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BOUNDEDNESS IN PERTURBED FUNCTIONAL DIFFERENTIAL SYSTEMS

SANG IL CHOI, DONG MAN IM AND YOON HOE GOO*

Abstract. In this paper, we investigate bounds for solutions of the nonlinear functional differential systems

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1. Introduction

The method incorporating integral inequalities takes an important place among the methods developed for the qualitative analysis of solutions to linear and nonlinear system of differential equations. The behavior of solutions of a perturbed system is determined in terms of the behavior of solutions of an unperturbed system. There are three useful methods for investigating the qualitative behavior of the solutions of perturbed nonlinear system of differential systems: the method of variation of constants formula, Lyapunov's second method, and the use of integral inequalities. In the presence the method of integral inequalities is as efficient as the direct Lyapunov's method.

The notion of h-stability (hS) was introduced by Pinto [13,14] with the intention of obtaining results about stability for a weakly stable system (at least, weaker than those given exponential asymptotic stability) under some perturbations. He obtained a general variational h-stability and some properties about asymptotic behavior of solutions of differential systems called h-systems. Also, he studied some general results about asymptotic integration and gave some important examples in [13]. Choi and Ryu [3], Choi, Koo [5], and Choi et al. [4] investigated bounds of solutions for nonlinear perturbed systems and nonlinear functional differential systems. Also, Goo [9,10] studied the boundedness of solutions for nonlinear functional perturbed systems.

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In this paper, we investigate bounds of solutions of the nonlinear functional perturbed differential systems.

2. Preliminaries

We consider the nonlinear functional differential equation

$$y' = f(t,y) + \int_{t_0}^{t} g(s,y(s),Ty(s))ds,\ y(t_0) = y_0, \tag{1}$$

where $t \in \mathbb{R}^+ = [0,\infty)$, $x \in \mathbb{R}^n$,$f \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $f(t,0) = 0$, the derivative $f_x \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $g(t,0,0) = 0$ and T is a continuous operator mapping from $C(\mathbb{R}^+, \mathbb{R}^n)$ into $C(\mathbb{R}^+, \mathbb{R}^n)$. The symbol $|\cdot|$ will be used to denote arbitrary vector norm in $\mathbb{R}^n$. We assume that for any two continuous functions $u, v \in C(I)$ where I is the closed interval, the operator T satisfies the following property:

$$u(t) \leq v(t), 0 \leq t \leq t_1, t_1 \in I,$$

implies $Tu(t) \leq Tv(t), 0 \leq t \leq t_1$, and $|Tu| \leq T|u|$.

Equation (1) can be considered as the perturbed equation of

$$x'(t) = f(t,x(t)), \quad x(t_0) = x_0, \tag{2}$$

Let $x(t,t_0,x_0)$ be denoted by the unique solution of (2) passing through the point $(t_0,x_0) \in \mathbb{R}^+ \times \mathbb{R}^n$ such that $x(t_0,t_0,x_0) = x_0$. Also, we can consider the associated variational systems around the zero solution of (2) and around $x(t)$, respectively,

$$v'(t) = f_x(t,0)v(t),\ v(t_0) = v_0 \tag{3}$$

and

$$z'(t) = f_x(t,x(t,t_0,x_0))z(t),\ z(t_0) = z_0. \tag{4}$$

The fundamental matrix $\Phi(t,t_0,x_0)$ of (4) is given by

$$\Phi(t,t_0,x_0) = \frac{\partial}{\partial x_0} x(t,t_0,x_0),$$

and $\Phi(t,t_0,0)$ is the fundamental matrix of (3).

We recall some notions of h-stability [13].

Definition 2.1. The system (2) (the zero solution $x = 0$ of (2)) is called an *h-system* if there exist a constant $c \geq 1$ and a positive continuous function h on $\mathbb{R}^+$ such that

$$|x(t)| \leq c\,|x_0|\,h(t)\,h(t_0)^{-1}$$

for $t \geq t_0 \geq 0$ and $|x_0|$ small enough (here $h(t)^{-1} = \frac{1}{h(t)}$).

Definition 2.2. The system (2) (the zero solution $x = 0$ of (2)) is called (hS) *h-stable* if there exists $\delta > 0$ such that (2) is an h-system for $|x_0| \leq \delta$ and h is bounded.

