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SOME NEW INEQUALITIES OF PERTURBED MIDPOINT RULE

FEIXIANG CHEN

ABSTRACT. In this paper, a generalized perturbed midpoint rule is established. Various error bounds for this generalization are also obtained.

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1. Introduction

In recent years a number of authors have considered error inequalities for some known and some new quadrature rules. Sometimes they have considered generalizations of these rules. For example, the well-known trapezoid and midpoint quadrature rules are considered in ([1, 2, 9, 10]). In [2], we can find

$$\int_a^b f(x)dx = (b-a)f(\frac{a+b}{2}) + \sum_{k=1}^{[\frac{n-1}{2}]} \frac{(b-a)^{2k+1}}{(2k+1)!2^{2k}} f^{(2k)}(\frac{a+b}{2}) + (-1)^n \int_a^b M_n(x) f^{(n)}(x)dx,$$

where $[\frac{n-1}{2}]$ denotes the integer part of $\frac{n-1}{2}$ and

$$M_n(x) := \begin{cases} \frac{(x-a)^n}{n!}, x \in [a, \frac{a+b}{2}], \\ \frac{(x-b)^n}{n!}, x \in (\frac{a+b}{2}, b]. \end{cases}$$

For $n = 1$, we get the midpoint rule

$$\int_a^b f(x)dx = (b-a)f(\frac{a+b}{2}) - \int_a^b M_1(x)f'(x)dx.$$

In [11], a generalized trapezoid rule is derived by Ujević as follows:

$$\int_a^b f(x)dx = \frac{b-a}{2}[f(a)+f(b)] - \sum_{k=1}^{[\frac{n-1}{2}]} \frac{k(b-a)^{2k+1}}{(2k+1)!2^{2k-1}} f^{(2k)}(\frac{a+b}{2}) + (-1)^n \int_a^b T_n(x) f^{(n)}(x)dx,$$

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where

$$T_n(x) := \begin{cases} \frac{(x-a)^n}{n!} - \frac{(b-a)(x-a)^{n-1}}{2(n-1)!}, & x \in [a, \frac{a+b}{2}], \\ \frac{(x-b)^n}{n!} + \frac{(b-a)(x-b)^{n-1}}{2(n-1)!}, & x \in (\frac{a+b}{2}, b]. \end{cases}$$

In [4] and [8], the following unified treatment for generalizations of the midpoint, trapezoid, averaged midpoint-trapezoid and Simpson type inequalities is obtained by Liu and Liu, respectively,

$$\int_a^b f(x)dx = \frac{b-a}{2}\Big[\theta f(a) + 2(1-\theta)f(\frac{a+b}{2}) + \theta f(b)\Big]$$
$$+ \sum_{k=1}^{[\frac{n-1}{2}]} \frac{[1-\theta(2k+1)](b-a)^{2k+1}}{(2k+1)!2^{2k}} f^{(2k)}(\frac{a+b}{2})$$
$$+(-1)^n \int_a^b K_n(x,\theta) f^{(n)}(x)dx,$$

where $\theta \in [0,1]$ and

$$K_n(x,\theta) := \begin{cases} \frac{(x-a)^n}{n!} - \frac{\theta(b-a)(x-a)^{n-1}}{2(n-1)!}, & x \in [a, \frac{a+b}{2}], \\ \frac{(x-b)^n}{n!} + \frac{\theta(b-a)(x-b)^{n-1}}{2(n-1)!}, & x \in (\frac{a+b}{2}, b]. \end{cases}$$

In [5], Liu established the following generalized perturbed trapezoid rule.

$$\int_a^b f(x)dx - \frac{b-a}{2}[f(a)+f(b)] + \frac{(b-a)^2}{12}[f'(b) - f'(a)]$$
$$- \sum_{k=1}^{[\frac{n-1}{2}]} \frac{k(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-2}} f^{(2k)}(\frac{a+b}{2}) \tag{1}$$
$$= (-1)^n \int_a^b K_n(x) f^{(n)}(x)dx,$$

where $K_n(x)$ is the kernel given by

$$K_n(x) := \begin{cases} \frac{(x-a)^n}{n!} - \frac{(b-a)(x-a)^{n-1}}{2(n-1)!} + \frac{(b-a)^2(x-a)^{n-2}}{12(n-2)!}, & x \in [a, \frac{a+b}{2}], \\ \frac{(x-b)^n}{n!} + \frac{(b-a)(x-b)^{n-1}}{2(n-1)!} + \frac{(b-a)^2(x-b)^{n-2}}{12(n-2)!}, & x \in (\frac{a+b}{2}, b]. \end{cases}$$

In [7] and [12], the following generalization of the perturbed midpoint-trapezoid rule is established by Liu and Ujević et al., respectively.

