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SOME NEW RESULTS ON IRREGULARITY OF GRAPHS†

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Abstract. Suppose G is a simple graph. The irregularity of G, $irr(G)$, is the summation of $imb(e)$ over all edges $uv = e \in G$, where $imb(e) = |deg(u) - deg(v)|$. In this paper, we investigate the behavior of this graph parameter under some old and new graph operations.

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1. Introduction

The degree−based graph invariants are parameters defined by degrees of vertices. The first of such graph parameters was introduced by Gutman and Trinajstić [10]. Suppose G is a graph $e = uv \in E(G)$. The **first Zagreb index** of G is defined as $M_1(G) = \sum_{v\in V(G)} deg(v)^2$. There are a lot of works on Zagreb group invariants and interested readers can be consulted [3, 8, 11, 16, 17] for more information on this topic. The *imbalance* of e is defined as $imb(e) = |deg(u) - deg(v)|$. The summation of imbalances over all edges of G is denoted by $irr(G)$. Albertson [2], named this parameter **"irregularity"** of the graph G. After this paper, there was a lot of research considering the irregularity index, see [12, 14, 15] for details. It is easy to see that $M_1(G) = \sum_{e=uv\in E(G)}[deg(u) + deg(v)]$. Fath-Tabar [9], unaware from the seminal paper of Albertson and because of similarity between M_1 and irr used the term "**third Zagreb index**" for "irregularity".

Albertson [2] computed the maximum irregularity of various classes of graphs. As a consequence, he proved that the irregularity of an arbitrary graph with n vertices is less than $\frac{4n^3}{27}$, and this bound is tight. Some of the present authors [20], characterized the graphs with minimum and maximum values of irregularity. Luo and Zhou [18] determined the maximum irregularity of trees and unicyclic graphs with a given number of vertices and matching number. They also characterized

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the extremal graphs with the mentioned property. Zhou and Luo [22], established an upper bound for $irr(G)$ in terms of n, m, and r, where n is the order of G, $m \geq 1$ is its size, and G is assumed to have no complete subgraph of order $r+1$ where $2 \leq r \leq n-1$. They also provided new upper bounds for the irregularity of trees and unicyclic graphs. These are both functions of the number of pendant vertices of the graph under consideration. For each of these three inequalities, the authors supplied a characterization of all graphs which attain the bound. Henning and Rautenbach [15] obtained the structure of bipartite graphs having maximum possible irregularity with given cardinalities of its bipartition and given number of edges. They derived a result for bipartite graphs with given cardinalities of its bipartition and presented an upper bound on the irregularity of these graphs. In particular, they shown that if G is a bipartite graph of order n with a bipartition of equal cardinalities, then $irr(G) \leq \frac{n^3}{27}$, while if G is a bipartite graph with partite sets of cardinalities n_1 and n_2, where $n_1 \geq 2n_2$, then $irr(G) \leq irr(K_{n_1,n_2})$.

Abdo and Dimitrov [1] introduced the concept of **total irregularity** of a graph G and obtain some exact formula for computing total irregularity of some old graph operations. The aim of this paper is to compute formulas for the regularity of graphs under some old and new graph invariants.

Throughout this paper the **path, complete** and **star** graphs of order n are denoted by by P_n, K_n and S_n, respectively. The degree of a vertex v is denoted by $deg_G(v)$. We denote by $\Delta(G)$ the maximum degree of vertices of G.