Let $\mathcal{M}$ denote the set of all $n \times n$ continuous matrices $A(t)$ defined on $\mathbb{R}^+$ and $\mathcal{N}$ be the subset of $\mathcal{M}$ consisting of those nonsingular matrices $S(t)$ that are of class C^1 with the property that $S(t)$ and $S^{-1}(t)$ are bounded. The notion of t_∞-similarity in $\mathcal{M}$ was introduced by Conti [6].

Definition 2.3. A matrix $A(t) \in \mathcal{M}$ is t_∞-*similar* to a matrix $B(t) \in \mathcal{M}$ if there exists an $n \times n$ matrix $F(t)$ absolutely integrable over $\mathbb{R}^+$, i.e.,

$$\int_0^\infty |F(t)|dt < \infty$$

such that

$$\dot{S}(t) + S(t)B(t) - A(t)S(t) = F(t) \tag{5}$$

for some $S(t) \in \mathcal{N}$.

We give some related properties that we need in the sequal.

Lemma 2.1 ([14]). *The linear system*

$$x' = A(t)x, \ x(t_0) = x_0, \tag{6}$$

where $A(t)$ is an $n \times n$ continuous matrix, is an h-system (h-stable, respectively,) if and only if there exist $c \geq 1$ and a positive and continuous (bounded, respectively,) function h defined on $\mathbb{R}^+$ such that

$$|\phi(t, t_0)| \leq c\, h(t)\, h(t_0)^{-1} \tag{7}$$

for $t \geq t_0 \geq 0$, where $\phi(t, t_0)$ is a fundamental matrix of (6).

We need Alekseev formula to compare between the solutions of (2) and the solutions of perturbed nonlinear system

$$y' = f(t, y) + g(t, y), \ y(t_0) = y_0, \tag{8}$$

where $g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$ and $g(t, 0) = 0$. Let $y(t) = y(t, t_0, y_0)$ denote the solution of (8) passing through the point (t_0, y_0) in $\mathbb{R}^+ \times \mathbb{R}^n$.

The following is a generalization to nonlinear system of the variation of constants formula due to Alekseev [1].

Lemma 2.2. *If $y_0 \in \mathbb{R}^n$, then for all t such that $x(t, t_0, y_0) \in \mathbb{R}^n$,*

$$y(t, t_0, y_0) = x(t, t_0, y_0) + \int_{t_0}^t \Phi(t, s, y(s))\, g(s, y(s))\, ds.$$

Theorem 2.3 ([3]). *If the zero solution of (2) is hS, then the zero solution of (3) is hS.*

Theorem 2.4 ([4]). *Suppose that $f_x(t, 0)$ is t_∞-similar to $f_x(t, x(t, t_0, x_0))$ for $t \geq t_0 \geq 0$ and $|x_0| \leq \delta$ for some constant $\delta > 0$. If the solution $v = 0$ of (3) is hS, then the solution $z = 0$ of (4) is hS.*

Lemma 2.5 ([5]). *Let $u, \lambda_1, \lambda_2, w \in C(\mathbb{R}^+)$ and $w(u)$ be nondecreasing in u such that $\frac{1}{v}w(u) \leq w(\frac{u}{v})$ for some $v > 0$. If ,for some $c > 0$,*

$$u(t) \leq c + \int_{t_0}^{t} \lambda_1(s)u(s)ds + \int_{t_0}^{t} \lambda_1(s)(\int_{t_0}^{s} \lambda_2(\tau)w(u(\tau))d\tau)ds, \;\; 0 \leq t_0 \leq t.$$

then

$$u(t) \leq W^{-1}\Big[W(c) + \int_{t_0}^{t} \lambda_2(s)ds\Big] exp \int_{t_0}^{t} \lambda_1(s)ds, \;\; t_0 \leq t < b_1,$$

where $W(u) = \int_{u_0}^{u} \frac{ds}{w(s)}$, $W^{-1}(u)$ is the inverse of $W(u)$, and

$$b_1 = \sup\Big\{t \geq t_0 : W(c) + \int_{t_0}^{t} \lambda_2(s)ds \in \mathrm{domW}^{-1}\Big\}.$$

