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SUPERCONVERGENCE AND A POSTERIORI ERROR ESTIMATES OF VARIATIONAL DISCRETIZATION FOR ELLIPTIC CONTROL PROBLEMS†

YUCHUN HUA AND YUELONG TANG*

ABSTRACT. In this paper, we investigate a variational discretization approximation of elliptic optimal control problems with control constraints. The state and the co-state are approximated by piecewise linear functions, while the control is not directly discretized. By using some proper intermediate variables, we derive a second-order convergence in L^2-norm and superconvergence between the numerical solution and elliptic projection of the exact solution in H^1-norm or the gradient of the exact solution and recovery gradient in L^2-norm. Then we construct a posteriori error estimates by using the superconvergence results and do some numerical experiments to confirm our theoretical results.

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1. Introduction

Optimal control problems have been extensively used in many aspects of the modern life such as social, economic, scientific and engineering numerical simulation. Finite element approximation seems to be the most widely used method in computing optimal control problems. A systematic introduction of finite element method for PDEs and optimal control problems can be found in [14, 17, 18, 23, 26].

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For a constrained optimal control problem, the control has lower regularity than the state and the co-state. So most researchers considered using piecewise linear functions to approximate the state and the co-state and using piecewise constant functions to approximate the control. They constructed a projection gradient algorithm in which the control is first-order convergent (see e.g., [16, 20]). Recently, Borzì considered a second-order discretization and multigrid solution of constrained nonlinear elliptic control problems in [4], Hinze introduced a variational discretization concept for optimal control problems and derived a second-order convergence for the control in [11, 12].

There has been extensive research on the superconvergence of finite element methods for optimal control problems in the literature, most of which focused upon elliptic control problems. Superconvergence properties of linear and semilinear elliptic control problems are established in [24] and [6], respectively. Superconvergence of finite element approximation for bilinear elliptic optimal control problems is studied in [32]. Recently, superconvergence of fully discrete finite element and variational discretization approximation for linear and semilinear parabolic control problems are derived in [27, 28, 29].

The literature on a posteriori error estimation of finite element method is huge. Some internationally known works can be found in [1, 2, 3, 5]. Concerning finite element methods of elliptic optimal control problems, a posteriori error estimates of residual type are investigated in [10, 21, 33, 34], a posteriori error estimates of recovery type are derived in [16, 20, 31]. Some error estimates and superconvergence results have been established in [6, 7, 36], and some adaptive finite element methods can be found in [3, 13, 15, 35]. For parabolic optimal control problems, residual type a posteriori error estimates of finite element methods are investigated in [22, 30] and an adaptive space-time finite element method is investigated in [25].

The purpose of this paper is to consider the superconvergence and recovery type a posteriori error estimates of variational discretization for elliptic optimal control problems with pointwise control constraints.

We are interested in the following quadratic optimal control problem:

$$\begin{cases} \min\limits_{u\in K}\left\{\dfrac{1}{2}\|y-y_d\|^2+\dfrac{\alpha}{2}\|u-u_d\|^2\right\}, \\ -\operatorname{div}(A\nabla y)=f+Bu, \quad \text{in}\,\Omega, \\ y=0, \quad \text{on}\,\partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^2$ with a Lipschitz boundary $\partial\Omega$, the coefficient $A=(a_{ij}(x))_{2\times 2}\in (W^{1,\infty}(\bar{\Omega}))^{2\times 2}$, such that $(A(x)\xi)\cdot\xi\geq c\mid\xi\mid^2$, $\forall\,\xi\in\mathbb{R}^2$, B is a linear continuous operator, y_d, u_d, $f\in L^2(\Omega)$, and K is defined by

$$K=\left\{v\in L^2(\Omega): a\leq v\leq b, \quad \text{a.e. in}\,\Omega\right\},$$

where a and b are constants.