Lemma 1.1. *Let G be a connected graph on $n, n > 2$ vertices. If G has exactly k pendant vertices then $irr(G) \geqslant k$, with equality if and only if $G \cong P_n$.*

Proof. Suppose u is a pendant vertex of G and $v \in V(G)$ is a vertex adjacent to u. Since $|V(G)| > 2$, $deg(v) \geqslant 2$ and so $|deg(u) - deg(v)| \geqslant 1$. But, G has exactly k pendant vertices, and so $irr(G) \geqslant k$. Notice that $irr(G) = k$ if and only if $deg_G(u) = 1$ or 2, for each vertex $u \in V(G)$. This implies that $G \cong P_n$. □

Corollary 1.2. *Let T be a tree with $\Delta(T) > 1$. Then $irr(T) \geqslant \Delta(T)$.*

Proof. By assumption, T has at least $\Delta(T)$ pendant vertices and so, by Lemma 1.1, $irr(T) \geqslant \Delta(T)$. □

Lemma 1.3. *Let T be a tree with $n > 2$ vertices. Then*

$$2 \leqslant irr(T) \leqslant (n-1)(n-2).$$

and the lower bound is attained if and only if $T \cong P_n$. The upper bound is attained if and only if $T \cong S_n$.

Proof. Since T is a tree with $n > 2$ vertices, it has at least two pendant vertices and so, by lemma 1.1, $2 \leqslant irr(T)$. On the other hand, it is not difficult to check that for each $uv \in E(T)$, $|deg(u) - deg(v)| \leqslant n-2$ and $E(T) = n-1$. So, $irr(T) \leqslant (n-1)(n-2)$. Notice that S_n is the unique graph with $n-2$ as difference of degrees between adjacent vertices. But by Lemma 1.1, P_n is the unique tree that its third Zagreb index is equal to 2, as desired. □

2. Main results

The **join** $G = G_1 + G_2$, Figure 1, of graphs G_1 and G_2 with disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2 is the graph union $G_1 \cup G_2$ together with all the edges joining V_1 and V_2. For example, $\bar{K}_n + \bar{K}_m = K_{n,m}$. The **composition** $G = G_1[G_2]$ of graphs G_1 and G_2 with disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2 is the graph with vertex set $V_1 \times V_2$ and $u = (u_1, v_1)$ is adjacent with $v = (u_2, v_2)$ whenever (u_1 is adjacent to u_2) or ($u_1 = u_2$ and v_1 is adjacent to v_2) [13]. For instance, $P_2[\bar{K}_n] = K_{n,n}$.

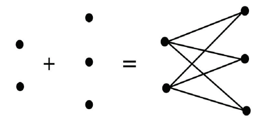

FIGURE 1. The Join of $\bar{K}_2$ and $\bar{K}_3$.

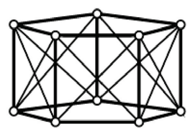

FIGURE 2. The Strong Product of C_5 and K_2.

The **Strong product** $G \boxtimes H$, Figure 2, of graphs G and H has the vertex set $V(G \boxtimes H) = V(G) \times V(H)$ and $(a, x)(b, y)$ is an edge of $G \boxtimes H$ if $a = b$ and $xy \in E(H)$, or $ab \in E(G)$ and $x = y$, or $ab \in E(G)$ and $xy \in E(H)$. As an example, $C_n \boxtimes K_2$ is the closed fence. The **tensor product** $G \otimes H$, Figure 3, is defined as the graph with vertex set $V(G) \times V(H)$ and $E(G \otimes H) = \{(u_1, u_2)(v_1, v_2) \mid u_1v_1 \in E(G) \textit{ and } u_2v_2 \in E(H)\}$. For example, $C_n \otimes P_2 = C_{2n}$. The **corona product** GoH, Figure 4, is obtained by taking one copy of G and $|V(G)|$ copies of H; and by joining each vertex of the i-th copy of H to the i-th vertex of G, $1 \leq i \leq |V(G)|$ [19]. For example, $K_1 o \bar{K}_n = S_n$. Finally, for a connected graph G, $R(G)$ is a graph obtained from G by adding a new vertex corresponding to each edge of G, then joining each new vertex to the end vertices of the corresponding edge [21]. For example, we can see $R(P_2) = K_3$.