Theorem 1.1. *Let $f : [a,b] \to \mathbb{R}$ be a function such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$. Then we have*

$$\int_a^b f(x)dx = (b-a)\frac{f(a) + 2f(\frac{a+b}{2}) + f(b)}{4} - \frac{(b-a)^2}{48}[f'(b) - f'(a)]$$

$$+\frac{1}{3}\sum_{k=2}^{[\frac{n-1}{2}]}\frac{(2k-2)(2k-3)(b-a)^{2k+1}}{(2k+1)!2^{2k+2}}f^{(2k)}(\frac{a+b}{2})+R(f),$$

where $[\frac{n-1}{2}]$ *denotes the integer part of* $\frac{n-1}{2}$ *and* $R(f)=(-1)^n\int_a^b Q_n(x)f^{(n)}(x)dx$,

$$Q_n(x):=\begin{cases}\frac{(x-a)^n}{n!}-\frac{(b-a)(x-a)^{n-1}}{4(n-1)!}+\frac{(b-a)^2(x-a)^{n-2}}{48(n-2)!}, & x\in[a,\frac{a+b}{2}],\\ \frac{(x-b)^n}{n!}+\frac{(b-a)(x-b)^{n-1}}{4(n-1)!}+\frac{(b-a)^2(x-b)^{n-2}}{48(n-2)!}, & x\in(\frac{a+b}{2},b].\end{cases}$$

Some sharp perturbed midpoint inequalities are proved by Liu in [6] based on the following identity:

$$\int_a^b f(x)dx-(b-a)f(\frac{a+b}{2})-\frac{(b-a)^2}{24}[f'(b)-f'(a)]=\int_a^b K_n(x)f^{(n)}(x)dx, \quad (2)$$

where

$$K_2(x):=\begin{cases}\frac{(x-a)^2}{2}-\frac{(b-a)^2}{24}, & x\in[a,\frac{a+b}{2}],\\ \frac{(x-b)^2}{2}-\frac{(b-a)^2}{24}, & x\in(\frac{a+b}{2},b]\end{cases}$$

and

$$K_3(x):=\begin{cases}\frac{1}{6}(x-a)(\frac{a+b}{2}-x)(x-\frac{3a-b}{2}), & x\in[a,\frac{a+b}{2}],\\ \frac{1}{6}(b-x)(x-\frac{a+b}{2})(x-\frac{3b-a}{2}), & x\in(\frac{a+b}{2},b].\end{cases}$$

Theorem 1.2 ([6])**.** *Let* $f:[a,b]\to\mathbb{R}$ *be a twice differentiable mapping such that* f'' *is integrable with* $\Gamma_2=\sup_{x\in(a,b)}f''(x)$ *and* $\gamma_2=\inf_{x\in(a,b)}f''(x)$*. Then we have*

$$\Big|\int_a^b f(x)dx-(b-a)f(\frac{a+b}{2})-\frac{(b-a)^2}{24}[f'(b)-f'(a)]\Big|\le\frac{\Gamma_2-\gamma_2}{36\sqrt{3}}(b-a)^3, \quad (3)$$

$$\Big|\int_a^b f(x)dx-(b-a)f(\frac{a+b}{2})-\frac{(b-a)^2}{24}[f'(b)-f'(a)]\Big|\le\frac{1}{12}(b-a)^2[f'(b)-f'(a)-\gamma_2(b-a)], \quad (4)$$

$$\Big|\int_a^b f(x)dx-(b-a)f(\frac{a+b}{2})-\frac{(b-a)^2}{24}[f'(b)-f'(a)]\Big|\le\frac{1}{12}(b-a)^2[\Gamma_2(b-a)-f'(b)+f'(a)]. \quad (5)$$

Theorem 1.3 ([6])**.** *Let* $f:[a,b]\to\mathbb{R}$ *be a third-order differentiable mapping such that* f''' *is integrable with* $\Gamma_3=\sup_{x\in(a,b)}f'''(x)$ *and* $\gamma_3=\inf_{x\in(a,b)}f'''(x)$*. Then we have*

$$\Big|\int_a^b f(x)dx-(b-a)f(\frac{a+b}{2})-\frac{(b-a)^2}{24}[f'(b)-f'(a)]\Big|\le\frac{\Gamma_3-\gamma_3}{384}(b-a)^4, \quad (6)$$

$$\Big|\int_a^b f(x)dx-(b-a)f(\frac{a+b}{2})-\frac{(b-a)^2}{24}[f'(b)-f'(a)]\Big|\le\frac{\sqrt{3}}{216}(b-a)^3[f''(b)-f''(a)-\gamma_3(b-a)], \quad (7)$$

$$\Big|\int_a^b f(x)dx-(b-a)f(\frac{a+b}{2})-\frac{(b-a)^2}{24}[f'(b)-f'(a)]\Big|\le\frac{\sqrt{3}}{216}(b-a)^3[\Gamma_3(b-a)-f''(b)+f''(a)]. \quad (8)$$

The purpose of this paper is to extend (2) to a more general version, that is, a generalized perturbed midpoint rule is established. Various error bounds for the generalizations are also given.