In this paper, we adopt the standard notation $W^{m,q}(\Omega)$ for Sobolev spaces on Ω with norm $\|\cdot\|_{W^{m,q}(\Omega)}$ and seminorm $|\cdot|_{W^{m,q}(\Omega)}$. We set $H_0^1(\Omega)\equiv$

$\{ v \in H^1(\Omega) : v|_{\partial\Omega} = 0 \}$ and denote $W^{m,2}(\Omega)$ by $H^m(\Omega)$. In addition, c or C denotes a generic positive constant.

The outline of this paper is as follows. In Section 2, we introduce a variational discretization approximation for the model problem. In Section 3, we derive the convergence. In Section 4, we obtain the superconvergence and a posteriori error estimates. We present some numerical examples to demonstrate our theoretical results in the last section.

2. variational discretization approximation for the model problem

We now consider a variational discretization approximation for the model problem (1). For ease of exposition, we set $W = H_0^1(\Omega)$, $U = L^2(\Omega)$, $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}$, $\|\cdot\|_m = \|\cdot\|_{H^m(\Omega)}$ and

$$a(y,w) = \int_\Omega (A\nabla y)\cdot\nabla w, \qquad \forall\, y,w \in W,$$
$$(u,w) = \int_\Omega u\cdot w, \qquad \forall\, u,w \in U.$$

It follows from the assumption on A that

$$a(y,y) \geq c\|y\|_1^2, \qquad |a(y,w)| \leq C\|y\|_{1,\Omega}\|w\|_1, \qquad \forall\, y,\, w \in W.$$

Then the model problem (1) can be restated as:

$$\begin{cases} \min\limits_{u\in K}\left\{\dfrac{1}{2}\|y-y_d\|^2 + \dfrac{\alpha}{2}\|u-u_d\|^2\right\}, \\ a(y,w) = (f+Bu,w), \qquad \forall\, w \in W. \end{cases} \tag{2}$$

It is well known (see e.g., [17]) that the control problem (2) has a unique solution $(y,u) \in W \times K$, and a pair $(y,u) \in W \times K$ is the solution of (2) if and only if there is a co-state $p \in W$ such that the triplet $(y,p,u) \in W \times W \times K$ satisfies the following optimality conditions:

$$a(y,w) = (f+Bu,w), \qquad \forall\, w \in W, \tag{3}$$
$$a(q,p) = (y_d - y, q), \qquad \forall\, q \in W, \tag{4}$$
$$(\alpha u - \alpha u_d - B^*p, v-u) \geq 0, \qquad \forall\, v \in K, \tag{5}$$

where B^* is the adjoint operator of B.

We introduce the following pointwise projection operator:

$$\Pi_{[a,b]}(g(x)) = \max(a, \min(b, g(x))).$$

It is clear that $\Pi_{[a,b]}(\cdot)$ is Lipschitz continuous with constant 1. As in [8], it is easy to prove the following lemma:

Lemma 2.1. *Let (y,p,u) be the solution of (3)-(5). Then*

$$u = \Pi_{[a,b]}\left(u_d + \frac{1}{\alpha}B^*p\right). \tag{6}$$

Remark 2.1. We should point out that (5) and (6) are equivalent. This theory can be used to another situation, for example, K is characterized by a bound on the integral on u over Ω, namely, $\int_\Omega u(x)dx \geq 0$, we have similar results.

Let $\mathcal{T}^h$ be a regular triangulation of Ω, such that $\bar{\Omega} = \cup_{\tau\in\mathcal{T}^h}\bar{\tau}$. Let $h = \max_{\tau\in\mathcal{T}^h}\{h_\tau\}$, where h_τ denotes the diameter of the element τ. Associated with $\mathcal{T}^h$ is a finite dimensional subspace S_h of $C(\bar{\Omega})$, such that $\chi|_\tau$ are polynomials of m-order($m = 1$) for all $\chi \in S_h$ and $\tau \in \mathcal{T}^h$. Let $W_h = \{v_h \in S_h : v_h|_{\partial\Omega} = 0\}$. It is easy to see that $W_h \subset W$. Then a possible variational discretization approximation scheme of (2) is as follows:

$$\begin{cases} \min\limits_{u_h\in K}\left\{\dfrac{1}{2}\|y_h - y_d\|^2 + \dfrac{\alpha}{2}\|u_h - u_d\|^2\right\}, \\ a(y_h, w_h) = (f + Bu_h, w_h), \qquad \forall\, w_h \in W_h. \end{cases} \tag{7}$$

