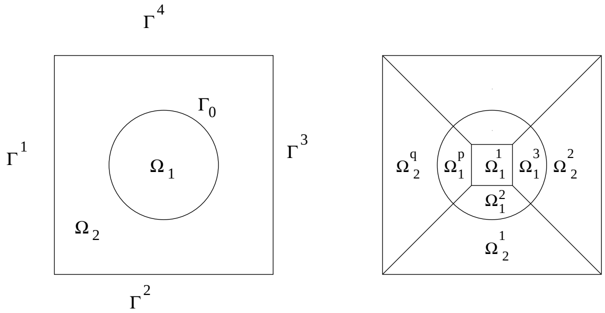


FIGURE 1. The domain Ω and discretization.

Each element is mapped to the master square $S=(-1,1)^2$. Define an analytic map M_i^l from the master square $S=(-1,1)^2$ to Ω_i^l by (see [1, 21])

$$x = X_i^l(\xi,\eta), \quad y = Y_i^l(\xi,\eta), \quad i = 1,2.$$

A brief description of the map is given in the Appendix A1. Here and in the rest of this section $l=1,...,p$ for $i=1$ and $l=1,...,q$ for $i=2$.

Define the spectral element functions [1] $\left\{\tilde{u}_1^{i,l}\right\}, \left\{\tilde{u}_2^{i,l}\right\}$ as the tensor product of polynomials of degree W in each variable ξ and η as

$$\tilde{u}_1^{i,l}(\xi,\eta) = \sum_{r=0}^{W}\sum_{s=0}^{W} g_{r,s}^{i,l}\xi^r\eta^s \text{ and } \tilde{u}_2^{i,l}(\xi,\eta) = \sum_{r=0}^{W}\sum_{s=0}^{W} h_{r,s}^{i,l}\xi^r\eta^s \text{ for } i=1,2.$$

Then [2] $\left\{u_1^{i,l}\right\}, \left\{u_2^{i,l}\right\}$ are given by

$$u_1^{i,l}(x,y) = \tilde{u}_1^{i,l}\left((M_i^l)^{-1}\right) \quad \text{and} \quad u_2^{i,l}(x,y) = \tilde{u}_2^{i,l}\left((M_i^l)^{-1}\right).$$

Now

$$\int_{\Omega_i^l} \left|\mathcal{L}\mathbf{u}_i^l\right|^2 dxdy = \int_S \left|\mathcal{L}\tilde{\mathbf{u}}_i^l\right|^2 J_i^l \; d\xi d\eta \text{ for } i=1,2.$$

Here $J_i^l(\xi,\eta)$ is the Jacobian of the mapping M_i^l from S to Ω_i^l.
Define [3] $\mathcal{L}_i^l(\xi,\eta) = \mathcal{L}(\xi,\eta)\sqrt{J_i^l}$ (the differential operator in the transformed coordinates in the domains Ω_1 and Ω_2 respectively). Then

$$\int_{\Omega_i^l} \left|\mathcal{L}\mathbf{u}_i^l\right|^2 dxdy = \int_S \left|\mathcal{L}_i^l\tilde{\mathbf{u}}_i^l\right|^2 \; d\xi d\eta \;\; for \;\; i=1,2.$$

Define $\mathbf{f}_1 = (f_1^1, f_2^1)^T = \mathbf{f}\mid_{\Omega_1}$ and $\mathbf{f}_2 = (f_1^2, f_2^2)^T = \mathbf{f}\mid_{\Omega_2}$. Let $\mathbf{f}_i^l(\xi,\eta) = \mathbf{f}_i(M_i^l(\xi,\eta))$ for $i=1,2$. Define [4]

$$\mathbf{F}_i^l(\xi,\eta) = \mathbf{f}_i^l(\xi,\eta)\sqrt{J_i^l(\xi,\eta)} \qquad \text{for } i=1,2.$$

Let γ_s be a side common to the two adjacent elements Ω_i^m and $\Omega_i^n, i=1,2$ (as shown in Fig. 2(a) for $i=2$). Assume that γ_s is the image of $\eta=-1$ under the mapping M_i^m which maps S to Ω_i^m and also the image of $\eta=1$ under the mapping M_i^n which maps S to Ω_i^n. By chain rule

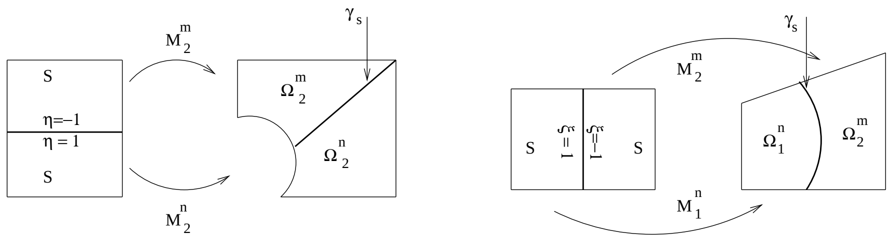
$$(\mathbf{u}_i^m)_x = (\tilde{\mathbf{u}}_i^m)_\xi\;\xi_x + (\tilde{\mathbf{u}}_i^m)_\eta\;\eta_x \text{ and } (\mathbf{u}_i^m)_y = (\tilde{\mathbf{u}}_i^m)_\xi\;\xi_y + (\tilde{\mathbf{u}}_i^m)_\eta\;\eta_y.$$

[1] The displacement components in the transformed coordinates, where i gives the domain status (Ω_1 or Ω_2) and l gives the number of the element in that domain.

[2] $\mathbf{u}_i^l = (u_1^{i,l}, u_2^{i,l})$ is vector $\mathbf{u}$ in element Ω_i^l

[3] $\mathcal{L}_i^l\tilde{\mathbf{u}}_i^l = (\mathcal{L}_1^{i,l}\tilde{\mathbf{u}}_i^l, \mathcal{L}_2^{i,l}\tilde{\mathbf{u}}_i^l)$

[4] $\mathbf{F}_i^l = (F_1^{i,l}, F_2^{i,l})$



(a) Inter element boundary (b) Along the interface

FIGURE 2. Elements with common edges

Then the jumps along the inter-element boundaries are defined as

$$\begin{aligned}
\|[\mathbf{u}_i]\|^2_{0,\gamma_s} &= \|\tilde{\mathbf{u}}_i^m(\xi,-1) - \tilde{\mathbf{u}}_i^n(\xi,1)\|^2_{0,I}\,, \\
\|[(\mathbf{u}_i)_x]\|^2_{1/2,\gamma_s} &= \|(\mathbf{u}_i^m)_x(\xi,-1) - (\mathbf{u}_i^n)_x(\xi,1)\|^2_{1/2,I}\,, \\
\|[(\mathbf{u_i})_y]\|^2_{1/2,\gamma_s} &= \|(\mathbf{u}_i^m)_y(\xi,-1) - (\mathbf{u}_i^n)_y(\xi,1)\|^2_{1/2,I}\,.
\end{aligned}$$

Here and in what follows, I is an interval $(-1,1)$.

As the division of the domain into subdomains match along the interface, we define the jump across the interface by taking it (a part of interface) as the common edge. Consider the elements Ω_1^n and Ω_2^m (as shown in Fig. 2(b)) which have the common edge $\gamma_s \subseteq \Gamma_0$. Let γ_s be the image of $\xi = 1$ under the mapping M_1^n which maps S to Ω_1^n and also the image of $\xi = -1$ under the mapping M_2^m which maps S to Ω_2^m. Define

$$\begin{aligned}
\|[\mathbf{u}]\|^2_{\frac{3}{2},\gamma_s} &= \|\mathbf{u}_2 - \mathbf{u}_1\|^2_{\frac{3}{2},\gamma_s} \\
&= \|\tilde{\mathbf{u}}_2^m(-1,\eta) - \tilde{\mathbf{u}}_1^n(1,\eta)\|^2_{0,I} + \left\|\frac{\partial \tilde{\mathbf{u}}_2^m}{\partial T}(-1,\eta) - \frac{\partial \tilde{\mathbf{u}}_1^n}{\partial T}(1,\eta)\right\|^2_{1/2,I}\,,
\end{aligned}$$

where $\frac{\partial \tilde{\mathbf{u}}_1}{\partial T}$ and $\frac{\partial \tilde{\mathbf{u}}_2}{\partial T}$ are the tangential derivatives of $\tilde{\mathbf{u}}_1$ and $\tilde{\mathbf{u}}_2$ respectively.