2. For differentiable mappings with bounded derivatives

Theorem 2.1. *Let $f : [a,b] \to \mathbb{R}$ be a mapping such that the derivative $f^{(n-1)}$ $(n \geq 2)$ is absolutely continuous on $[a,b]$ and $M_n = \sup_{x\in(a,b)} |f^{(n)}(x)| < \infty$. Then we have*

$$\Bigg| \int_a^b f(x)dx - (b-a)f(\frac{a+b}{2}) - \frac{(b-a)^2}{24}[f'(b) - f'(a)] + \sum_{k=1}^{[\frac{n-1}{2}]} \frac{(k-1)(2k+3)(b-a)^{2k+1}}{3(2k+1)!2^{2k}} f^{(2k)}(\frac{a+b}{2}) \Bigg| \leq M_n \times \begin{cases} \frac{\sqrt{3}(b-a)^3}{54}, & n = 2, \\ \frac{(n+3)(n-2)(b-a)^{n+1}}{3(n+1)!2^{n+1}}, & n \geq 3, \end{cases} \tag{9}$$

where $[\frac{n-1}{2}]$ denotes the integer part of $\frac{n-1}{2}$.

Proof. It is not difficult to find the identity

$$\int_a^b f(x)dx - (b-a)f(\frac{a+b}{2}) - \frac{(b-a)^2}{24}[f'(b) - f'(a)] + \sum_{k=1}^{[\frac{n-1}{2}]} \frac{(k-1)(2k+3)(b-a)^{2k+1}}{3(2k+1)!2^{2k}} f^{(2k)}(\frac{a+b}{2}) = (-1)^n \int_a^b S_n(x) f^{(n)}(x)dx, \tag{10}$$

where $S_n(x)$ is the kernel given by

$$S_n(x) := \begin{cases} \frac{(x-a)^n}{n!} - \frac{(b-a)^2(x-a)^{n-2}}{24(n-2)!}, & x \in [a, \frac{a+b}{2}], \\ \frac{(x-b)^n}{n!} - \frac{(b-a)^2(x-b)^{n-2}}{24(n-2)!}, & x \in (\frac{a+b}{2}, b]. \end{cases}$$

Using the above identity, we get

$$\Bigg| \int_a^b f(x)dx - (b-a)f(\frac{a+b}{2}) - \frac{(b-a)^2}{24}[f'(b) - f'(a)] + \sum_{k=1}^{[\frac{n-1}{2}]} \frac{(k-1)(2k+3)(b-a)^{2k+1}}{3(2k+1)!2^{2k}} f^{(2k)}(\frac{a+b}{2}) \Bigg| = \int_a^b |S_n(x) f^{(n)}(x)|dx \leq M_n \int_a^b |S_n(x)|dx. \tag{11}$$

Now, we put

$$P_n(x) = \frac{(x-a)^n}{n!} - \frac{(b-a)^2(x-a)^{n-2}}{24(n-2)!}, \qquad x \in [a, \frac{a+b}{2}],$$

$$Q_n(x) = \frac{(x-b)^n}{n!} - \frac{(b-a)^2(x-b)^{n-2}}{24(n-2)!}, \qquad x \in (\frac{a+b}{2}, b].$$

It is clear that $P_n(x)$ and $Q_n(x)$ are symmetric with respect to the line $x = \frac{a+b}{2}$ for n even and symmetric with respect to the point $(\frac{a+b}{2}, 0)$ for n odd. Therefore,

$$\int_a^b |S_n(x)|dx = 2\int_a^{\frac{a+b}{2}} |P_n(x)|dx = \frac{(b-a)^{n+1}}{2^n n!}\int_0^1 t^{n-2}\Big|t^2 - \frac{n(n-1)}{6}\Big|dt.$$

By substitution $x = a + \frac{b-a}{2}t$, we find that $r_n(t) := t^2 - \frac{n(n-1)}{6}$ is always negative on $[0,1]$ for $n \geq 3$. Thus

$$\begin{aligned}\int_a^b |S_n(x)|dx &= 2\int_a^{\frac{a+b}{2}} |P_n(x)|dx \\ &= \frac{(b-a)^{n+1}}{2^n n!}\int_0^1 t^{n-2}\Big|t^2 - \frac{n(n-1)}{6}\Big|dt \\ &= \frac{(n+3)(n-2)(b-a)^{n+1}}{3(n+1)!2^{n+1}}\end{aligned}$$

for $n \geq 3$, and

$$\begin{aligned}\int_a^b |S_2(x)|dx &= 2\int_a^{\frac{a+b}{2}} |P_2(x)|dx \\ &= \frac{(b-a)^3}{8}\int_0^1 \Big|t^2 - \frac{1}{3}\Big|dt \\ &= \frac{\sqrt{3}(b-a)^3}{54}.\end{aligned}$$

Hence,

$$\int_a^b |S_n(x)|dx = \begin{cases} \frac{\sqrt{3}(b-a)^3}{54}, & n = 2, \\ \frac{(n+3)(n-2)(b-a)^{n+1}}{3(n+1)!2^{n+1}}, & n \geq 3. \end{cases} \tag{12}$$

Consequently, inequalities (9) follow from (11) and (12). □

Remark. Applying (10) for $n = 2, 3$ respectively, we get the identity (2).