It is well known (see e.g., [21]) that the control problem (7) has a unique solution $(y_h, u_h) \in W_h \times K$, and that if the pair $(y_h, u_h) \in W_h \times K$ is the solution of (7), if and only if there is a co-state $p_h \in W_h$ such that the triplet $(y_h, p_h, u_h) \in W_h \times W_h \times K$ satisfies the following optimality conditions:

$$a(y_h, w_h) = (f + Bu_h, w_h), \qquad \forall\, w_h \in W_h, \tag{8}$$

$$a(q_h, p_h) = (y_d - y_h, q_h), \qquad \forall\, q_h \in W_h, \tag{9}$$

$$(\alpha u_h - \alpha u_d - B^* p_h, v - u_h) \geq 0, \qquad \forall\, v \in K. \tag{10}$$

Similar to Lemma 2.1, it is easy to show the following lemma:

Lemma 2.2. *Suppose (y_h, p_h, u_h) be the solution of (8)-(10). Then , we have*

$$u_h = \Pi_{[a,b]}\left(u_d + \frac{1}{\alpha}B^* p_h\right). \tag{11}$$

Remark 2.2. According to (11), we can replace u_h by $\max(a, \min(b, u_d + \frac{1}{\alpha}B^* p_h)$ in our program. Thus the control need not be discretized directly.

3. Convergence analysis

We first introduce the following intermediate variables. For any $u_h \in K$, let $(y(u_h), p(u_h)) \in W \times W$ satisfies the following system:

$$a(y(u_h), w) = (f + Bu_h, w), \qquad \forall\, w \in W, \tag{12}$$

$$a(q, p(u_h)) = (y_d - y(u_h), q), \qquad \forall\, q \in W. \tag{13}$$

The following lemmas are very important in deriving the convergence.

Lemma 3.1 ([9])**.** *Let π_h be the standard Lagrange interpolation operator. For $m = 0$ or 1, $q > \frac{n}{2}$ and $\forall\, v \in W^{2,q}(\Omega)$, we have*

$$|v - \pi_h v|_{W^{m,q}(\Omega)} \leq Ch^{2-m}|v|_{W^{2,q}(\Omega)}. \tag{14}$$

Lemma 3.2. *Let (y_h, p_h, u_h) and $(y(u_h), p(u_h))$ be the solutions of (8)-(10) and (12)-(13), respectively. Assume that $p(u_h), y(u_h) \in H^2(\Omega)$. Then there exists a constant C independent of h such that*

$$\|y(u_h) - y_h\|_1 + \|p(u_h) - p_h\|_1 \le Ch. \tag{15}$$

Proof. From (9), (13)-(14) and Young's inequality, we have

$$\begin{aligned}
&c\|p(u_h) - p_h\|_1^2\\
\le&a(p(u_h) - p_h, p(u_h) - p_h)\\
=&a(p(u_h) - \pi_h p(u_h), p(u_h) - p_h) + (y_h - y(u_h), \pi_h p(u_h) - p_h)\\
\le&C\|p(u_h) - p_h\|_1\|p(u_h) - \pi_h p(u_h)\|_1 + C\|y_h - y(u_h)\|\|\pi_h p(u_h) - p_h\|\\
\le&C(\delta)\|p(u_h) - \pi_h p(u_h)\|_1^2 + C(\delta)\|y(u_h) - y_h\|^2 + C\delta\|p(u_h) - p_h\|_{1,\Omega}^2\\
\le&C(\delta)h^2\|p(u_h)\|_2^2 + C(\delta)\|y(u_h) - y_h\|^2 + C\delta\|p(u_h) - p_h\|_1^2.
\end{aligned} \tag{16}$$