Now along the boundary $\Gamma = \cup_{j=1}^4 \Gamma^j$, let $\gamma_s \subseteq \Gamma^j$ (for some j) be the image of $\xi = 1$ under the mapping M_2^m which maps S to Ω_2^m. Then

$$\|\mathbf{u}_2\|^2_{0,\gamma_s} + \left\|\frac{\partial \mathbf{u}_2}{\partial T}\right\|^2_{1/2,\gamma_s} = \|\tilde{\mathbf{u}}_2^m(1,\eta)\|^2_{0,I} + \left\|\frac{\partial \tilde{\mathbf{u}}_2^m}{\partial T}(1,\eta)\right\|^2_{1/2,I}\,.$$

As defined earlier $\mathbf{u}_1 = \mathbf{u}\mid_{\Omega_1}$ and $\mathbf{u}_2 = \mathbf{u}\mid_{\Omega_2}$, so the boundary condition $\mathbf{u} = \mathbf{g}$ on Γ in the discrete form will be $\mathbf{u}_2 = \mathbf{g}$ on $\Gamma^j \cap \partial\Omega_2^m$. Let $\Gamma^j \cap \partial\Omega_2^m = c_2^m$ be the image of the mapping M_2^m of S onto Ω_2^m corresponding to the side $\xi = 1$ and

$$\mathbf{o}_2^m(\eta) = \mathbf{g}(M_2^m(1,\eta)),$$

where $-1 \leq \eta \leq 1$.

On the interface Γ_0 we have $[\mathbf{u}] = 0$ and $[\sigma\mathbf{n}] = \mathbf{q}$. Let $\gamma_s \subseteq \Gamma_0$ be the image of $\xi = 1$ under the mapping M_1^n which maps S to Ω_1^n and also the image of $\xi = -1$ under the mapping M_2^m which maps S to Ω_2^m. Let

$$\mathbf{l}_1^{m,n}(\eta) = \mathbf{q}(-1,\eta) = \mathbf{q}(1,\eta) \text{ for } -1 \le \eta \le 1.$$

Let $\left\{\left\{\tilde{\mathbf{u}}_1^k(\xi,\eta)\right\}_k, \left\{\tilde{\mathbf{u}}_2^l(\xi,\eta)\right\}_l\right\} \in \Pi^W$, the space of spectral element functions. Define the functional

$$\begin{aligned}
&\mathbf{r}^W\left(\left\{\tilde{\mathbf{u}}_1^k(\xi,\eta)\right\}_k, \left\{\tilde{\mathbf{u}}_2^l(\xi,\eta)\right\}_l\right)\\
&= \sum_{k=1}^{p} \left\|(\mathcal{L}_1^k)\tilde{\mathbf{u}}_1^k(\xi,\eta) - \mathbf{F}_1^k(\xi,\eta)\right\|_{0,S}^2 + \sum_{l=1}^{q} \left\|(\mathcal{L}_2^l)\tilde{\mathbf{u}}_2^l(\xi,\eta) - \mathbf{F}_2^k(\xi,\eta)\right\|_{0,S}^2\\
&\quad + \sum_{i=1}^{2}\sum_{\gamma_s\subseteq\Omega_i}\left(\|[\mathbf{u}_i]\|_{0,\gamma_s}^2 + \|[(\mathbf{u}_i)_x]\|_{1/2,\gamma_s}^2 + \|[(\mathbf{u}_i)_y]\|_{1/2,\gamma_s}^2\right) \qquad (7)\\
&\quad + \sum_{\gamma_s\subseteq\Gamma_0}\left(\|[\mathbf{u}]\|_{3/2,\gamma_s}^2 + \|[\sigma\mathbf{n}] - \mathbf{l}_1^{m,n}\|_{1/2,\gamma_s}^2\right)\\
&\quad + \sum_{\gamma_s\subseteq\Gamma}\left(\|\mathbf{u}_2 - \mathbf{o}_2^m(\eta)\|_{0,\gamma_s}^2 + \left\|\left(\frac{\partial\mathbf{u}_2}{\partial T}\right) - \left(\frac{\partial\mathbf{o}_2^m}{\partial T}\right)\right\|_{1/2,\gamma_s}^2\right).
\end{aligned}$$

The approximate solution is chosen as the unique $\left\{\left\{\tilde{\mathbf{z}}_1^k(\xi,\eta)\right\}_k, \left\{\tilde{\mathbf{z}}_2^l(\xi,\eta)\right\}_l\right\} \in \Pi^W$, which minimizes the functional $\mathbf{r}^W\left(\left\{\tilde{\mathbf{u}}_1^k(\xi,\eta)\right\}_k, \left\{\tilde{\mathbf{u}}_2^l(\xi,\eta)\right\}_l\right)$ over all $\left\{\left\{\tilde{\mathbf{u}}_1^k(\xi,\eta)\right\}_k, \left\{\tilde{\mathbf{u}}_2^l(\xi,\eta)\right\}_l\right\}$.

The minimization problem leads to a symmetric and positive definite linear system $\mathbf{A}Z = \mathbf{b}$. Where Z be a vector assembled from the values of $\left\{\left\{\tilde{\mathbf{z}}_1^k(\xi,\eta)\right\}_k, \left\{\tilde{\mathbf{z}}_2^l(\xi,\eta)\right\}_l\right\}$ at the Gauss-Lobatto-Legendre points arranged in lexicographic order for $1 \le k \le p$, $1 \le l \le q$. The solution is obtained by preconditioned conjugate gradient method. The action of a matrix on a vector in each iteration is obtained efficiently and inexpensively without storing the matrix $\mathbf{A}$ (since PCGM requires the action of a matrix on a vector). The details are shown in Appendix A2.

We used a preconditioner which was proposed in [15]. The preconditioner[5] is block diagonal matrix, where each diagonal block corresponds to the H^2 norm of the spectral element function representation of each component of the vector on a particular element which is mapped onto the master square S. The obtained solution of the preconditioned system is nonconforming. A set of corrections are made to the solution so that the corrected solution is conforming (see A3).

Let $\mathbf{u}$ be the exact solution and $\mathbf{z}$ be the approximate solution which is conforming. Let $e = \mathbf{u} - \mathbf{z}$. Then for W large enough we have the following error estimate $\mathbf{H}^1$ norm (since the jump in displacement across the interface is zero,

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[5] $\sum_{k=1}^{p} \|\tilde{u}_1^k\|_{2,S}^2 + \sum_{l=1}^{q} \|\tilde{u}_2^l\|_{2,S}^2$.

$\mathbf{u} \in \mathbf{H}^1(\Omega)$)

$$\|e\|_{1,\Omega} = \|\mathbf{u} - \mathbf{z}\|_{1,\Omega} \leq C\, e^{-bW}$$

holds, where C and b are constants. Proof is very similar to the one proven in [26].

4. Numerical Results

To prove the effectiveness of the method we present the numerical results for the problem defined in Section 3. The relative error $\|e\|_{ER}$ is defined as $\|e\|_{ER} = \dfrac{\|e\|_{1,\Omega}}{\|\mathbf{u}\|_{1,\Omega}}$. In all the examples, degree of the approximation polynomial is denoted by [6] W, 'DOF' means the number of degrees of freedom and 'Iters' means the total number of iterations required to compute the solution using PCGM. We have used the relative residual norm as a stopping criteria in PCGM. That is, the iteration process is stopped when the relative residual norm $\dfrac{\|\mathbf{r}_i\|_2}{\|\mathbf{b}\|_2}$ ($\mathbf{r}_i$ is the residual in i^{th} iteration, $\|.\|_2$ is vector norm) is less than the tolerance ϵ.