It is well-known that the Cartesian product of graphs can be recognized efficiently, in time $O(mlogn)$ for a graph with n vertices and m edges [22]. This operation is commutative and associative as an operation on isomorphism classes of graphs, but it is not commutative. The Cartesian product is not a product in the category of graphs, but the tensor product is the categorical product.

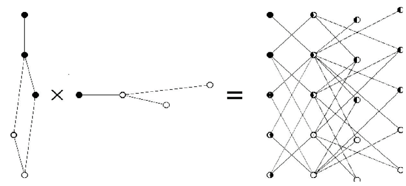

FIGURE 3. The Tensor Product of Two Graphs.

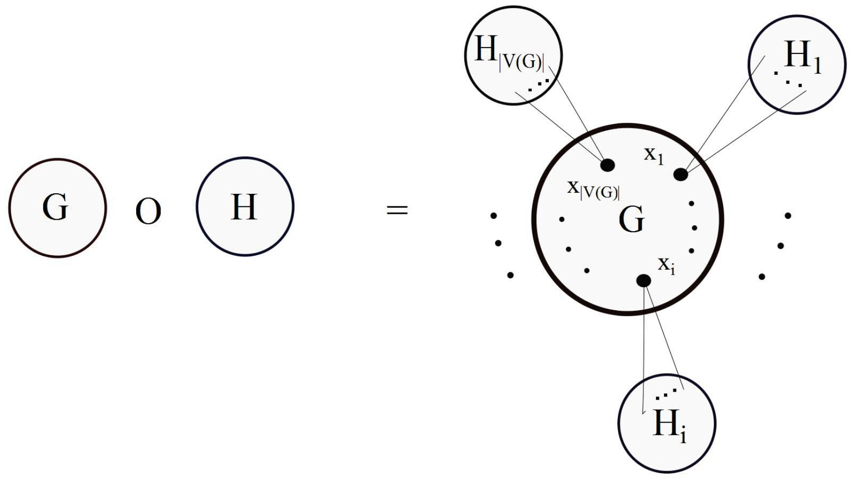

FIGURE 4. The Corona Product of Two Graphs.

Suppose G and H are graphs with disjoint vertex sets. Following Došlić [8], for given vertices $y \in V(G)$ and $z \in V(H)$ a splice of G and H by vertices y and z, $(G \cdot H)(y; z)$, is defined by identifying the vertices y and z in the union of G and H. Similarly, a link of G and H by vertices y and z is defined as the graph $(G \sim H)(y; z)$ obtained by joining y and z by an edge in the union of these graphs. Let H is a tree of progressive degree p and generation r that whose root vertex is r_1. Also, let $DD_{p,r}$ be the graph of the regular dicentric dendrimer, Figure 6. So, it is clear that $DD_{p,r} = (H \sim H)(r_1; r_1)$.

Lemma 2.1. *Suppose G and H are rooted graphs with respect to the rooted vertices of a and b, respectively. Then*

$$\begin{aligned} irr(G) + irr(H) - 2deg_G(a)deg_H(b) &\leqslant irr((G \cdot H)(a; b)) \\ &\leqslant irr(G) + irr(H) + 2deg_G(a)deg_H(b). \end{aligned}$$

and the upper bound is attained if and only if for every vertex $u \in V(G)$ that $ua \in E(G)$, $deg_G(a) \geqslant deg_G(u)$ and for every vertex $v \in V(H)$ that $vb \in E(H)$, $deg_H(b) \geqslant deg_H(v)$. Moreover, the lower bound is attained if and only if for each *edge $au \in E(G)$, $deg_G(u) - deg_G(a) \geqslant deg_H(b)$ and for each edge $bv \in E(H)$, $deg_H(v) - deg_H(b) \geqslant deg_G(a)$.*

Proof. This follows immediately from the definition of splice of two graphs. $\square$

In a similar way, by definition of link of two graphs, we have:

Lemma 2.2. *Suppose G and H are rooted graphs with respect to the rooted vertices of a and b, respectively. Then*

$$\begin{aligned} irr(G) + irr(H) + \Big||V(G)| - |V(H)|\Big| - deg_G(a) - deg_H(b) &\leqslant irr((G \sim H)(a;b)) \\ &\leqslant irr(G) + irr(H) + \Big||V(G)| - |V(H)|\Big| + deg_G(a) + deg_H(b). \end{aligned}$$

and the upper bound is attained if and only if for every vertex $u \in V(G)$ that $ua \in E(G)$, $deg_G(a) \geqslant deg_G(u)$ and for every vertex $v \in V(H)$ that $vb \in E(H)$, $deg_H(b) \geqslant deg_H(v)$. Moreover, the lower bound is attained if and only if for every vertex $u \in V(G)$ that $ua \in E(G)$, $deg_G(a) < deg_G(u)$ and for every vertex $v \in V(H)$ that $vb \in E(H)$, $deg_H(b) < deg_H(v)$.

Lemma 2.3. *Let G be a connected graph. Then $irr(G) = 0$ if and only if G is regular.*

The line graph $L(G)$ of a graph G is defined as follows: each vertex of $L(G)$ represents an edge of G, and any two vertices of $L(G)$ are adjacent if and only if their corresponding edges share a common endpoint in G [21].

Lemma 2.4. *Let G be a connected graph. Then $irr(L(S(G))) = irr(G)$.*

Proof. We assume that uv is an edge of $L(G)$ that u and v are vertices corresponding to edges wx and wy of G, respectively. Then $|deg_{L(G)}(u) - deg_{L(G)}(v)| = |deg_G(x) - deg_G(y)|$. To compute the third Zagreb index of $L(G)$, it is enough to calculate the summation of all degree differences of vertices of distance 2 in $L(G)$. Therefore, $irr(L(S(G))) = irr(G)$. $\square$

In the next result, the relationship between strong and tensor products of two graphs under the third Zagreb index is investigated.

Lemma 2.5. *Let G and H be graphs. Then*

$$irr(G \boxtimes H) - irr(G \otimes H) \leqslant (|V(G)| + 4|E(G)|)irr(H) + (|V(H)| + 4|E(H)|)irr(G).$$

Proof. The summation of $|deg_{G\boxtimes H}(u) - deg_{G\boxtimes H}(v)|$ over all edges $(a,x)(b,y)$ such that ($a = b$ and $xy \in E(H)$) or ($x = y$ and $ab \in E(G)$), is equal to:

$$(|V(G)| + 2|E(G)|)irr(H) + (|V(H)| + 2|E(H)|)irr(G).$$

On the other hand, for each edge $(a,x)(b,y)$ of $G \boxtimes H$ that $ab \in E(G)$ and $xy \in E(H)$, we have:

$$\begin{aligned} &|deg_{G\boxtimes H}((a,x)) - deg_{G\boxtimes H}((b,y))| \\ &= |deg_G(a) + deg_H(x) + deg_G(a)deg_H(x) - deg_G(b) - deg_H(y) - deg_G(b)deg_H(y)| \end{aligned}$$

$$\leqslant |deg_G(a) - deg_G(b)| + |deg_H(x) - deg_H(y)| + |deg_G(a)deg_H(x) - deg_G(b)deg_H(y)|$$
$$\leqslant |deg_G(a) - deg_G(b)| + |deg_H(x) - deg_H(y)| + |deg_{G\otimes H}((a,x)) - deg_{G\otimes H}((b,y))|,$$

and hence

$$\sum_{(a,x)(b,y)\ s.t\ ab\in E(G), xy\in E(H)} |deg_{G\boxtimes H}((a,x)) - deg_{G\boxtimes H}((b,y))|$$
$$\leqslant 2|E(H)|irr(G) + 2|E(G)|irr(H) + irr(G\otimes H),$$

which completes the proof. □

Lemma 2.6. *Let G and H be two connected graphs. Then*

$$irr(GoH) = irr(G) + |V(G)|irr(H) + |V(G)||V(H)|(|V(H)| - 1) + 2|V(H)||E(G)| - 2|V(G)||E(H)|.$$