Let δ be small enough, we obtain

$$\|p(u_h) - p_h\|_1 \le Ch + C\|y(u_h) - y_h\|. \tag{17}$$

Similarly, we can prove that

$$\|y(u_h) - y_h\|_1 \le Ch\|y(u_h)\|_2 \le Ch. \tag{18}$$

Then (15) follows from (16)-(18). □

We introduce the following auxiliary problems:

$$\begin{aligned}
-\mathrm{div}(A^*\nabla\xi) =&F_1, \qquad \text{in}\,\Omega,\\
\xi|_{\partial\Omega} =&0,
\end{aligned} \tag{19}$$

$$\begin{aligned}
-\mathrm{div}(A\nabla\zeta) =&F_2, \qquad \text{in}\,\Omega,\\
\zeta|_{\partial\Omega} =&0.
\end{aligned} \tag{20}$$

From the regularity estimates (see e.g., [9]), we obtain

$$\|\xi\|_2 \le C\|F_1\|, \qquad \|\zeta\|_2 \le C\|F_2\|.$$

Lemma 3.3. *Let (y_h, p_h, u_h) be the solution of (8)-(10). Suppose that $y(u_h), p(u_h) \in H^2(\Omega)$. Then there exists a constant C independent of h such that*

$$\|y(u_h) - y_h\| + \|p(u_h) - p_h\| \le Ch^2. \tag{21}$$

Proof. Let $F_1 = y(u_h) - y_h$ in (19) and $\xi_h = \pi_h\xi$. From (8), (12) and (15), we have

$$\begin{aligned}
\|y(u_h) - y_h\|^2 =&a(y(u_h) - y_h, \xi)\\
=&a(y(u_h) - y_h, \xi - \xi_h)\\
\le&C\|y(u_h) - y_h\|_1\|\xi - \xi_h\|_1.
\end{aligned} \tag{22}$$

Note that

$$\|\xi - \xi_h\|_1 \le Ch\|\xi\|_2 \le Ch\|y(u_h) - y_h\|. \tag{23}$$

Thus,

$$\|y(u_h) - y_h\| \leq Ch\|y(u_h) - y_h\|_{1,\Omega} \leq Ch^2. \tag{24}$$

Similarly, let $F_2 = p(u_h) - p_h$ in (20) and $\zeta_h = \pi_h\zeta$. From (9), (13) and (15), we obtain

$$\|p(u_h) - p_h\| \leq Ch^2. \tag{25}$$

From (24) and (25), we get (21). □

Lemma 3.4. *Let (y, p, u) and (y_h, p_h, u_h) be the solutions of (3)-(5) and (8)-(10), respectively. Assume that all the conditions in Lemma 3.3 are valid. Then there exists a constant C independent of h such that*

$$\|u - u_h\| \leq Ch^2. \tag{26}$$

Proof. By selecting $v = u_h$ and $v = u$ in (5) and (10), respectively. From (3)-(4) and (12)-(13), we have

$$\begin{aligned}
\alpha\|u - u_h\|^2 \leq &(B^*p - B^*p_h, u - u_h) \\
= &(B^*p - B^*p(u_h), u - u_h) + (B^*p(u_h) - B^*p_h, u - u_h) \\
= &-\|y - y(u_h)\|^2 + (B^*p(u_h) - B^*p_h, u - u_h) \\
\leq &C\|p(u_h) - p_h\|\|u - u_h\|.
\end{aligned} \tag{27}$$

From (21) and (27), we derive (26). □

Now we combine lemmas 3.2-3.4 to come up with the following main result.