For convenience in further discussions, we collect some technical results which are not difficult to obtain by elementary calculus as:

$$\int_a^b S_n(x)dx = -\frac{[1+(-1)^n](b-a)^{n+1}(n-2)(n+3)}{3(n+1)!2^{n+2}} = \begin{cases} 0, & n \ \ odd, \\ -\frac{(b-a)^{n+1}(n-2)(n+3)}{3(n+1)!2^{n+1}}, & n \ even, \end{cases} \tag{13}$$

$$\int_a^b S_n^2(x)dx = \frac{(4n^6 - 8n^5 - 45n^4 + 98n^3 + 131n^2 - 324n + 108)(b-a)^{2n+1}}{36(n!)^2(2n-3)(4n^2-1)2^{2n}}, \tag{14}$$

$$\sup_{x\in[a,b]} |S_n(x)| = \begin{cases} \frac{(b-a)^2}{12}, & n = 2, \\ \frac{\sqrt{3}(b-a)^3}{216}, & n = 3, \\ \frac{(b-a)^n(n+2)(n-3)}{3n!2^{n+1}}, & n \geq 4. \end{cases} \tag{15}$$

Before we end this section, we introduce the notations

$$I = \int_a^b f(x)dx,$$

$$F_n = (b-a)f(\frac{a+b}{2}) + \frac{(b-a)^2}{24}[f'(b)-f'(a)] - \sum_{k=1}^{[\frac{n-1}{2}]} \frac{(k-1)(2k+3)(b-a)^{2k+1}}{3(2k+1)!2^{2k}} f^{(2k)}(\frac{a+b}{2}).$$

3. For functions whose $(n-1)$th derivatives are Lipschitzian type

Recall that a function $f : [a, b] \to R$ is said to be L-Lipschitzian on $[a, b]$ if

$$|f(x) - f(y)| \leq L|x - y|$$

for all $x, y \in [a, b]$,where $L > 0$ is given, and, it is said to be (l, L) -Lipschitzian on $[a, b]$ if

$$l(x - y) \leq f(x) - f(y) \leq L(x - y)$$

for all $a \leq x \leq y \leq b$ where $l, L \in R$ with $l < L$.

From [3], we get that if $h, g : [a, b] \to \mathbb{R}$ are such that h is Riemann-integral on $[a, b]$ and g is L-Lipschitzian on $[a, b]$, then $\int_a^b h(t)dg(t)$ exists and

$$\Big|\int_a^b h(t)dg(t)\Big| \leq L\int_a^b |h(t)|dt. \tag{16}$$

Theorem 3.1. *Let $f : [a, b] \to \mathbb{R}$ be a mapping such that derivative $f^{(n-1)}$ $(n \geq 2)$ is (l, L)-Lipschitzian on $[a, b]$. Then we have*

$$\begin{aligned} &\Big|I - F_n + \frac{(-1)^n(L+l)[1+(-1)^n](b-a)^{n+1}(n-2)(n+3)}{3(n+1)!2^{n+3}}\Big| \\ &\leq \frac{L-l}{2} \times \begin{cases} \frac{\sqrt{3}(b-a)^3}{54}, & n = 2, \\ \frac{(n+3)(n-2)(b-a)^{n+1}}{3(n+1)!2^{n+1}}, & n \geq 3. \end{cases} \end{aligned} \tag{17}$$

Proof. By (10) and (13) we get

$$\begin{aligned} &I - F_n + \frac{(-1)^n(L+l)[1+(-1)^n](b-a)^{n+1}(n-2)(n+3)}{3(n+1)!2^{n+3}} \\ &= (-1)^n \int_a^b S_n(x)d\Big[f^{(n-1)}(x) - \frac{L+l}{2}x\Big]. \end{aligned}$$

Then notice that $f^{(n-1)}(x) - \frac{L+l}{2}x$ is $\frac{L-l}{2}$-Lipschitzian on $[a, b]$ and by using (16), we have

$$\begin{aligned} &\Big|I - F_n + \frac{(-1)^n(L+l)[1+(-1)^n](b-a)^{n+1}(n-2)(n+3)}{3(n+1)!2^{n+3}}\Big| \\ &\leq \frac{L-l}{2} \times \int_a^b |S_n(x)|dx. \end{aligned}$$