Proof. Let H_i be the i-th copy of H, $1 \leq i \leq |V(G)|$, and let G' be the copy of G in GoH. We partition edges of GoH into the following three subsets:

$$A = \{uv \in E(GoH) \mid u, v \in V(H_i), i = 1, 2, ..., |V(G)|\},$$
$$B = \{uv \in E(GoH) \mid u, v \in V(G')\},$$
$$C = \{uv \in E(GoH) \mid u \in V(G'),\ v \in V(H_i), i = 1, 2, ..., |V(G)|\}.$$

If $u'v' \in A$, then $deg_{GoH}(u') = deg_H(u) + 1$ and $deg_{GoH}(v') = deg_H(v) + 1$ that $u', v' \in V(H_i)$ are corresponding to $u, v \in V(H)$, respectively. Thus $|deg_{GoH}(u') - deg_{GoH}(v')| = |deg_H(u) - deg_H(v)|$. Since $|V(G)|$ edges $u'v'$ in $E(GoH)$ are corresponding to each edge $uv \in E(H)$, $irr_1 = \sum_{uv\in A} |deg_{GoH}(u) - deg_{GoH}(v)| = |V(G)|irr(H)$.

It is clear that for each $u'v' \in B$, $deg_{GoH}(u') = deg_G(u) + |V(H)|$ and $deg_{GoH}(v') = deg_G(v) + |V(H)|$, where $u', v' \in V(G')$ are corresponding to $u, v \in V(G)$, respectively. Hence $irr_2 = \sum_{uv\in B} |deg_{GoH}(u) - deg_{GoH}(v)| = irr(G)$.

Finally, if $u'v' \in C$, then $deg_{GoH}(u') = deg_G(u) + |V(H)|$ and $deg_{GoH}(v') = deg_H(v) + 1$, where $u' \in V(G'), v' \in V(H_i)$ are corresponding to $u \in V(G), v \in V(H)$, respectively. Hence $|deg_{GoH}(u') - deg_{GoH}(v')| = deg_G(u) + |V(H)| - deg_H(v) - 1$. Consequently,

$$irr_3 = \sum_{uv\in C} |deg_{GoH}(u) - deg_{GoH}(v)| = |V(G)||V(H)|(|V(H)| - 1) + 2|V(H)||E(G)| - 2|V(G)||E(H)|.$$

By summation of irr_1, irr_2 and irr_3, the result can be proved. □

As in the proof of Lemma 2.6, $|deg_{GoH}(u') - deg_{GoH}(v')| = deg_G(u) + |V(H)| - deg_H(v) - 1$, where for each edge $u'v' \in E(GoH)$, vertices $u' \in V(G')$ and $v' \in V(H_i)$ are corresponding to $u \in V(G)$ and $v \in V(H)$, respectively. Thus,

$$2|V(H)|(|V(G)| - 1) \leqslant \sum_{uv\in E(GoH} |deg_{GoH}(u) - deg_{GoH}(v)|$$

$$\leqslant |V(G)|(|V(H)|^2 + |V(G)||V(H)| - 4|V(H)| + 2).$$

Corollary 2.7. *Let G and H be two connected graphs. Then*

$$\begin{aligned}&irr(G) + |V(G)|irr(H) + 2|V(H)|(|V(G)| - 1) \leqslant irr(GoH)\\ &\qquad \leqslant irr(G) + |V(G)|irr(H) + |V(G)|(|V(H)|^2 + |V(G)||V(H)| - 4|V(H)| + 2)\end{aligned}$$

and the upper bound is attained if and only if H is a tree and $G \cong K_n$. Moreover, the lower bound is attained if and only if G is a tree and $H \cong K_n$.