Lemma 2.6 ([2]). *Let $u, \lambda_1, \lambda_2, \lambda_3 \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$, and $w(u)$ be nondecreasing in u. Suppose that for some $c > 0$,*

$$u(t) \leq c + \int_{t_0}^{t} \lambda_1(s)w(u(s))ds + \int_{t_0}^{t} \lambda_2(s)(\int_{t_0}^{s} \lambda_3(\tau)w(u(\tau))d\tau)ds, \;\; 0 \leq t_0 \leq t.$$

Then

$$u(t) \leq W^{-1}\Big[W(c) + \int_{t_0}^{t} (\lambda_1(s) + \lambda_2(s) \int_{t_0}^{s} \lambda_3(\tau)d\tau)ds\Big], \;\; t_0 \leq t < b_1,$$

where W, W^{-1} are the same functions as in Lemma 2.5, and

$$b_1 = \sup\Big\{t \geq t_0 : W(c) + \int_{t_0}^{t} (\lambda_1(s) + \lambda_2(s) \int_{t_0}^{s} \lambda_3(\tau)d\tau)ds \in \mathrm{domW}^{-1}\Big\}.$$

Lemma 2.7 ([10]). *Let $u, p, q, w, r \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$, and $w(u)$ be nondecreasing in u. Suppose that for some $c > 0$,*

$$u(t) \leq c + \int_{t_0}^{t} (p(s) \int_{t_0}^{s} (q(\tau)w(u(\tau)) + v(\tau) \int_{t_0}^{\tau} r(a)w(u(a))da)d\tau)ds, \; t \geq t_0. \quad (9)$$

Then

$$u(t) \leq W^{-1}\Big[W(c) + \int_{t_0}^{t} (p(s) \int_{t_0}^{s} (q(\tau) + v(\tau) \int_{t_0}^{\tau} r(a)da)d\tau)ds\Big], \; t_0 \leq t < b_1, \quad (10)$$

where W, W^{-1} are the same functions as in Lemma 2.5, and

$$b_1 = \sup\Big\{t \geq t_0 : W(c) + \int_{t_0}^{t} (p(s) \int_{t_0}^{s} (q(\tau) + v(\tau) \int_{t_0}^{\tau} r(a)da)d\tau)ds \in \mathrm{domW}^{-1}\Big\}.$$

3. Main results

In this section, we investigate the bounded property for the nonlinear functional differential systems.

Theorem 3.1. *Let $a, c, u, w \in C(\mathbb{R}^+)$, $w(u)$ be nondecreasing in u and $\frac{1}{v}w(u) \leq w(\frac{u}{v})$ for some $v > 0$. Suppose that $f_x(t, 0)$ is t_∞-similar to $f_x(t, x(t, t_0, x_0))$ for $t \geq t_0 \geq 0$ and $|x_0| \leq \delta$ for some constant $\delta > 0$, the solution $x = 0$ of (2) is hS with the increasing function h, and g in (1) satisfies*

$$\left| \int_{t_0}^{s} g(\tau, y(\tau), Ty(\tau)) d\tau \right| \leq a(s)(|y(s)| + |Ty(s)|),\ t \geq t_0 \geq 0,$$

and

$$|Ty| \leq \int_{t_0}^{t} c(s) w(|y(s)|) ds,$$

where $\int_{t_0}^{\infty} a(s)ds < \infty$ and $\int_{t_0}^{\infty} c(s)ds < \infty$. Then, any solution $y(t) = y(t, t_0, y_0)$ of (1) is bounded on $[t_0, \infty)$ and it satisfies

$$|y(t)| \leq h(t) W^{-1}\Big[W(k) + \int_{t_0}^{t} c(s)ds\Big] \exp\Big(\int_{t_0}^{t} \beta(s)ds\Big),\ t_0 \leq t < b_1,$$

where W, W^{-1} are the same functions as in Lemma 2.5 , $\beta(t) = c_2 a(t)$, k is a positive constant, and

$$b_1 = \sup\Big\{t \geq t_0 : W(k) + \int_{t_0}^{t} c(s)ds \in \mathrm{dom}W^{-1}\Big\}.$$