Hence, the inequality (17) follows from (16) and (12). □

Corollary 3.2. *Let $f : [a,b] \to \mathbb{R}$ be a mapping such that derivative $f^{(n-1)}$ $(n \geq 2)$ is L-Lipschitzian on $[a,b]$. Then we have*

$$\Big|I - F_n\Big| \leq L \times \begin{cases} \frac{\sqrt{3}(b-a)^3}{54}, & n = 2, \\ \frac{(n+3)(n-2)(b-a)^{n+1}}{3(n+1)!2^{n+1}}, & n \geq 3. \end{cases}$$

4. Bounds in terms of some Lebesgue norms

Theorem 4.1. *Let $f : [a,b] \to \mathbb{R}$ be a mapping such that the $(n-1)$th derivative $f^{(n-1)}$ $(n \geq 2)$ is absolutely continuous on $[a,b]$. If $f^{(n)} \in L_\infty[a,b]$, then we have*

$$\Big|I - F_n\Big| \leq \|f^{(n)}\|_\infty \times \begin{cases} \frac{\sqrt{3}(b-a)^3}{54}, & n = 2, \\ \frac{(n+3)(n-2)(b-a)^{n+1}}{3(n+1)!2^{n+1}}, & n \geq 3. \end{cases}$$

where $\|f^{(n)}\|_\infty := ess\sup_{x\in[a,b]} |f^{(n)}(x)|$ is the usual Lebesgue norm on $L_\infty[a,b]$.

Proof. We can obtain the result by taking $L = \|f^{(n)}\|_\infty$ in Corollary 3.2. □

Theorem 4.2. *Let $f : [a,b] \to \mathbb{R}$ be a mapping such that the $(n-1)$th derivative $f^{(n-1)}$ $(n \geq 2)$ is absolutely continuous on $[a,b]$. If $f^{(n)} \in L_1[a,b]$, then we have*

$$\Big|I - F_n\Big| \leq \|f^{(n)}\|_1 \times \begin{cases} \frac{(b-a)^2}{12}, & n = 2, \\ \frac{\sqrt{3}(b-a)^3}{216}, & n = 3, \\ \frac{(b-a)^n(n-3)(n+2)}{3n!2^{n+1}}, & n \geq 4. \end{cases}$$

where $\|f^{(n)}\|_1 := \int_a^b |f^{(n)}(x)|dx$ is the usual Lebesgue norm on $L_1[a,b]$.

Proof. By using the identity (10) we get

$$\Big|I - F_n\Big| = \Big|\int_a^b S_n(x)f^{(n)}(x)dx\Big| \leq \sup_{x\in[a,b]} |S_n(x)| \int_a^b |f^{(n)}(x)|dx.$$

Then the conclusion follows from (15). □

Theorem 4.3. *Let $f : [a,b] \to \mathbb{R}$ be a mapping such that the $(n-1)$th derivative $f^{(n-1)}$ $(n \geq 2)$ is absolutely continuous on $[a,b]$. If $f^{(n)} \in L_2[a,b]$, then we have*

$$\Big|I - F_n\Big| \leq \frac{\|f^{(n)}\|_2(b-a)^{n+\frac{1}{2}}}{6n!2^n}\sqrt{\frac{4n^6 - 8n^5 - 45n^4 + 98n^3 + 131n^2 - 324n + 108}{(2n-3)(4n^2-1)}}.$$

where $\|f^{(n)}\|_2 := \Big\{\int_a^b [f^{(n)}(x)]^2dx\Big\}^{\frac{1}{2}}$ is the usual Lebesgue norm on $L_2[a,b]$.

Proof. By using the identity (10) we get

$$\Big|I - F_n\Big| = \Big|\int_a^b S_n(x)f^{(n)}(x)dx\Big| \leq \|f^{(n)}\|_2\Big\{\int_a^b |S_n^2(x)|dx\Big\}^{\frac{1}{2}}.$$

Then the conclusion follows from (14). □

5. Non symmetric bounds

Theorem 5.1. *Let $f : [a,b] \to \mathbb{R}$ be a mapping such that the $(n-1)$th derivative $f^{(n-1)}$ $(n \geq 2)$ is absolutely continuous with $\gamma_n \leq f^{(n)}(x) \leq \Gamma_n$ a.e. on $[a,b]$, where γ_n, $\Gamma_n \in \mathbb{R}$ are constants, then we have*

$$\Big| I - F_n + \frac{(-1)^n(\Gamma_n+\gamma_n)[1+(-1)^n](b-a)^{n+1}(n-2)(n+3)}{3(n+1)!2^{n+3}} \Big|$$
$$\leq \frac{\Gamma_n-\gamma_n}{2} \times \begin{cases} \frac{\sqrt{3}(b-a)^3}{54}, & n=2, \\ \frac{(n+3)(n-2)(b-a)^{n+1}}{3(n+1)!2^{n+1}}, & n \geq 3. \end{cases}$$