Let $G_i = (V_i, E_i)$ be N graphs with each vertex set V_i, $1 \leq i \leq N$, having a distinguished or root vertex, labeled 0. The hierarchical product $H = G_N \sqcap ... \sqcap G_2 \sqcap G_1$ is the graph with vertices the $N-$tuples $x_N...x_3x_2x_1$, $x_i \in V_i$, and edges defined by the adjacencies:

$$x_N...x_3x_2x_1 \sim \begin{cases} x_N...x_3x_2y_1 & \text{if } y_1 \sim x_1 \text{ in } G_1,\\ x_N...x_3y_2x_1 & \text{if } y_2 \sim x_2 \text{ in } G_2 \text{ and } x_1 = 0,\\ x_N...y_3x_2x_1 & \text{if } y_3 \sim x_3 \text{ in } G_3 \text{ and } x_1 = x_2 = 0,\\ \vdots & \vdots \\ y_N...x_3x_2x_1 & \text{if } y_N \sim x_N \text{ in } G_N \text{ and } x_1 = x_2 = ... = x_N = 0.\end{cases}$$

We encourage the reader to consult [5, 6] for the mathematical properties of this new graph operation.

Lemma 2.8. *Let G and H be connected rooted graphs and r is the root vertex of H. Then*

$$\begin{aligned}irr(G) + |V(G)|irr(H) - 2deg_H(r)|E(G)| &\leqslant irr(G \sqcap H)\\ &\leqslant irr(G) + |V(G)|irr(H) + 2deg_H(r)|E(G)|.\end{aligned}$$

The upper bound is attained if and only if for each $ur \in E(H)$, $deg_H(r) \geqslant deg_H(u)$. Moreover, the lower bound is attained if and only if for each $ur \in E(H)$ and $v \in V(G)$, $deg_H(r) + deg_G(v) \leqslant deg_H(u)$.

Proof. Let $ur \in E(H)$ and $v \in V(G)$, then $(v, r)(v, u) \in E(G \sqcap H)$ and so

$$|deg_{G\sqcap H}((v, r)) - deg_{G\sqcap H}((v, u))| = |deg_H(r) + deg_G(v) - deg_H(u)|.$$

Consequently, if $deg_H(r) \geqslant deg_H(u)$ then $|deg_{G\sqcap H}((v, r)) - deg_{G\sqcap H}((v, u))| = (deg_H(r) - deg_H(u)) + deg_G(v)$ and if $deg_H(r) + deg_G(v) \leqslant deg_H(u)$ then $|deg_{G\sqcap H}((v, r)) - deg_{G\sqcap H}((v, u))| = (deg_H(u) - deg_H(r)) - deg_G(v)$. On the other hand for each edge $(u, r)(v, r)$ of $G\sqcap H$ that $uv \in E(G)$, we have $|deg_{G\sqcap H}((u, r)) - deg_{G\sqcap H}((v, r))| = |deg_G(u) - deg_G(v)|$ and for each edge $(w, u)(w, v)$ of $G \sqcap H$ that $u \neq v \neq r$, we have $|deg_{G\sqcap H}((w, u)) - deg_{G\sqcap H}((w, v))| = |deg_H(u) - deg_H(v)|$, which proves the result. □

In what follows, let $\prod_i^j f_i = 1$ for each $i, j \in \{0, 1, 2, ...\}$, that $i - j = 1$. Let $G_1, G_2, ..., G_n$ be connected rooted graphs with root vertices $r_1, r_2, ..., r_n$, respectively. We set $|V_{i,j}| = \prod_{k=i}^j |V(G_k)|$. Also, if $G = G_n \sqcap ... \sqcap G_2 \sqcap G_1$ then we will use $deg_G(r)$ to denote $deg_{G_1}(r_1) + deg_{G_2}(r_2) + ... + deg_{G_n}(r_n)$.