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (2) and (1), respectively. By Theorem 2.3, since the solution $x = 0$ of (2) is hS, the solution $v = 0$ of (3) is hS. Therefore, by Theorem 2.4, the solution $z = 0$ of (4) is hS. By Lemma 2.1, Lemma 2.2 and the increasing property of the function h, we have

$$\begin{aligned} |y(t)| &\leq |x(t)| + \int_{t_0}^{t} |\Phi(t, s, y(s))| \left| \int_{t_0}^{s} g(\tau, y(\tau), Ty(\tau)) d\tau \right| ds \\ &\leq c_1 |y_0| h(t)\, h(t_0)^{-1} + \int_{t_0}^{t} c_2 h(t) a(s) \frac{|y(s)|}{h(s)} ds \\ &+ \int_{t_0}^{t} c_2 h(t) a(s) \int_{t_0}^{s} c(\tau) w(\frac{|y(\tau)|}{h(\tau)}) d\tau ds. \end{aligned}$$

Set $u(t) = |y(t)| h(t)^{-1}$. Then, by Lemma 2.5, we obtain

$$|y(t)| \leq h(t) W^{-1}\Big[W(k) + \int_{t_0}^{t} c(s)ds\Big] \exp\Big(\int_{t_0}^{t} \beta(s)ds\Big),\ t_0 \leq t < b_1,$$

where $k = c_1 |y_0| h(t_0)^{-1}$ and $\beta(t) = c_2 a(t)$. This completes the proof. □

Remark 3.1. Letting $c(\tau) = 0$ in Theorem 3.1, we have the similar result as that of Theorem 3.3 in [7].

Theorem 3.2. *Let $a, b, c, u, w \in C(\mathbb{R}^+)$, $w(u)$ be nondecreasing in u and $\frac{1}{v}w(u) \leq w(\frac{u}{v})$ for some $v > 0$. Suppose that the solution $x = 0$ of (2) is hS with a nondecreasing function h and the perturbed term g in (1) satisfies*

$$|\Phi(t,s,y)g(t,y,Ty)| \leq a(s)w(|y|) + b(s)|Ty|,\ t \geq t_0 \geq 0,$$

and

$$|Ty| \leq \int_{t_0}^{t} c(s)w(|y(s)|)ds,$$

where $\int_{t_0}^{\infty} a(s)ds < \infty$, $\int_{t_0}^{\infty} b(s)ds < \infty$,and $\int_{t_0}^{\infty} c(s)ds < \infty$. Then any solution $y(t) = y(t, t_0, y_0)$ of (1) is bounded on $[t_0, \infty)$ and it satisfies

$$|y(t)| \leq h(t)W^{-1}\Big[W(k) + \int_{t_0}^{t}(a(s) + b(s)\int_{t_0}^{s} c(\tau)d\tau)ds\Big],\ t_0 \leq t < b_1.$$

where W, W^{-1} are the same functions as in Lemma 2.5, k is a positive constant, and

$$b_1 = \sup\Big\{t \geq t_0 : W(k) + \int_{t_0}^{t}(a(s) + b(s)\int_{t_0}^{s} k(\tau)d\tau)ds \in \mathrm{domW}^{-1}\Big\}.$$

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (2) and (1), respectively. By Lemma 2.2, we obtain

$$\begin{aligned} |y(t)| &\leq |x(t)| + \int_{t_0}^{t} |\Phi(t,s,y(s))g(s,y(s),Ty(s))|ds \\ &\leq c_1|y_0|h(t)h(t_0)^{-1} + \int_{t_0}^{t}(a(s)w(|y(s)|) + b(s)\int_{t_0}^{s} c(\tau)w(|y(\tau)|)d\tau)ds \\ &\leq c_1|y_0|h(t)h(t_0)^{-1} + \int_{t_0}^{t} a(s)h(t)w(\frac{|y(s)|}{h(s)})ds \\ &+ \int_{t_0}^{t} b(s)\int_{t_0}^{s} h(t)c(\tau)w(\frac{|y(\tau)|}{h(\tau)})d\tau ds, \end{aligned}$$

since h is nondecreasing. Set $u(t) = |y(t)|h(t)^{-1}$. Then, by Lemma 2.6, we have

$$|y(t)| \leq h(t)W^{-1}\Big[W(k) + \int_{t_0}^{t}(a(s) + b(s)\int_{t_0}^{s} c(\tau)d\tau)ds\Big],\ t_0 \leq t < b_1,$$

where $k = c_1|y_0|h(t_0)^{-1}$. Therefore, we obtain the result. □

Remark 3.2. Letting $c(\tau) = 0$ in Theorem 3.2, we have the similar result as that of Theorem 3.1 in [8].