Theorem 3.5. *Let (y, p, u) and (y_h, p_h, u_h) be the solutions of (3)-(5) and (8)-(10), respectively. Assume that all the conditions in lemmas 3.2-3.4 are valid. Then we have*

$$\|u - u_h\| + \|y - y_h\| + \|p - p_h\| \leq Ch^2. \tag{28}$$

4. Superconvergence and a posteriori error estimates

We now derive the superconvergence and a posteriori error estimates for the variational discretization approximation. To begin with, let us introduce the elliptic projection operator $P_h : W \to W_h$, which satisfies: for any $\phi \in W$

$$a(\phi - P_h\phi, w_h) = 0, \qquad \forall\, w_h \in W_h.$$

It has the following approximation properties:

$$\|\phi - P_h\phi\| \leq Ch^2\|\phi\|_2, \qquad \forall\, \phi \in H^2(\Omega). \tag{29}$$

Theorem 4.1. *Let (y, p, u) and (y_h, p_h, u_h) be the solutions of (3)-(5) and (8)-(10), respectively. Assume that all the conditions in Theorem 3.5 are valid. Then we have*

$$\|P_h y - y_h\|_1 + \|P_h p - p_h\|_1 \leq Ch^2. \tag{30}$$

Proof. From (3)-(8), we have the following error equation:

$$a(y - y_h, w_h) = (Bu - Bu_h, w_h), \qquad \forall\, w_h \in W_h. \tag{31}$$

By using the definition of P_h and choosing $w_h = P_h y - y_h$, we have

$$a(P_h y - y_h, P_h y - y_h) = (Bu - Bu_h, P_h y - y_h). \tag{32}$$

Let us note that B is linear continuous operator. From (26), (29) and Holder inequality, we get

$$\|P_h y - y_h\|_1 \le Ch^2. \tag{33}$$

Similarly, we can derive

$$\|P_h p - p_h\|_1 \le Ch^2. \tag{34}$$

Thus, (30) follows from (33) and (34). □

In the second, let us recall recovery operators R_h and G_h, respectively. Let $R_h v$ be a continuous piecewise linear function (without zero boundary constraint). Similar to the Z-Z patch recovery in [37, 38], the value of $R_h v$ on the nodes are defined by least-squares argument on an element patches surrounding the nodes. The gradient recovery operator $G_h v = (R_h v_{x_1}, R_h v_{x_2})$, where R_h is the recovery operator defined above for the recovery of the control. The details can be found in [16].

Theorem 4.2. *Let (y, p, u) and (y_h, p_h, u_h) be the solutions of (3)-(5) and (8)-(10), respectively. Suppose that all the conditions in Theorem 4.1 are valid and $y, p \in H^3(\Omega)$. Then*

$$\|G_h y_h - \nabla y\| + \|G_h p_h - \nabla p\| \le Ch^2. \tag{35}$$

Proof. Let y_I be the piecewise linear Lagrange interpolation of y. According to Theorem 2.1.1 in [19], we have

$$\|P_h y - y_I\|_1 \le Ch^2 \|y\|_3. \tag{36}$$

From the standard interpolation error estimate technique (see, e.g., [9]) that

$$\|G_h y_I - \nabla y\| \le Ch^2 |y|_3. \tag{37}$$

By using (36)-(37), we get

$$\begin{aligned} \|G_h y_h - \nabla y\| &= \|G_h y_h - G_h P_h y\| + \|G_h P_h y - G_h y_I\| + \|G_h y_I - \nabla y\| \\ &\le C\,\|y_h - P_h y\|_1 + C\,\|P_h y - y_I\|_1 + \|G_h y_I - \nabla y\| \\ &\le C\,\|y_h - P_h y\|_1 + Ch^2 \|y\|_3. \end{aligned} \tag{38}$$

Therefore,

$$\|G_h y_h - \nabla y\|^2 \le C\|y_h - P_h y\|_1^2 + Ch^4 \|y\|_3^2. \tag{39}$$

From Theorem 4.1 and (39), we derive

$$\|G_h y_h - \nabla y\| \le Ch^2. \tag{40}$$

Similarly, we can prove that

$$\|G_h p_h - \nabla p\| \leq Ch^2. \tag{41}$$

Then (36) follows from (40)-(41). □

By using the superconvergence results above, we obtain the following a posteriori error estimates of variational discretization approximation for the elliptic optimal control problems.