Corollary 2.9. *Let $G_1, G_2, ..., G_n$ be connected rooted graphs with root vertices $r_1, r_2, ..., r_n$, respectively. Then*

$$\sum_{i=1}^{n}|V_{i+1,n}|irr(G_i) - 2\sum_{i=1}^{n-1}|V_{i+2,n}||E(G_{i+1})|\sum_{k=1}^{i}deg_{G_k}(r_k) \leqslant irr(G_n \sqcap ... \sqcap G_2 \sqcap G_1)$$
$$\leqslant \sum_{i=1}^{n}|V_{i+1,n}|irr(G_i) + 2\sum_{i=1}^{n-1}|V_{i+2,n}||E(G_{i+1})|\sum_{k=1}^{i}deg_{G_k}(r_k).$$

The upper bound is attained if and only if for each $ur_i \in E(G_i)$, $deg_{G_i}(r_i) \geqslant deg_{G_i}(u)$, $i = 1, 2, ..., n-1$. Moreover, the lower bound is attained if and only if for each $ur_i \in E(G_i)$ and $v \in V(G_j)$, $\sum_{k=i}^{j-1} deg_{G_i}(r_i) + deg_{G_j}(v) \leqslant deg_{G_i}(u)$, $i = 1, 2, ..., n-1$, $j = i+1, i+2, ..., n$.

Proof. Use induction on n. By Lemma 2.8, the result is valid for $n = 2$. Let $n \geqslant 3$ and assume the result holds for n. Set $G = G_n \sqcap ... \sqcap G_2 \sqcap G_1$. Thus $G_{n+1} \sqcap ... \sqcap G_2 \sqcap G_1 = G_{n+1} \sqcap G$. Then by our assumption,

$$irr(G_{n+1}) + |V(G_{n+1})|irr(G) - 2deg_G(r)|E(G_{n+1})| \leqslant irr(G_{n+1} \sqcap G)$$
$$\leqslant irr(G_{n+1}) + |V(G_{n+1})|irr(G) + 2deg_G(r)|E(G_{n+1})|.$$

On the other hand, again by our assumption,

$$\sum_{i=1}^{n}|V_{i+1,n}|irr(G_i) - 2\sum_{i=1}^{n-1}|V_{i+2,n}||E(G_{i+1})|\sum_{j=1}^{i}deg_{G_k}(r_k) \leqslant irr(G)$$
$$\leqslant \sum_{i=1}^{n}|V_{i+1,n}|irr(G_i) + 2\sum_{i=1}^{n-1}|V_{i+2,n}||E(G_{i+1})|\sum_{j=1}^{i}deg_{G_k}(r_k),$$

which completes our argument. □

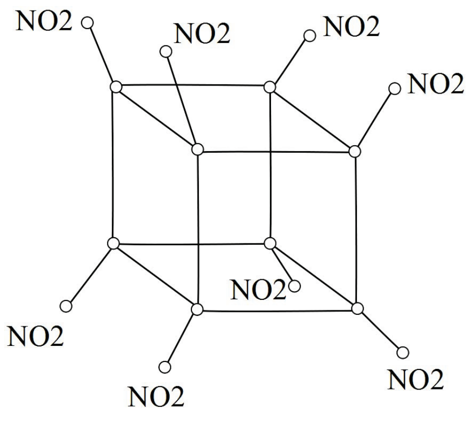

FIGURE 5. The Molecular Graph of Octanitrocubane.

Example 2.10. Octanitrocubane is the most powerful chemical explosive with formula $C_8(NO_2)_8)$, Figure 5. Let Γ be the graph of this molecule. Then obviously $\Gamma = Q_3 \sqcap P_2$. If r is the root vertex of P_2, one can easily see that $irr(Q_3) = irr(P_2) = 0$, $|E(Q_3)| = 12$, $deg_{P_2}(r) = 1$ and for each $ur \in E(P_2)$, $deg_{P_2}(r) = deg_{P_2}(u)$ and so, by Lemma 2.8, we have

$$irr(\Gamma) = irr(Q_3 \sqcap P_2) = irr(Q_3) + |V(Q_3)|irr(P_2) + 2deg_{P_2}(r)|E(Q_3)| = 24.$$