Example 1. ***Interface problem with homogeneous jump conditions:*** Consider the linear elasticity interface problem (5) (plane strain, see (4)) stated in Section 2 on a square domain $[-0.75, 0.75]^2$ with a circle centered at the origin of radius s as the interface as shown in Fig. 1. The coefficients λ, μ are given by

$$\begin{aligned} \lambda_1 = \mu_1 = 1 \; if \; r \leq s \\ \lambda_2 = \mu_2 = b \; if \; r > s. \end{aligned} \tag{8}$$

Chosen the data such that the given interface problem has the exact solution $\mathbf{u} = (u_1, u_2)$

$$u_1 = u_2 = \begin{cases} r^2 & r \leq s \\ \frac{r^2}{b} + (1 - \frac{1}{b})s^2 & r > s, \end{cases}$$

where $r = \sqrt{x^2 + y^2}$. Here we choose the radius of the circle $s = \frac{1}{2}$. Note that the solution $\mathbf{u}$ satisfies homogeneous jump conditions across the interface.

This problem have been studied in [39]. We discretized the domain into 9 quadrilateral elements as shown in Fig. 1. The conforming numerical solution has been obtained for various values of b (see (8)) for different degree of the approximating polynomial W. The relative error $\|e\|_{ER}$ in percent, the iterations are tabulated for different values of b (see (8)) in Table 1 and Table 2.

For smaller values of b (see (8)) the iteration count is less but the iteration count is large when b is large. More efficient preconditioner is under investigation. The log of relative error against the degree of the approximating polynomial W is drawn in Fig. 3 for $b = 50, 100$. The relation is almost linear. This shows the exponential accuracy of the method. The error decays exponentially for all values of b.

[6] For the simplicity of the programming W is assumed to be uniform in all elements.

TABLE 1. The relative error in percent and iterations for different W

		$b=5$		$b=10$	
W	DOF	$\|e\|_{ER}\%$	Iters	$\|e\|_{ER}\%$	Iters
2	162	1.63229E+01	19	2.534144E+01	28
3	288	3.820257E-00	63	2.926786E-00	85
4	450	2.897170E-01	93	3.201887E-01	123
5	648	5.828495E-02	185	6.129304E-02	262
6	882	2.468277E-03	225	1.844171E-03	342
7	1152	2.818127E-04	318	4.349615E-04	490
8	1458	2.885878E-05	370	2.411179E-05	620
9	1800	2.842893E-06	441	2.780419E-06	768

TABLE 2. The relative error in percent and iterations for different W

		$b=50$		$b=100$	
W	DOF	$\|e\|_{ER}\%$	Iters	$\|e\|_{ER}\%$	Iters
2	162	4.72332E+01	42	4.9527E+01	60
3	288	2.735828E-00	150	2.38944E-00	223
4	450	2.707493E-01	193	2.01159E-01	405
5	648	3.331489E-02	622	2.92799E-02	767
6	882	2.134931E-03	959	3.03820E-03	1435
7	1152	2.632882E-04	1468	2.63029E-04	2246
8	1458	2.824413E-05	1908	1.57197E-05	3061
9	1800	2.885666E-06	2557	2.94435E-06	4290

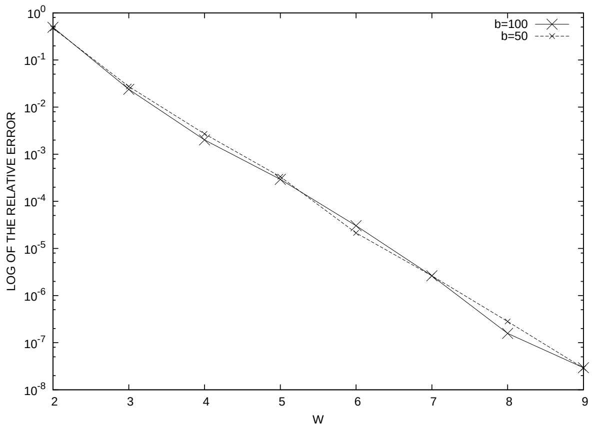

FIGURE 3. Log of the relative error against W for $b = 50, 100$

Example 2. ***Interface problem with nonhomogeneous jump in the flux:*** Consider the linear elasticity problem as defined in (5) (plane strain problem) on the same domain as considered in Example 1 with a circle centered at the origin of radius s as the interface. The coefficients λ, μ are given by

$$\lambda_1 = \mu_1 = 1 \text{ if } r \leq s \text{ and } \lambda_2 = \mu_2 = b \text{ if } r > s.$$

Chosen the data such that the given interface problem has the exact solution $\mathbf{u} = (u_1, u_2)$

$$u_1 = u_2 = \begin{cases} r^2 + \log(1 + r^2), & r \leq s \\ \frac{r^2}{b/2} + (1 - \frac{1}{b/2})s^2 + \frac{\log(1+r^2)}{b} + (1 - \frac{1}{b})\log(1 + s^2), & r > s, \end{cases}$$

where $r = \sqrt{x^2 + y^2}$. Here we choose the radius of the circle $s = \frac{1}{2}$.

TABLE 3. The relative error in percent and iterations for different W

		$b = 10$		$b = 0.1$	
W	DOF	$\Vert e\Vert_{ER}$ %	Iters	$\Vert e\Vert_{ER}$ %	Iters
2	162	3.133049E+01	23	1.79670E+01	21
3	288	3.876138E-00	69	2.85257E-00	51
4	450	3.218966E-01	124	6.62517E-01	117
5	648	8.120330E-02	250	1.77620E-01	147
6	882	1.075311E-02	317	1.58008E-02	232
7	1152	6.768839E-03	406	7.64950E-03	287
8	1458	1.106595E-03	529	1.08173E-03	301
9	1800	6.699847E-04	640	4.80593E-04	330

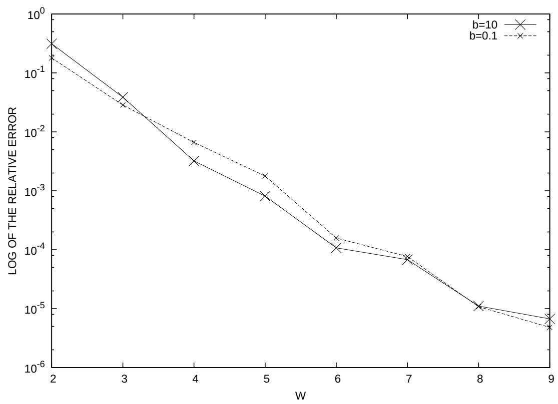

FIGURE 4. Log of the relative error against W for $b = 0.1, 10$

For any b, the solution $\mathbf{u}$ is continuous across the interface and $[\sigma\mathbf{n}] = \mathbf{q} = (q_1, q_2)$,

$$q_1 = \frac{2(3x^2 + 2xy + y^2)}{\sqrt{x^2 + y^2}}, q_2 = \frac{2(3y^2 + 2xy + x^2)}{\sqrt{x^2 + y^2}}.$$

The domain is discretized as in the above example and the conforming solution is obtained for $b = 10, 0.1$. The relative error $\|e\|_{ER}$ in percent, the iterations are tabulated in Table 3. The log of relative error against W is drawn in Fig. 4 for $b = 10, 0.1$. Here one can see that the iteration count is large compared to the count in previous example (look at Table 1 for $b = 10$).

5. Conclusions

The proposed method is nonconforming and exponentially accurate. The interface is resolved exactly using blending elements. A small data has to be interchanged in between the elements for each iteration of the PCGM and the residuals in the normal equations can be obtained efficiently and inexpensively. The proposed method is efficient even when the jump in the coefficient is large. The numerical results shows that large differences in the coefficients leads to increase in the number of iterations of the PCGM. A more efficient preconditioner is under investigation. This method is also efficient on parallel computers. The method is applicable to arbitrary smooth interfaces too and the method can be extended to the singular case which is ongoing work.