Proof. By (10) and (13) we get

$$I - F_n + \frac{(-1)^n(\Gamma_n+\gamma_n)[1+(-1)^n](b-a)^{n+1}(n-2)(n+3)}{3(n+1)!2^{n+3}}$$
$$= (-1)^n \int_a^b S_n(x)\Big[f^{(n)}(x) - \frac{\Gamma_n+\gamma_n}{2}\Big]dx,$$

then notice that $\Big|f^{(n)}(x) - \frac{\Gamma_n+\gamma_n}{2}\Big| \leq \frac{\Gamma_n-\gamma_n}{2}$ a.e. on $[a,b]$, we have

$$\Big| I - F_n + \frac{(-1)^n(\Gamma_n+\gamma_n)[1+(-1)^n](b-a)^{n+1}(n-2)(n+3)}{3(n+1)!2^{n+3}} \Big|$$
$$\leq \frac{\Gamma_n-\gamma_n}{2} \int_a^b |S_n(x)|dx.$$

We complete the proof from (12). □

Remark. Applying Theorem 5.1 for $n = 2, 3$, we get (3), (6), respectively.

Theorem 5.2. *Let $f : [a,b] \to \mathbb{R}$ be a mapping such that the $(n-1)$th derivative $f^{(n-1)}$ $(n \geq 2)$ is absolutely continuous with $\gamma_n \leq f^{(n)}(x)$ a.e. on $[a,b]$, where $\gamma_n \in \mathbb{R}$ is a constant, then we have*

$$\Big| I - F_n + \frac{(-1)^n\gamma_n[1+(-1)^n](b-a)^{n+1}(n-2)(n+3)}{3(n+1)!2^{n+2}} \Big|$$
$$\leq (D_n - \gamma_n) \times \begin{cases} \frac{(b-a)^3}{12}, & n=2, \\ \frac{\sqrt{3}(b-a)^4}{216}, & n=3, \\ \frac{(b-a)^{n+1}(n+2)(n-3)}{3n!2^{n+1}}, & n \geq 4, \end{cases}$$

where

$$D_n := \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a}.$$

Proof. By (10) and (13) we get

$$I - F_n + \frac{(-1)^n\gamma_n[1+(-1)^n](b-a)^{n+1}(n-2)(n+3)}{3(n+1)!2^{n+2}}$$

$$= (-1)^n \int_a^b S_n(x)\Big[f^{(n)}(x) - \gamma_n\Big]dx,$$

then notice that $f^{(n)}(x) - \gamma_n \geq 0$ a.e. on $[a,b]$, we have

$$\Big|I - F_n + \frac{(-1)^n\gamma_n[1+(-1)^n](b-a)^{n+1}(n-2)(n+3)}{3(n+1)!2^{n+2}}\Big|$$
$$\leq \sup_{x\in[a,b]} |S_n(x)| \int_a^b \Big[f^{(n)}(x) - \gamma_n\Big]dx$$
$$= \sup_{x\in[a,b]} |S_n(x)| \Big[f^{(n-1)}(b) - f^{(n-1)}(a) - \gamma_n(b-a)\Big].$$

From (15), we get the desired result. □

Remark. Applying Theorem 5.2 for $n = 2, 3$, we get (4), (7), respectively.

Theorem 5.3. *Let $f : [a,b] \to \mathbb{R}$ be a mapping such that the $(n-1)$th derivative $f^{(n-1)}$ $(n \geq 2)$ is absolutely continuous with $f^{(n)}(x) \leq \Gamma_n$ a.e. on $[a,b]$, where $\Gamma_n \in \mathbb{R}$ is a constant, then we have*

$$\Big|I - F_n - \frac{(-1)^n\Gamma_n[1+(-1)^n](b-a)^{n+1}}{2^{n+1}} \frac{(n-2)(n+3)}{6}\Big|$$
$$\leq (\Gamma_n - D_n) \times \begin{cases} \frac{(b-a)^3}{12}, & n = 2, \\ \frac{\sqrt{3}(b-a)^4}{216}, & n = 3, \\ \frac{(b-a)^{n+1}(n+2)(n-3)}{3n!2^{n+1}}, & n \geq 4, \end{cases} \tag{18}$$

where D_n is defined in Theorem 5.2.

Proof. The proof of inequalities (18) is similar to the proof of Theorem 5.2 and so is omitted. □

Remark. Applying Theorem 5.3 for $n = 2, 3$, we get (5), (8), respectively.

6. Another sharp bound

In this section, we derive two sharp error inequalities when n is an odd and an even integer, respectively.