Theorem 4.3. *Assume that all the conditions in Theorem 4.1 and Theorem 4.2 are valid. Then*

$$\eta_1 := \|G_h y_h - \nabla y_h\| = \|\nabla(y - y_h)\| + \mathcal{O}(h^2), \tag{42}$$

$$\eta_2 := \|G_h p_h - \nabla p_h\| = \|\nabla(p - p_h)\| + \mathcal{O}(h^2). \tag{43}$$

Proof. From Theorem 4.1 and Theorem 4.2, it is easy to obtain the above results. □

5. Numerical experiments

In this section, we present some numerical examples which is solved numerically with codes developed based on AFEPack. The package provide a freely available tool of finite element approximation for PDEs and the details can be found in [15].

We solve the following optimal control problems:

$$\begin{cases} \min\limits_{u(x)\in K} \left\{ \dfrac{1}{2}\|y(x) - y_d(x)\| + \dfrac{\alpha}{2}\|u(x) - u_d(x)\| \right\}, \\ -\operatorname{div}(A(x)\nabla y(x)) = f(x) + Bu(x), \quad x \in \Omega, \\ y(x) = 0, \quad x \in \partial\Omega, \end{cases}$$

where

$$K = \left\{ v(x) \in L^2(\Omega) : a \leq v(x) \leq b \right\},$$

and the domain Ω is the unit square $[0,1] \times [0,1]$ and $B = I$ is the identity operator and E is the 2×2 identity matrix.

Example 1. The data are as follows:

$$\begin{aligned} &\alpha = 1,\ a = 0,\ b = 1,\ A(x) = E, \\ &p(x) = -x_1x_2(1 - x_1)(1 - x_2), \\ &y(x) = p(x), \\ &u_d(x) = 1 - \sin(\pi x_1/2) - \sin(\pi x_2/2), \\ &u(x) = \max(0, \min(1, u_d(x) + p(x))), \\ &f(x) = -\operatorname{div}(A(x)\nabla y(x)) - u(x), \\ &y_d(x) = y(x) - \operatorname{div}(A^*(x)\nabla p(x)). \end{aligned}$$

The numerical results are listed in Table 1. In Figure 1, we show the profiles of the exact solution u alongside the solution error.

TABLE 1. Numerical results, Example 1.

Mesh	$\|u-u_h\|$	$\|y-y_h\|$	$\|p-p_h\|$
16×16	2.14058e-04	6.06285e-04	6.33941e-04
32×32	5.39805e-05	1.52186e-04	1.59139e-04
64×64	1.34857e-05	3.81582e-05	3.98295e-05
128×128	3.36858e-06	9.61916e-06	9.96385e-06
256×256	8.42796e-07	2.48396e-06	2.49504e-06

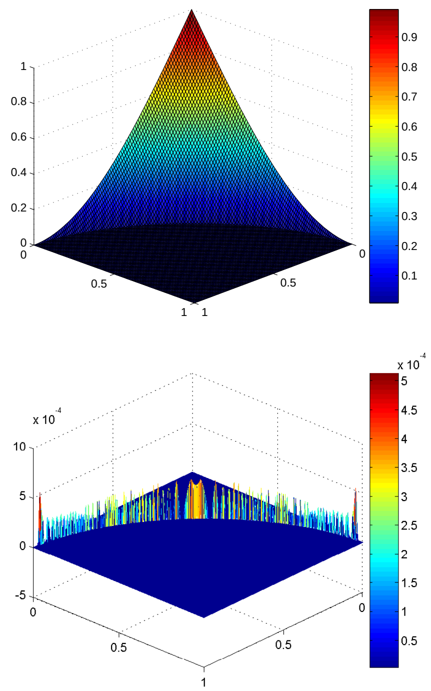

FIGURE 1. The exact solution u (top) and the error $u_h - u$ (bottom), Example 1.