A Appendix

For the better understanding here we give the details of some of the stated results in the article.

A1. Analytic Map by blending function method

Consider an element of the domain (as shown in Fig. 1). We define an analytic map M_2^l from $S = (-1, 1)^2$ to Ω_2^l

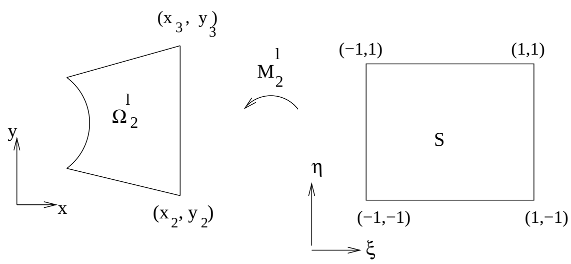

FIGURE 5. Analytic map from S to Ω_2^l

The left edge γ_s of Ω_2^l is a part of the interface Γ_0. Let the curved side γ_s be parametrized by $(r\,cos\,\theta, r\,sin\,\theta), \theta_1 \leq \theta \leq \theta_2$. Then we can define the map

$$x = r\left(\frac{1-\xi}{2}\right)\cos\left(\theta_1\left(\frac{1-\eta}{2}\right) + \theta_2\left(\frac{1+\eta}{2}\right)\right) + x_2\left(\frac{1-\eta}{2}\right)\left(\frac{1+\xi}{2}\right) + x_3\left(\frac{1+\eta}{2}\right)\left(\frac{1+\xi}{2}\right)$$

$$y = r\left(\frac{1-\xi}{2}\right)\sin\left(\theta_1\left(\frac{1-\eta}{2}\right)+\theta_2\left(\frac{1+\eta}{2}\right)\right)+y_2\left(\frac{1-\eta}{2}\right)\left(\frac{1+\xi}{2}\right)+y_3\left(\frac{1+\eta}{2}\right)\left(\frac{1+\xi}{2}\right).$$

A2. Residual calculations

Here we briefly describe how to obtain the action of the matrix $\mathbf{A}$ on a vector at each iteration of the preconditioned conjugate gradient method without storing the matrix $\mathbf{A}$. This has been described for scalar equation case in [36]. Here we describe for the linear elasticity system. In the least-squares minimization of the functional defined in (7) we get different integrals on element domain, on the interelement boundaries and on the boundaries of the domain.

We first show how to compute the integrals on the element domain (the variation of the first term in (7) gives the following integral)

$$\int_S\int ((\mathcal{L}_i^l)\tilde{\mathbf{v}}_i^l)^T\,((\mathcal{L}_i^l)\tilde{\mathbf{u}}_i^l - \mathbf{F}_i^l)\,d\xi\,d\eta$$

where $(\mathcal{L}_i^l)\tilde{\mathbf{u}}_i^l = ((\mathcal{L}_1^{i,l})\tilde{\mathbf{u}}_i^l, (\mathcal{L}_2^{i,l})\tilde{\mathbf{u}}_i^l)^T,\ \mathbf{F}_i^l = (F_1^{i,l}, F_2^{i,l})$. So we have

$$\int_S\int (\mathcal{L}_1^{i,l})\tilde{\mathbf{v}}_i^l((\mathcal{L}_1^{i,l})\tilde{\mathbf{u}}_i^{i,l} - F_1^{i,l})\,d\xi\,d\eta + \int_S\int (\mathcal{L}_2^{i,l})\tilde{\mathbf{v}}_i^l((\mathcal{L}_2^{i,l})\tilde{\mathbf{u}}_i^l - F_2^{i,l})\,d\xi\,d\eta. \quad (9)$$

For simplicity we shall drop the subscripts and superscripts and denote $(\mathcal{L}_1^{i,l})\tilde{\mathbf{u}}_i^l$ and $(\mathcal{L}_2^{i,l})\tilde{\mathbf{u}}_i^l$ by [7]

$$\begin{aligned}(\mathcal{L}_1)\tilde{\mathbf{u}} = &(A_1(\tilde{u}_1)_{\xi\xi} + B_1(\tilde{u}_1)_{\xi\eta} + C_1(\tilde{u}_1)_{\eta\eta} + D_1(\tilde{u}_1)_\xi + E_1(\tilde{u}_1)_\eta)\\ &+ (A_2(\tilde{u}_2)_{\xi\xi} + B_2(\tilde{u}_2)_{\xi\eta} + C_2(\tilde{u}_2)_{\eta\eta} + D_2(\tilde{u}_2)_\xi + E_2(\tilde{u}_2)_\eta)\,,\\ (\mathcal{L}_2)\tilde{\mathbf{u}} = &(A_3(\tilde{u}_1)_{\xi\xi} + B_3(\tilde{u}_1)_{\xi\eta} + C_3(\tilde{u}_1)_{\eta\eta} + D_3(\tilde{u}_1)_\xi + E_3(\tilde{u}_1)_\eta)\\ &+ (A_4(\tilde{u}_2)_{\xi\xi} + B_4(\tilde{u}_2)_{\xi\eta} + C_4(\tilde{u}_2)_{\eta\eta} + D_4(\tilde{u}_2)_\xi + E_4(\tilde{u}_2)_\eta)\,.\end{aligned}$$

By rearranging the integral (9) we get

$$\begin{aligned}&\int_S\int \big(A_1(\tilde{v}_1)_{\xi\xi} + B_1(\tilde{v}_1)_{\xi\eta} + C_1(\tilde{v}_1)_{\eta\eta} + D_1(\tilde{v}_1)_\xi + E_1(\tilde{v}_1)_\eta\big)\,((\mathcal{L}_1)\tilde{\mathbf{u}} - F_1)\,d\xi\,d\eta\\ &+ \int_S\int \big(A_3(\tilde{v}_1)_{\xi\xi} + B_3(\tilde{v}_1)_{\xi\eta} + C_3(\tilde{v}_1)_{\eta\eta} + D_3(\tilde{v}_1)_\xi + E_3(\tilde{v}_1)_\eta\big)\,((\mathcal{L}_2)\tilde{\mathbf{u}} - F_2)\,d\xi\,d\eta\\ &+ \int_S\int \big(A_2(\tilde{v}_2)_{\xi\xi} + B_2(\tilde{v}_2)_{\xi\eta} + C_2(\tilde{v}_2)_{\eta\eta} + D_2(\tilde{v}_2)_\xi + E_2(\tilde{v}_2)_\eta\big)\,((\mathcal{L}_1)\tilde{\mathbf{u}} - F_1)\,d\xi\,d\eta\\ &+ \int_S\int \big(A_4(\tilde{v}_2)_{\xi\xi} + B_4(\tilde{v}_2)_{\xi\eta} + C_4(\tilde{v}_2)_{\eta\eta} + D_4(\tilde{v}_2)_\xi + E_4(\tilde{v}_2)_\eta\big)\,((\mathcal{L}_2)\tilde{\mathbf{u}} - F_2)\,d\xi\,d\eta.\end{aligned} \quad (10)$$

Consider first term in the above integral and denote

$L\tilde{v}_1 = A_1(\tilde{v}_1)_{\xi\xi} + B_1(\tilde{v}_1)_{\xi\eta} + C_1(\tilde{v}_1)_{\eta\eta} + D_1(\tilde{v}_1)_\xi + E_1(\tilde{v}_1)_\eta$ and $\mathfrak{r}_1 = ((\mathcal{L}_1)\tilde{\mathbf{u}} - F_1)$.