Theorem 6.1. *Let $f : [a,b] \to \mathbb{R}$ be a mapping such that the $(n-1)$th derivative $f^{(n-1)}$ $(n \geq 2)$ is absolutely continuous on $[a,b]$. If $f^{(n)} \in L_2[a,b]$ and n is an odd integer. Then we have*

$$\Big|I - F_n\Big| \leq \frac{(b-a)^{n+\frac{1}{2}}}{6n!2^n}\sqrt{\frac{4n^6 - 8n^5 - 45n^4 + 98n^3 + 131n^2 - 324n + 108}{(2n-3)(4n^2-1)}}\sqrt{\sigma(f^{(n)})}. \tag{19}$$

where $\sigma(\cdot)$ is defined by $\sigma(f) = \|f^{(n)}\|_2^2 - \frac{1}{b-a}\Big(\int_a^b f(x)dx\Big)^2$. Inequality (19) is the best possible in the sense that the constant

$$\frac{1}{6n!2^n}\sqrt{\frac{4n^6 - 8n^5 - 45n^4 + 98n^3 + 131n^2 - 324n + 108}{(2n-3)(4n^2-1)}}$$

can not be replaced by a smaller one.

Proof. By using the identity (10) and (13) we get

$$\Big|I - F_n\Big| = \Big|\int_a^b S_n(x) f^{(n)}(x)dx\Big| = \Big|\int_a^b S_n(x)\Big[f^{(n)}(x) - \frac{1}{b-a}\int_a^b f^{(n)}(t)dt\Big]dx\Big|$$

$$\leq \Big(\int_a^b S_n^2(x)dx\Big)^{\frac{1}{2}} \left(\int_a^b \Big[f^{(n)}(x) - \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a}\Big]^2 dx\right)^{\frac{1}{2}}$$

$$= \Big(\frac{(4n^6 - 8n^5 - 45n^4 + 98n^3 + 131n^2 - 324n + 108)(b-a)^{2n+1}}{36(n!)^2(2n-3)(4n^2-1)2^{2n}}\Big)^{\frac{1}{2}}$$

$$\Big(\|f^{(n)}\|_2^2 - \frac{[f^{(n-1)}(b) - f^{(n-1)}(a)]^2}{b-a}\Big)^{\frac{1}{2}}$$

$$= \frac{(b-a)^{n+\frac{1}{2}}}{6n!2^n}\sqrt{\frac{4n^6 - 8n^5 - 45n^4 + 98n^3 + 131n^2 - 324n + 108}{(2n-3)(4n^2-1)}}\sqrt{\sigma(f^{(n)})}.$$

To prove the sharpness of (19), we suppose that (19) holds with a constant $C > 0$ as

$$\Big|I - F_n\Big| \leq C(b-a)^{n+\frac{1}{2}}\sqrt{\sigma(f^{(n)})}. \tag{20}$$

We may find a function $f : [a, b] \to \mathbb{R}$ such that the $(n-1)$th derivative $f^{(n-1)}$ $(n \geq 2)$ is absolutely continuous on $[a, b]$ as

$$f^{(n-1)}(x) := \begin{cases} \frac{(x-a)^{n+1}}{(n+1)!} - \frac{(b-a)^2(x-a)^{n-1}}{24(n-1)!}, & x \in [a, \frac{a+b}{2}], \\ \frac{(x-b)^{n+1}}{(n+1)!} - \frac{(b-a)^2(x-b)^{n-1}}{24(n-1)!}, & x \in (\frac{a+b}{2}, b]. \end{cases}$$

It follows that

$$f^{(n)}(x) := \begin{cases} \frac{(x-a)^{n}}{n!} - \frac{(b-a)^2(x-a)^{n-2}}{24(n-2)!}, & x \in [a, \frac{a+b}{2}], \\ \frac{(x-b)^{n}}{n!} - \frac{(b-a)^2(x-b)^{n-2}}{24(n-2)!}, & x \in (\frac{a+b}{2}, b]. \end{cases}$$

Then we can find that the left-hand side of inequality (20) becomes

$$L.H.S(20) = \frac{(4n^6 - 8n^5 - 45n^4 + 98n^3 + 131n^2 - 324n + 108)(b-a)^{2n+1}}{36(n!)^2(2n-3)(4n^2-1)2^{2n}} \tag{21}$$

and the right-hand side of inequality (20) becomes

$$R.H.S(20) = \Big(\frac{4n^6 - 8n^5 - 45n^4 + 98n^3 + 131n^2 - 324n + 108}{36(n!)^2(2n-3)(4n^2-1)2^{2n}}\Big)^{\frac{1}{2}} C(b-a)^{2n+1}. \tag{22}$$

From (20), (21) and (22), we get

$$C \geq \frac{1}{6n!2^n}\sqrt{\frac{4n^6 - 8n^5 - 45n^4 + 98n^3 + 131n^2 - 324n + 108}{(2n-3)(4n^2-1)}},$$

which proving that the constant $\frac{1}{6n!2^n}\sqrt{\frac{4n^6-8n^5-45n^4+98n^3+131n^2-324n+108}{(2n-3)(4n^2-1)}}$ is the best possible in (19). □