The results in Table 1 indicate that $\|u-u_h\| = O(h^2)$, $\|y-y_h\| = O(h^2)$ and $\|p-p_h\| = O(h^2)$. It is consistent with our theoretical result obtained in the Theorem 3.5.

Example 2. The data are as follows:

$$\alpha = 1,\ a = -0.5,\ b = 0,$$

$$A(x) = \begin{cases} E, & x_1 + x_2 \geq 1, \\ 2E, & x_1 + x_2 < 1, \end{cases}$$

$$p(x) = \begin{cases} -2\sin(\pi x_1)\sin(\pi x_2), & x_1 + x_2 \geq 1 \\ -\sin(\pi x_1)\sin(\pi x_2), & x_1 + x_2 < 1, \end{cases}$$
$$y(x) = p(x),\ u_d(x) = 0,$$
$$u(x) = \max(-0.5, \min(0, u_d(x) + p(x))),$$
$$f(x) = -\mathrm{div}(A(x)\nabla y(x)) - u(x),$$
$$y_d(x) = y(x) - \mathrm{div}(A^*(x)\nabla p(x)).$$

The numerical results based on adaptive mesh and uniform mesh are presented in Table 2. In Figure 2, we show the profiles of the exact solution u alongside the solution error. From Table 2, it is clear that the adaptive mesh generated via the error indicators in Theorem 4.3 are able to save substantial computational work, in comparison with the uniform mesh. Our numerical results confirm our theoretical results.

TABLE 2. Numerical results, Example 2.

Mesh	Nodes	Sides	Elements	Dofs	$\|u - u_h\|$
Uniform mesh	513	1456	944	513	6.19419e-02
Adaptive mesh	145	392	248	145	6.04841e-02

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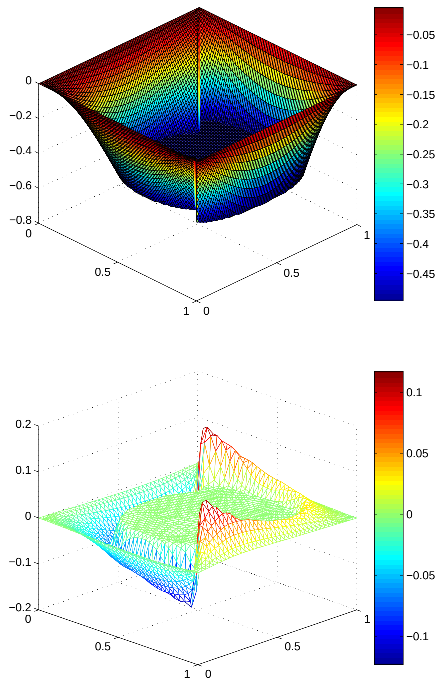

FIGURE 2. The exact solution u (top) and the error $u_h - u$ (bottom), Example 2.

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Yuchun Hua received M.Sc. form Yunnan University. His research interests include differential equations and dynamical system.

Department of Mathematics and Computational Science, Hunan University of Science and Engineering, Yongzhou 425100, Hunan, China.

e-mail: huayuchun306@sina.com

Yuelong Tang received M.Sc. and Ph.D from Xiangtan University. His research interests include numerical optimization and numerical methods for partial differential equations.

Department of Mathematics and Computational Science, Hunan University of Science and Engineering, Yongzhou 425100, Hunan, China.
e-mail: tangyuelonga@163.com