Let L^T denote the formal adjoint of the differential operator L. Then

$$L^t\tilde{v}_1 = (A_1\tilde{v}_1)_{\xi\xi} + (B_1\tilde{v}_1)_{\xi\eta} + (C_1\tilde{v}_1)_{\eta\eta} - (D_1\tilde{v}_1)_\xi - (E_1\tilde{v}_1)_\eta\,.$$

Integrating by parts and rearranging some terms, we obtain

[7] General form of these operators in terms of coordinates ξ and η

$$
\begin{aligned}
&\int_{(-1,1)^2}\int L\tilde{v}_1\mathfrak{r}_1\,d\xi d\eta\\
&=\int_{(-1,1)^2}\int \tilde{v}_1 L^t\mathfrak{r}_1\,d\xi d\eta+\int_{(-1,1)}\Big(\big(A_1(\tilde{v}_1)_\xi+D_1\tilde{v}_1\big)\,\mathfrak{r}_1-\tilde{v}_1\,(B_1\,\mathfrak{r}_1)_\eta-\tilde{v}_1\,(A_1\,\mathfrak{r}_1)_\xi\Big)\,(1,\eta)\,d\eta\\
&-\int_{(-1,1)}\Big(\big(A_1(\tilde{v}_1)_\xi+D_1\tilde{v}_1\big)\,\mathfrak{r}_1-\tilde{v}_1\,(B_1\,\mathfrak{r}_1)_\eta-\tilde{v}_1\,(A_1\,\mathfrak{r}_1)_\xi\Big)\,(-1,\eta)\,d\eta\\
&+\tilde{v}_1\,B_1\mathfrak{r}_1\,(1,\eta)|_{-1}^{1}-\tilde{v}_1B_1\,\mathfrak{r}_1\,(-1,\eta)|_{-1}^{1}\\
&+\int_{(-1,1)}\Big((C_1(\tilde{v}_1)_\eta+E_1\tilde{v}_1)\mathfrak{r}_1-\tilde{v}_1\,(C_1\,\mathfrak{r}_1)_\eta-\tilde{v}_1\,(B_1\,\mathfrak{r}_1)_\xi\Big)\,(\xi,1)\,d\xi\\
&-\int_{(-1,1)}\Big((C_1(\tilde{v}_1)_\eta+E_1\tilde{v}_1)\mathfrak{r}_1-\tilde{v}_1\,(C_1\,\mathfrak{r}_1)_\eta-\tilde{v}_1\,(B_1\,\mathfrak{r}_1)_\xi\Big)\,(\xi,-1)\,d\xi.
\end{aligned}
$$

The integral is evaluated by the Gauss-Lobatto-Legendre (GLL) quadrature formula with $2W+1$ points. Let $\xi_0^{2W},\ldots,\xi_{2W}^{2W}$ and $\eta_0^{2W},\ldots,\eta_{2W}^{2W}$ represent the $(2W+1)$ quadrature points in each direction and $w_0^{2W},\ldots,w_{2W}^{2W}$ the corresponding weights. Let the matrix $D^{2W}=d_{i,j}^{2W}$ denotes the differentiation matrix. Thus

$$
\frac{dl}{d\eta}\left(\eta_i^{2W}\right)=\sum_{j=0}^{2W} d_{i,j}^{2W}\,l\left(\eta_j^{2W}\right) \tag{11}
$$

if l is a polynomial of degree less than or equal to $2W$. Then

$$
\begin{aligned}
\int_{(-1,1)^2}\int Lv_1\mathfrak{r}_1 d\xi d\eta \cong &\sum_{i=0}^{2W}\sum_{j=0}^{2W}\tilde{v}_1\left(\xi_i^{2W},\eta_j^{2W}\right)\left(w_i^{2W}w_j^{2W}L^t\mathfrak{r}_1\left(\xi_i^{2W},\eta_j^{2W}\right)\right)\\
&+\sum_{i=0}^{2W}\sum_{j=0}^{2W}\tilde{v}_1\left(\xi_i^{2W},\eta_j^{2W}\right)\left(w_j^{2W}d_{2W,i}^{2W}A_1\left(1,\eta_j^{2W}\right)\mathfrak{r}_1\left(1,\eta_j^{2W}\right)\right)\\
&+\sum_{j=0}^{2W}\tilde{v}_1\left(1,\eta_j^{2W}\right)w_j^{2W}\left(D_1\mathfrak{r}_1-(B_1\mathfrak{r}_1)_\eta-(A_1\mathfrak{r}_1)_\xi\right)\left(1,\eta_j^{2W}\right)\\
&-\sum_{i=0}^{2W}\sum_{j=0}^{2W}\tilde{v}_1\left(\xi_i^{2W},\eta_j^{2W}\right)\left(w_j^{2W}d_{0,i}^{2W}A_1\left(-1,\eta_j^{2W}\right)\mathfrak{r}_1\left(-1,\eta_j^{2W}\right)\right)\\
&-\sum_{j=0}^{2W}\tilde{v}_1\left(-1,\eta_j^{2W}\right)w_j^{2W}\left(D_1\mathfrak{r}_1-(B_1\mathfrak{r}_1)_\eta-(A_1\mathfrak{r}_1)_\xi\right)\left(-1,\eta_j^{2W}\right)\\
&+\sum_{i=0}^{2W}\sum_{j=0}^{2W}\tilde{v}_1\left(\xi_i^{2W},\eta_j^{2W}\right)\left(w_i^{2W}d_{2W,j}^{2W}C_1\left(\xi_i^{2W},1\right)\mathfrak{r}_1\left(\xi_i^{2W},1\right)\right)\\
&+\sum_{i=0}^{2W}\tilde{v}_1\left(\xi_i^{2W},1\right)w_i^{2W}\left(E_1\mathfrak{r}_1-(B_1\mathfrak{r}_1)_\xi-(C_1\mathfrak{r}_1)_\eta\right)\left(\xi_i^{2W},1\right)\\
&-\sum_{i=0}^{2W}\sum_{j=0}^{2W}\tilde{v}_1\left(\xi_i^{2W},\eta_j^{2W}\right)\left(w_i^{2W}d_{0,j}^{2W}C_1\left(\xi_i^{2W},-1\right)\mathfrak{r}_1\left(\xi_i^{2W},-1\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&-\sum_{i=0}^{2W} \tilde{v}_1\left(\xi_i^{2W},-1\right) w_i^{2W}\left(E_1\mathfrak{r}_1-\left(B_1\mathfrak{r}_1\right)_\xi-\left(C_1\mathfrak{r}_1\right)_\eta\right)\left(\xi_i^{2W},-1\right)\\
&+\tilde{v}_1(1,1)B_1\mathfrak{r}_1\,(1,1)-\tilde{v}_1(1,-1)B_1\mathfrak{r}_1\,(1,-1)\\
&+\tilde{v}_1(1,-1)B_1\mathfrak{r}_1\,(-1,-1)-\tilde{v}_1(-1,1)B_1\mathfrak{r}_1\,(-1,1)\,.
\end{aligned}
$$

Remark: Of course, in writing the above we commit an error. It can be argued as in [16] that this error is spectrally small. Similarly we can write the other integral terms in (10).

Now Rewrite $\tilde{u}_1\left(\xi_i^W,\eta_j^W\right)$ and $\tilde{u}_2\left(\xi_i^W,\eta_j^W\right)$ by arranging then in lexicographic order and denote

$$
\begin{aligned}
U^W_{1;(W+1)i+j+1} &= \tilde{u}_1\left(\xi_i^W,\eta_j^W\right) &&\text{for } 0\le i\le W, 0\le j\le W,\\
U^W_{2;(W+1)i+j+1} &= \tilde{u}_2\left(\xi_i^W,\eta_j^W\right) &&\text{for } 0\le i\le W, 0\le j\le W,\\
\mathbf{U}^W &= \begin{bmatrix} U_1^W \\ U_2^W \end{bmatrix}
\end{aligned}
$$

and let

$$
\begin{aligned}
U^{2W}_{1;(2W+1)i+j+1} &= \tilde{u}_1\left(\xi_i^{2W},\eta_j^{2W}\right) &&\text{for } 0\le i\le 2W, 0\le j\le 2W,\\
U^{2W}_{2;(2W+1)i+j+1} &= \tilde{u}_2\left(\xi_i^{2W},\eta_j^{2W}\right) &&\text{for } 0\le i\le 2W, 0\le j\le 2W,\\
\mathbf{U}^{2W} &= \begin{bmatrix} U_1^{2W} \\ U_2^{2W} \end{bmatrix}.
\end{aligned}
$$