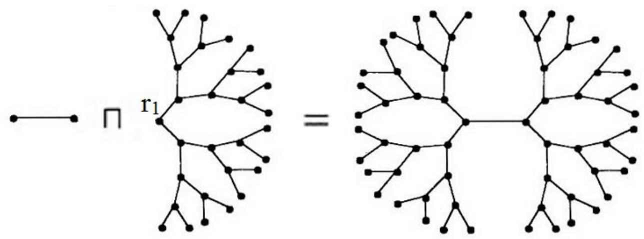

FIGURE 6. Regular Dicentric ($DD_{2.4}$) Dendrimer.

Example 2.11. Dendrimers are branched molecules have a high degree of molecular uniformity. The molecular graph of this molecules is constructed from a core and some branches connecting to the core. Let $DD_{p,r}$ be the graph of the regular dicentric dendrimer, see [7] for more information. Then $DD_{p,r} = P_2 \sqcap H$, where H is a tree of progressive degree p and generation r, Figure 6. One can see that $irr(P_2) = 0$, $irr(H) = p^{r+1}+p$, and for each $ur \in E(H)$ and $v \in V(P_2)$, $deg_H(r) + deg_{P_2}(v) \leqslant deg_H(u)$. Therefore, by Lemma 2.8, we have:

$$irr(DD_{p,r}) = irr(P_2 \sqcap H) = irr(P_2) + |V(P_2)|irr(H) - 2deg_H(r)|E(P_2)| = 2p^{r+1}.$$

Example 2.12. Consider the graph S_4 whose root vertex is 0. If $deg_{S_4}(0) = 3$ then by Lemma 2.8 we have:

$$irr(P_n \sqcap S_4) = irr(P_n) + |V(P_n)|irr(S_4) + 2deg_{S_4}(0)|E(P_n)| = 12n - 4, \quad (n > 2);$$

and if $deg_{S_4}(0) = 1$ then again by Lemma 2.8 we have:

$$irr(P_n \sqcap S_4) = irr(P_n) + |V(P_n)|irr(S_4) - 2deg_{S_4}(0)|E(P_n)| = 4n + 4, \quad (n > 2).$$

Lemma 2.13. *Let G and H be connected graphs. Then*

(1) *If G is regular, then we have $irr(G[H]) = |E(G)|irr(2H) + (|V(G)| - 2|E(G)|)irr(H)$.*
(2) *If H is regular, then $irr(G[H]) = |V(H)|^3 irr(G)$.*

Proof. **1).** Let G be regular. Clearly, for each vertex $(u, v) \in V(G[H])$, $deg_{G[H]}((u, v)) = |V(H)|deg_G(u) + deg_H(v)$. Since G is regular, for each edge $(u, x)(v, y) \in G[H]$ that $uv \in E(G)$, we have $|deg_{G[H]}((u, x)) - deg_{G[H]}((v, y))| = |deg_H(x) - deg_H(y)|$ and for every edge $(u, x)(u, y)$ that $xy \in E(H)$, we have $|deg_{G[H]}((u, x)) - deg_{G[H]}((u, y))| = |deg_H(x) - deg_H(y)|$, which proves the result.

2). Let H be regular. Then for every edge $(u,x)(u,y)$ that $xy \in E(H)$, we have $|deg_{G[H]}((u,x)) - deg_{G[H]}((u,y))| = 0$ and for every edge $(u,x)(v,y)$ of $G[H]$ that $uv \in E(G)$, we have $|deg_{G[H]}((u,x)) - deg_{G[H]}((v,y))| = |V(H)||deg_G(u) - deg_G(v)|$. On the other hand, to each edge $uv \in E(G)$, there correspond $|V(H)|^2$ edges $(u,x)(v,y)$ in $E(GoH)$, so $irr(G[H]) = |V(H)|^3 irr(G)$. $\square$

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