Theorem 3.3. *Let $a, b, c, u, w \in C(\mathbb{R}^+)$, $w(u)$ be nondecreasing in u and $\frac{1}{v}w(u) \leq w(\frac{u}{v})$ for some $v > 0$. Suppose that $f_x(t, 0)$ is t_∞-similar to $f_x(t, x(t, t_0, x_0))$ for $t \geq t_0 \geq 0$ and $|x_0| \leq \delta$ for some constant $\delta > 0$, the solution $x = 0$ of (2) is hS with the increasing function h, and g in (1) satisfies*

$$\Big|\int_{t_0}^{s} g(\tau, y(\tau), Ty(\tau))d\tau\Big| \leq a(s)w(|y(s)|) + b(s)|Ty(s)|,$$

and

$$|Ty(t)| \le \int_{t_0}^{t} c(s)w(|y(s)|)ds,$$

where $\int_{t_0}^{\infty} a(s)ds < \infty$ *and* $\int_{t_0}^{\infty} b(s)ds < \infty$*, and* $\int_{t_0}^{\infty} c(s)ds < \infty$*. Then, any solution* $y(t) = y(t, t_0, y_0)$ *of (1) is bounded on* $[t_0, \infty)$ *and it satisfies*

$$|y(t)| \le h(t)W^{-1}\Big[W(k) + c_2 \int_{t_0}^{t}(a(s) + b(s)\int_{t_0}^{s} c(\tau)d\tau)ds\Big],$$

where W*,* W^{-1} *are the same functions as in Lemma 2.5,* k *is a positive constant, and*

$$b_1 = \sup\Big\{t \ge t_0 : W(k) + c_2 \int_{t_0}^{t}(a(s) + b(s)\int_{t_0}^{s} c(\tau)d\tau)ds \in \mathrm{domW}^{-1}\Big\}.$$

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (2) and (1), respectively. By Theorem 2.3, since the solution $x = 0$ of (2) is hS, the solution $v = 0$ of (3) is hS. Therefore, by Theorem 2.4, the solution $z = 0$ of (4) is hS. By Lemma 2.1, Lemma 2.2 and the increasing property of the function h, we obtain

$$\begin{aligned} |y(t)| &\le |x(t)| + \int_{t_0}^{t} |\Phi(t, s, y(s))| \Big| \int_{t_0}^{s} g(\tau, y(\tau), Ty(\tau))d\tau \Big| ds \\ &\le c_1|y_0|h(t)\, h(t_0)^{-1} + \int_{t_0}^{t} c_2h(t)a(s)w(\frac{|y(s)|}{h(s)})ds \\ &+ \int_{t_0}^{t} c_2h(t)b(s)\int_{t_0}^{s} c(\tau)w(\frac{|y(\tau)|)}{h(\tau)})d\tau ds. \end{aligned}$$

Set $u(t) = |y(t)|h(t)^{-1}$. Then, by Lemma 2.6, we have

$$|y(t)| \le h(t)W^{-1}\Big[W(k) + c_2 \int_{t_0}^{t}(a(s) + b(s)\int_{t_0}^{s} c(\tau)d\tau)ds\Big],\ t_0 \le t < b_1,$$

where $k = c_1|y_0|h(t_0)^{-1}$. Hence, the proof is complete. □

Remark 3.3. Letting $c(\tau) = 0$ in Theorem 3.3, we have the similar result as that of Theorem 3.2 in [8].

Theorem 3.4. *Let* $b, c, u, w \in C(\mathbb{R}^+)$*,* $w(u)$ *be nondeacreasing in* u *and* $\frac{1}{v}w(u) \le w(\frac{u}{v})$ *for some* $v > 0$*. Suppose that* $f_x(t, 0)$ *is* t_∞*-similar to* $f_x(t, x(t, t_0, x_0))$ *for* $t \ge t_0 \ge 0$ *and* $|x_0| \le \delta$ *for some constant* $\delta > 0$*. If the solution* $x = 0$ *of (2) is an h-system with a positive continuous function* h *and* g *in (1) satisfies*

$$|g(t, y, Ty)| \le a(t)w(|y(t)|) + b(t)|Ty(t)|,\ t \ge t_0,\ y \in \mathbb{R}^n$$