Similarly

$$
\begin{aligned}
\mathfrak{r}^{2W}_{1;(2W+1)i+j+1} &= \mathcal{L}_1\tilde{\mathbf{u}}\left(\xi_i^{2W},\eta_j^{2W}\right)-F_1\left(\xi_i^{2W},\eta_j^{2W}\right),\\
\mathfrak{r}^{2W}_{2;(2W+1)i+j+1} &= \mathcal{L}_2\tilde{\mathbf{u}}\left(\xi_i^{2W},\eta_j^{2W}\right)-F_2\left(\xi_i^{2W},\eta_j^{2W}\right)
\end{aligned}
$$

Then we may write

$$
\int_{(-1,1)^2}\!\!\int L\tilde{v}_1\left((\mathcal{L}_1)\tilde{\mathbf{u}}-F_1\right)d\xi d\eta=\left(V_1^{2W}\right)^t R_1\mathfrak{r}_1^{2W}
$$

where R_1 is a matrix such that $R_1\mathfrak{r}_1^{2W}$ is easily computed. In similar way we calculate the other terms in integral (10) and finally we may write

$$
\begin{aligned}
&\int_{(-1,1)^2}\!\!\int ((\mathcal{L}_i^l)\tilde{\mathbf{v}}_i^l)^T\left((\mathcal{L}_i^l)\tilde{\mathbf{u}}_i^l-\mathbf{F}_i^l\right)d\xi\,d\eta\\
&=\left(V_1^{2W}\right)^t R_1\mathfrak{r}_1^{2W}+\left(V_1^{2W}\right)^t R_2\mathfrak{r}_2^{2W}+\left(V_2^{2W}\right)^t R_3\mathfrak{r}_1^{2W}+\left(V_2^{2W}\right)^t R_4\mathfrak{r}_2^{2W}.
\end{aligned}
$$

(ii) Integrals on the boundary of the elements:

We now show how to evaluate the integral involving the jump in the flux across the interface. For this we have to examine the norm $H^{1/2}(-1,1)$. Now

$$
\|l\|^2_{1/2,(-1,1)}\cong\int_{-1}^{1} l^2(\eta)\,d\eta+\int_{-1}^{1}\int_{-1}^{1}\frac{\left(l(x)-l(y)\right)^2}{(x-y)^2}dxdy.
$$

Let $l(\eta)$ be a polynomial of degree less than or equal to $2W$. Then $\frac{(l(x)-l(y))}{(x-y)}$ is polynomial of degree less than or equal to $2W$ in x and y. And so we may define

$$\|l\|^2_{1/2,(-1,1)} = \sum_{i=0}^{2W} w_i^{2W} l^2\left(\eta_i^{2W}\right) + \sum_{j=0}^{2W} \sum_{i\neq j, i=0}^{2W} w_i^{2W} w_j^{2W} \left(\frac{l\left(\eta_i^{2W}\right) - l\left(\eta_j^{2W}\right)}{\eta_i^{2W} - \eta_j^{2W}}\right)^2 + \sum_{i=0}^{2W} \left(w_i^{2W}\right)^2 \left(\frac{dl}{d\eta}\left(\eta_i^{2W}\right)\right)^2.$$

Thus there is a symmetric positive definite matrix H^{2W} such that

$$\|l\|^2_{1/2,(-1,1)} = \sum_{i=0}^{2W}\sum_{j=0}^{2W} l\left(\eta_i^{2W}\right) H_{i,j}^{2W} l\left(\eta_j^{2W}\right). \tag{12}$$

Now consider the jump in the flux across the interface. Consider the elements Ω_1^k and Ω_2^m which have the common edge $\gamma_s \subseteq \Gamma_0$. Let γ_s be the image of $\eta = -1$ under the mapping M_1^k which maps S to Ω_1^k and also the image of $\eta = 1$ under the mapping M_2^m which maps S to Ω_2^m. From equation (7), we need to evaluate

$$\left\|[\sigma\mathbf{n}] - \mathbf{l}_1^{m,k}\right\|^2_{\frac{1}{2},\gamma_s} = \left\|\left((\sigma_2^m\mathbf{n})(\xi,1) - (\sigma_1^k\mathbf{n})(\xi,-1)\right) - \mathbf{l}_1^{m,k}\right\|^2_{\frac{1}{2},(-1,1)} \tag{13}$$

where $\sigma_2^m\mathbf{n} = ((\sigma_{11}n_1 + \sigma_{12}n_2), (\sigma_{12}n_1 + \sigma_{22}n_2))^T = (T_1^m\tilde{\mathbf{u}}_2^m, T_2^m\tilde{\mathbf{u}}_2^m)^T$, $\sigma_2^k\mathbf{n} = ((\sigma_{11}n_1 + \sigma_{12}n_2), \ (\sigma_{12}n_1 + \sigma_{22}n_2))^T = (T_1^k\tilde{\mathbf{u}}_1^k, T_2^k\tilde{\mathbf{u}}_1^k)^T$ in transformed coordinates ξ and η and

$$\mathbf{l}_1^{m,k}(\xi) = (q_1^{m,k}(\xi,1), q_2^{m,k}(\xi,1))^T = (q_1^{m,k}(\xi,-1), q_2^{m,k}(\xi,-1))^T.$$

The variation of the boundary term in (13) is given by

$$\cong \sum_{i=0}^{2W}\sum_{j=0}^{2W}(T_1^m\tilde{\mathbf{v}}_2^m(\xi_i^{2W},1) - T_1^k\tilde{\mathbf{v}}_1^k(\xi_i^{2W},-1))\, H_{i,j}^{2W}\,(I) + \sum_{i=0}^{2W}\sum_{j=0}^{2W}(T_2^m\tilde{\mathbf{v}}_2^m(\xi_i^{2W},1) - T_2^k\tilde{\mathbf{v}}_1^k(\xi_i^{2W},-1))\, H_{i,j}^{2W}\,(II),$$

where $I = \Big((T_1^m\tilde{\mathbf{u}}_2^m(\xi_j^{2W},1) - T_1^k\tilde{\mathbf{u}}_1^k(\xi_j^{2W},-1)) - q_1^{m,k}(\xi_j^{2W},-1)\Big)$ and $II = \Big((T_2^m\tilde{\mathbf{u}}_2^m(\xi_j^{2W},1) - T_2^k\tilde{\mathbf{u}}_1^k(\xi_j^{2W},-1)) - q_2^{m,k}(\xi_j^{2W},-1)\Big)$.

Now this can be written as

$$\begin{aligned} &= \sum_{i=0}^{2W}\sum_{j=0}^{2W} T_1^m\tilde{\mathbf{v}}_2^m(\xi_i^{2W},1)\ H_{i,j}^{2W}\ (I) + \sum_{i=0}^{2W}\sum_{j=0}^{2W} T_2^m\tilde{\mathbf{v}}_2^m(\xi_i^{2W},1)\ H_{i,j}^{2W}\ (II) \\ &\quad + \sum_{i=0}^{2W}\sum_{j=0}^{2W} T_1^k\tilde{\mathbf{v}}_1^k(\xi_i^{2W},-1))\ H_{i,j}^{2W}\ (-I) + \sum_{i=0}^{2W}\sum_{j=0}^{2W} T_2^k\tilde{\mathbf{v}}_1^k(\xi_i^{2W},-1))\ H_{i,j}^{2W}\ (-II) \end{aligned} \tag{14}$$