and

$$|Ty(t)| \le \int_{t_0}^{t} c(s)w(|y(s)|)ds,$$

where $a : \mathbb{R}^+ \to \mathbb{R}^+$ *is continuous with*

$$\int_{t_0}^{\infty} \frac{1}{h(s)} \int_{t_0}^{s} (h(\tau)a(\tau) + b(\tau) \int_{t_0}^{\tau} h(r)c(r)dr) d\tau ds < \infty, \tag{11}$$

for all $t_0 \geq 0$*, then any solution* $y(t) = y(t, t_0, y_0)$ *of (1) satisfies*

$$|y(t)| \leq h(t) W^{-1} \Big[W(k) + \int_{t_0}^{t} \frac{c_2}{h(s)} \int_{t_0}^{s} (h(\tau)a(\tau) + b(\tau) \int_{t_0}^{\tau} h(r)c(r)dr) d\tau ds \Big]$$

, $t_0 \leq t < b_1$*, where* W*,* W^{-1} *are the same functions as in Lemma 2.5, k is a positive constant, and*

$$b_1 = \sup \Big\{ t \geq t_0 : W(k) + \int_{t_0}^{t} \frac{c_2}{h(s)} \int_{t_0}^{s} (h(\tau)a(\tau) + b(\tau) \int_{t_0}^{\tau} h(r)c(r)dr) d\tau ds \in \mathrm{dom} W^{-1} \Big\}.$$

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (2) and (1), respectively. By Theorem 2.3, since the solution $x = 0$ of (2) is an h-system, the solution $v = 0$ of (3) is an h-system. Therefore, by Theorem 2.4, the solution $z = 0$ of (4) is an h-system. By Lemma 2.2, we have

$$\begin{aligned} |y(t)| &\leq |x(t)| + \int_{t_0}^{t} |\Phi(t, s, y(s))| \int_{t_0}^{s} |g(\tau, y(\tau), Ty(\tau))| d\tau ds \\ &\leq c_1 |y_0| h(t)\, h(t_0)^{-1} + \int_{t_0}^{t} c_2 \frac{h(t)}{h(s)} \int_{t_0}^{s} h(\tau)a(\tau) w(\frac{|y(\tau)|}{h(\tau)}) d\tau ds \\ &+ \int_{t_0}^{t} c_2 \frac{h(t)}{h(s)} \int_{t_0}^{s} b(\tau) \int_{t_0}^{\tau} h(r)c(r) w(\frac{|y(r)|}{h(r)}) dr d\tau ds. \end{aligned}$$

Setting $u(t) = |y(t)| h(t)^{-1}$ and using Lemma 2.7, we obtain

$$|y(t)| \leq h(t) W^{-1} \Big[W(k) + \int_{t_0}^{t} \frac{c_2}{h(s)} \int_{t_0}^{s} (h(\tau)a(\tau) + b(\tau) \int_{t_0}^{\tau} h(r)c(r)dr) d\tau ds \Big]$$

, $t_0 \leq t < b_1$, where $k = c_1 |y_0| h(t_0)^{-1}$. Hence, the proof is complete. □

Remark 3.4. Letting $c(\tau) = 0$ in Theorem 3.4, we have the similar result as that of Theorem 3.5 in [8].

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Sang Il Choi received the BS from Korea University and Ph.D at North Carolina State University under the direction of J. Silverstein . Since 1995 he has been at Hanseo University as a professor. His research interests focus on Analysis and Probability theory.

Department of Mathematics, Hanseo University, Seasan 356-706, Republic of Korea
e-mail: schoi@hanseo.ac.kr

Dong Man Im received the BS and Ph.D at Inha University. Since 1982 he has been at Cheongju University as a professor. His research interests focus on Algebra and differential equations.

Department of Mathematics Education Cheongju University Cheongju Chungbuk 360-764, Republic of Korea
e-mail: dmim@cheongju.ac.kr

Yoon Hoe Goo received the BS from Cheongju University and Ph.D at Chungnam National University under the direction of Chin-Ku Chu. Since 1993 he has been at Hanseo University as a professor. His research interests focus on topological dynamical systems and differential equations.

Department of Mathematics, Hanseo University, Seasan 356-706, Republic of Korea
e-mail: yhgoo@hanseo.ac.kr