Consider first line in the above equation. For simplicity we shall drop the subscripts and superscripts and represent $T_1\mathbf{v}$ and $T_2\mathbf{v}$ in general form

$$\begin{aligned}T_1\mathbf{v} &= \tilde{P}_1(\tilde{v}_1)_\xi + \tilde{Q}_1(\tilde{v}_1)_\eta + \tilde{R}_1(\tilde{v}_2)_\xi + \tilde{S}_1(\tilde{v}_2)_\eta,\\ T_2\mathbf{v} &= \tilde{P}_2(\tilde{v}_1)_\xi + \tilde{Q}_2(\tilde{v}_1)_\eta + \tilde{R}_2(\tilde{v}_2)_\xi + \tilde{S}_2(\tilde{v}_2)_\eta\end{aligned}$$

and denote $\mathcal{J}_1^{2W}(j) = (T_1^m \tilde{\mathbf{u}}_2^m(\xi_j^{2W},1) - T_1^k \tilde{\mathbf{u}}_1^k(\xi_j^{2W},-1))$
and $\mathcal{J}_2^{2W}(j) = (T_2^m \tilde{\mathbf{u}}_2^m(\xi_j^{2W},1) - T_2^k \tilde{\mathbf{u}}_1^k(\xi_j^{2W},-1))$. Therefore first line of (14) becomes

$$\begin{aligned}&\sum_{i=0}^{2W}\sum_{j=0}^{2W} T_1\mathbf{v}\,(\xi_i^{2W},1)\,H_{i,j}^{2W}\,\left(\mathcal{J}_1^{2W}(j) - q_1(\xi_j^{2W},1)\right)\\ &+\sum_{i=0}^{2W}\sum_{j=0}^{2W} T_2\mathbf{v}\,(\xi_i^{2W},1)\,H_{i,j}^{2W}\,\left(\mathcal{J}_2^{2W}(j) - q_2(\xi_j^{2W},1)\right).\end{aligned} \tag{15}$$

By rearranging the terms, we get

$$\begin{aligned}=&\sum_{i=0}^{2W}\sum_{j=0}^{2W}\left((\tilde{P}_1(\tilde{v}_1)_\xi + \tilde{Q}_1(\tilde{v}_1)_\eta)(\xi_i^{2W},1)\right)\,H_{i,j}^{2W}\,\left(\mathcal{J}_1^{2W}(j) - q_1(\xi_j^{2W},1)\right)\\ &+\sum_{i=0}^{2W}\sum_{j=0}^{2W}\left((\tilde{P}_2(\tilde{v}_1)_\xi + \tilde{Q}_2(\tilde{v}_1)_\eta)(\xi_i^{2W},1)\right)\,H_{i,j}^{2W}\,\left(\mathcal{J}_2^{2W}(j) - q_2(\xi_j^{2W},1)\right)\\ &+\sum_{i=0}^{2W}\sum_{j=0}^{2W}\left((\tilde{R}_1(\tilde{v}_2)_\xi + \tilde{S}_1(\tilde{v}_2)_\eta)(\xi_i^{2W},1)\right)\,H_{i,j}^{2W}\,\left(\mathcal{J}_1^{2W}(j) - q_1(\xi_j^{2W},1)\right)\\ &+\sum_{i=0}^{2W}\sum_{j=0}^{2W}\left((\tilde{R}_2(\tilde{v}_2)_\xi + \tilde{S}_2(\tilde{v}_2)_\eta)(\xi_i^{2W},1)\right)\,H_{i,j}^{2W}\,\left(\mathcal{J}_2^{2W}(j) - q_2(\xi_j^{2W},1)\right).\end{aligned} \tag{16}$$

Let

$$\rho_{i,1}^{2W} = \sum_{j=0}^{2W} H_{i,j}^{2W}\left(\mathcal{J}_1^{2W}(j) - q_1(\xi_j^{2W},1)\right) \text{ and } \rho_{i,2}^{2W} = \sum_{j=0}^{2W} H_{i,j}^{2W}\left(\mathcal{J}_2^{2W}(j) - q_2(\xi_j^{2W},1)\right).$$

Then we may write (16) as (using 11)

$$\begin{aligned}=&\sum_{j=0}^{2W}\tilde{v}_1(\xi_j^{2W},1)\left(\sum_{i=0}^{2W} d_{i,j}^{2W}\,\tilde{P}_1(\xi_i^{2W},1)\right)\rho_{i,1}^{2W} + \sum_{i=0}^{2W}\sum_{j=0}^{2W}\tilde{v}_1\left(\xi_i^{2W},\eta_j^{2W}\right)\left(\tilde{Q}_1(\xi_i^{2W},1)d_{2W,j}^{2W}\rho_{i,1}^{2W}\right)\\ &+\sum_{j=0}^{2W}\tilde{v}_1(\xi_j^{2W},1)\left(\sum_{i=0}^{2W} d_{i,j}^{2W}\,\tilde{P}_2(\xi_i^{2W},1)\right)\rho_{i,2}^{2W} + \sum_{i=0}^{2W}\sum_{j=0}^{2W}\tilde{v}_1\left(\xi_i^{2W},\eta_j^{2W}\right)\left(\tilde{Q}_2(\xi_i^{2W},1)d_{2W,j}^{2W}\rho_{i,2}^{2W}\right)\\ &+\sum_{j=0}^{2W}\tilde{v}_2(\xi_j^{2W},1)\left(\sum_{i=0}^{2W} d_{i,j}^{2W}\,\tilde{R}_1(\xi_i^{2W},1)\right)\rho_{i,1}^{2W} + \sum_{i=0}^{2W}\sum_{j=0}^{2W}\tilde{v}_2\left(\xi_i^{2W},\eta_j^{2W}\right)\left(\tilde{S}_1(\xi_i^{2W},1)d_{2W,j}^{2W}\rho_{i,1}^{2W}\right)\\ &+\sum_{j=0}^{2W}\tilde{v}_2(\xi_j^{2W},1)\left(\sum_{i=0}^{2W} d_{i,j}^{2W}\,\tilde{R}_2(\xi_i^{2W},1)\right)\rho_{i,2}^{2W} + \sum_{i=0}^{2W}\sum_{j=0}^{2W}\tilde{v}_2\left(\xi_i^{2W},\eta_j^{2W}\right)\left(\tilde{S}_2(\xi_i^{2W},1)d_{2W,j}^{2W}\rho_{i,2}^{2W}\right)\\ =&\left(V_1^{2W}\right)^t T_1X_1^{2W} + \left(V_1^{2W}\right)^t T_2X_2^{2W} + \left(V_2^{2W}\right)^t T_3X_1^{2W} + \left(V_2^{2W}\right)^t T_4X_2^{2W}.\end{aligned}$$

Here T_1, T_2, T_3, T_4 are $(2W+1)^2 \times (2W+1)$ matrices and $T_1 X_1^{2W}, T_2 X_2^{2W}$, $T_3 X_1^{2W}$ and $T_4 X_2^{2W}$ can be easily computed. Similarly terms of the second line in (14) can be obtained.

Similarly one can evaluate the integrals involving the inter element jumps and the integrals on the boundaries of the domain.

Combining the all the integral terms in an element

Consider an element Ω_2^m with a part of the interface $\gamma_s \subseteq \Gamma_0$(as we have considered above) as one edge and the other edges may be part of the boundary of the domain or the common edges shared by neighbouring elements.

Adding all the terms we obtain

$$\begin{aligned}
&\int_{(-1,1)^2}\int ((\mathcal{L}_2^m)\tilde{\mathbf{v}}_2^m)^T\,((\mathcal{L}_2^m)\tilde{\mathbf{u}}_2^m - \mathbf{F}_2^m)\,d\xi\,d\eta \\
&\quad + \sum_{i=0}^{2W}\sum_{j=0}^{2W} T_1^m \tilde{\mathbf{v}}_2^m(\xi_i^{2W},1)\;H_{i,j}^{2W}\;(I) + \sum_{i=0}^{2W}\sum_{j=0}^{2W} T_2^m \tilde{\mathbf{v}}_2^m(\xi_i^{2W},1)\;H_{i,j}^{2W}\;(II) + \ldots.. \\
&= \left(V_1^{2W}\right)^t R_1 \mathfrak{r}_1^{2W} + \left(V_1^{2W}\right)^t R_2 \mathfrak{r}_2^{2W} + \left(V_2^{2W}\right)^t R_3 \mathfrak{r}_1^{2W} + \left(V_2^{2W}\right)^t R_4 \mathfrak{r}_2^{2W} \\
&\quad + \left(V_1^{2W}\right)^t T_1 X_1^{2W} + \left(V_1^{2W}\right)^t T_2 X_2^{2W} + \left(V_2^{2W}\right)^t T_3 X_1^{2W} + \left(V_2^{2W}\right)^t T_4 X_2^{2W} + \ldots. \\
&= \left(V_1^{2W}\right)^t O_1^{2W} + \left(V_2^{2W}\right)^t O_2^{2W}
\end{aligned}$$

where $O_1^{2W} = R_1\mathfrak{r}_1^{2W} + R_2\mathfrak{r}_2^{2W} + T_1X_1^{2W} + T_2X_2^{2W} + \ldots..$ and $O_2^{2W} = R_3\mathfrak{r}_1^{2W} + R_4\mathfrak{r}_2^{2W} + T_3X_1^{2W} + T_4X_2^{2W} + \ldots...$ are $(2W+1)^2$ vectors which can be easily computed. Now there exists a matrix G^W such that

$$V_1^{2W} = G^W V_1^W \,\text{and}\, V_2^{2W} = G^W V_2^W.$$

Hence

$$\left(V_1^{2W}\right)^t O_1^{2W} = \left(V_1^W\right)^t\left(\left(G^W\right)^t O_1^{2W}\right) \text{ and } \left(V_2^{2W}\right)^t O_2^{2W} = \left(V_2^W\right)^t\left(\left(G^W\right)^t O_2^{2W}\right).$$

In [14] it has been shown how $\left(G^W\right)^t O_1^{2W}$ can be computed. Let $J_1 = \left(G^W\right)^t O_1^{2W}$. Similarly we obtain J_2. Now $J = \begin{bmatrix} J_1 \\ J_2 \end{bmatrix}$ is $2(W+1)^2$ vector. Finally we obtain J vector corresponding to V^tJ. Thus we see to compute J we do not need to compute and store any matrices such as the mass and stiffness matrices. The evaluation of the residuals on each element requires the interchange of the boundary values between neighbouring elements.

A3. Corrections to the nonconforming solution

We can construct a set of corrections to the spectral element functions $\left\{\left\{\tilde{\mathbf{z}}_1^k(\xi,\eta)\right\}_k, \left\{\tilde{\mathbf{z}}_2^l(\xi,\eta)\right\}_l\right\}$ so that the corrected solution $\left\{\left\{\hat{\mathbf{z}}_1^k(\xi,\eta)\right\}_k, \left\{\hat{\mathbf{z}}_2^l(\xi,\eta)\right\}_l\right\}$ is conforming and belongs to $H^1(\Omega)$. These corrections are defined below (see [37]).

We do this in two steps :

1. First, we make a bilinear correction $\left\{\left\{\tilde{\mathfrak{s}}_1^k(\xi,\eta)\right\}_k, \left\{\tilde{\mathfrak{s}}_2^l(\xi,\eta)\right\}_l\right\}$ so that

$$\left\{\left\{\tilde{\mathbf{z}}_1^k(\xi,\eta)+\tilde{\mathfrak{s}}_1^k(\xi,\eta)\right\}_k, \left\{\tilde{\mathbf{z}}_2^l(\xi,\eta)+\tilde{\mathfrak{s}}_2^l(\xi,\eta)\right\}_l\right\}$$

are continuous at the vertices of the rectangles on which they are defined.

Consider an element Ω_2^m. M_2^m is the map from $S=(-1,1)^2$ to Ω_2^m. Let P_1, P_2, P_3 and P_4 be the four vertices of S. These vertices are common to some of the neighbouring elements. Now consider the average values $\overline{\mathbf{z}}\,(P_i)$ of the solutions at each vertex P_i $(i=1,2,3,4)$ from all the elements which have P_i as common vertex. Define $\mathbf{a}_i=\overline{\mathbf{z}}\,(P_i)-\tilde{\mathbf{z}}_2^m(P_i)$ for $i=1,2,3,4$. Now we define bilinear correction $\tilde{\mathfrak{s}}_2^m(\xi,\eta)$

$$\begin{aligned}\tilde{\mathfrak{s}}_2^m(\xi,\eta)=&\mathbf{a}_1\left(\frac{(1-\xi)(1-\eta)}{4}\right)+\mathbf{a}_2\left(\frac{(1+\xi)(1-\eta)}{4}\right)\\&+\mathbf{a}_3\left(\frac{(1+\xi)(1+\eta)}{4}\right)+\mathbf{a}_4\left(\frac{(1-\xi)(1+\eta)}{4}\right).\end{aligned}$$

such that $\left(\tilde{\mathbf{z}}_2^m+\tilde{\mathfrak{s}}_2^m\right)(P_i)=\overline{\mathbf{z}}\,(P_i)$. Similarly one can define in other elements.

2. Next, we make a correction $\left\{\left\{\tilde{\mathfrak{t}}_1^k(\xi,\eta)\right\}_k, \left\{\tilde{\mathfrak{t}}_2^l(\xi,\eta)\right\}_l\right\}$ so that

$$\left\{\left\{\tilde{\mathbf{z}}_1^k(\xi,\eta)+\tilde{\mathfrak{s}}_1^k(\xi,\eta)+\tilde{\mathfrak{t}}_1^k(\xi,\eta)\right\}_k, \left\{\tilde{\mathbf{z}}_2^l(\xi,\eta)+\tilde{\mathfrak{s}}_2^l(\xi,\eta)+\tilde{\mathfrak{t}}_2^l(\xi,\eta)\right\}_l\right\}$$

are conforming.

Consider Ω_2^m. If γ is a side of S then we choose $\tilde{\mathfrak{t}}_2^m$ so that $\left(\tilde{\mathbf{z}}_2^m+\tilde{\mathfrak{s}}_2^m+\tilde{\mathfrak{t}}_2^m\right)(P)=(\overline{\mathbf{z}}+\overline{\mathfrak{s}})(P)$ for $P\in\gamma$. Now $\tilde{\mathfrak{t}}_2^m$ has it's traces defined on the sides of the square S. The traces of $\tilde{\mathfrak{t}}_2^m$ are polynomials on the sides of S. Let

$$\tilde{\mathfrak{t}}_2^m(\xi,-1)=\phi_1(\xi),\ \ \tilde{\mathfrak{t}}_2^m(\xi,1)=\phi_3(\xi);\ \ \tilde{\mathfrak{t}}_2^m(-1,\eta)=\phi_4(\eta),\ \ \tilde{\mathfrak{t}}_2^m(1,\eta)=\phi_2(\eta)$$

where $\phi_i(.)=(\phi_i^1(.),\phi_i^2(.))^T$ for i=1, 2, 3, 4. We define a lifting of $\tilde{\mathfrak{t}}_2^m(\xi,\eta)$ onto S as

$$\tilde{\mathfrak{t}}_2^m(\xi,\eta)=\frac{1}{2}\left(\phi_1(\xi)(1-\eta)+\phi_3(\eta)(1+\eta)+\phi_2(\eta)(1+\xi)+\phi_4(\eta)(1-\xi)\right).$$

We now define the corrected set of spectral element functions as

$$\begin{aligned}&\left\{\left\{\hat{\mathbf{z}}_1^k(\xi,\eta)\right\}_k, \left\{\hat{\mathbf{z}}_2^l(\xi,\eta)\right\}_l\right\}\\&=\left\{\left\{\tilde{\mathbf{z}}_1^k(\xi,\eta)+\tilde{\mathfrak{s}}_1^k(\xi,\eta)+\tilde{\mathfrak{t}}_1^k(\xi,\eta)\right\}_k, \left\{\tilde{\mathbf{z}}_2^l(\xi,\eta)+\tilde{\mathfrak{s}}_2^l(\xi,\eta)+\tilde{\mathfrak{t}}_2^l(\xi,\eta)\right\}_l\right\}.\end{aligned}$$

These constructions are similar to Lemma 4.57 in [34].

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