Theorem 6.2. *Let $f:[a,b]\to\mathbb{R}$ be a mapping such that the $(n-1)$th derivative $f^{(n-1)}$ $(n\geq 2)$ is absolutely continuous on $[a,b]$. If $f^{(n)}\in L_2[a,b]$ and n is an even integer. Then we have*

$$\Big|I-F_n+\frac{(b-a)^n(n-2)(n+3)}{3(n+1)!2^{n+1}}[f^{(n)}(b)-f^{(n)}(a)]\Big| \leq \frac{(b-a)^{n+\frac{1}{2}}}{6(n+1)!2^n}\sqrt{\frac{4n^8-8n^7-61n^6+114n^5+247n^4-424n^3+32n^2}{(2n-3)(4n^2-1)}}\sqrt{\sigma(f^{(n)})}. \tag{23}$$

where $\sigma(\cdot)$ is defined in Theorem 6.1. Inequality (23) is the best possible in the sense that the constant $\frac{1}{6(n+1)!2^n}\sqrt{\frac{4n^8-8n^7-61n^6+114n^5+247n^4-391n^3+32n^2}{(2n-3)(4n^2-1)}}$ can not be replaced by a smaller one.

Proof. By using the identity (10) and (13) we get

$$\begin{aligned}
&\Big|I-F_n+\frac{(b-a)^n(n-2)(n+3)}{3(n+1)!2^{n+1}}[f^{(n-1)}(b)-f^{(n-1)}(a)]\Big|\\
&=\Big|\int_a^b S_n(x)f^{(n)}(x)dx-\frac{1}{b-a}\int_a^b S_n(x)dx\int_a^b f^{(n)}(x)dx\Big|\\
&=\frac{1}{2(b-a)}\Big|\int_a^b\int_a^b[S_n(x)-S_n(t)][f^{(n)}(x)-f^{(n)}(t)]dxdt\Big|\\
&\leq\frac{1}{2(b-a)}\Big(\int_a^b\int_a^b[S_n(x)-S_n(t)]^2dxdt\Big)^{\frac{1}{2}}\Big(\int_a^b\int_a^b[f^{(n)}(x)-f^{(n)}(t)]^2dxdt\Big)^{\frac{1}{2}}\\
&=\left(\int_a^b S_n^2(x)dx-\frac{1}{b-a}\Big(\int_a^b S_n(x)dx\Big)^2\right)^{\frac{1}{2}}\left(\int_a^b[f^{(n)}(x)]^2dx-\frac{1}{b-a}\Big(\int_a^b f^{(n)}(x)dx\Big)^2\right)^{\frac{1}{2}}\\
&=\left(\frac{(4n^8-8n^7-61n^6+114n^5+247n^4-424n^3+32n^2)(b-a)^{2n+1}}{36[(n+1)!]^2(2n-3)(4n^2-1)2^{2n}}\right)^{\frac{1}{2}}\\
&\left(\|f^{(n)}\|_2^2-\frac{[f^{(n-1)}(b)-f^{(n-1)}(a)]^2}{b-a}\right)^{\frac{1}{2}}.
\end{aligned}$$

We now suppose that (23) holds with a constant $C>0$ as

$$\Big|I-F_n+\frac{(b-a)^n(n-2)(n+3)}{3(n+1)!2^{n+1}}[f^{(n-1)}(b)-f^{(n-1)}(a)]\Big|\leq C(b-a)^{n+\frac{1}{2}}\sqrt{\sigma(f^{(n)})}. \tag{24}$$

We may find a function $f:[a,b]\to\mathbb{R}$ such that the $(n-1)$th derivative $f^{(n-1)}$ $(n\geq 2)$ is absolutely continuous on $[a,b]$ as

$$f^{(n-1)}(x):=\begin{cases}\frac{(x-a)^{n+1}}{(n+1)!}-\frac{(b-a)^2(x-a)^{n-1}}{24(n-1)!}+\frac{(b-a)^{n+1}(n-2)(n+3)}{3(n+1)!2^{n+2}}, & x\in[a,\frac{a+b}{2}],\\ \frac{(x-b)^{n+1}}{(n+1)!}-\frac{(b-a)^2(x-b)^{n-1}}{24(n-1)!}-\frac{(b-a)^{n+1}(n-2)(n+3)}{3(n+1)!2^{n+2}}, & x\in(\frac{a+b}{2},b].\end{cases}$$

It follows that

$$f^{(n)}(x):=\begin{cases}\frac{(x-a)^n}{n!}-\frac{(b-a)^2(x-a)^{n-2}}{24(n-2)!}, & x\in[a,\frac{a+b}{2}],\\ \frac{(x-b)^n}{n!}-\frac{(b-a)^2(x-b)^{n-2}}{24(n-2)!}, & x\in(\frac{a+b}{2},b].\end{cases}$$

Then we can find that the left-hand side of inequality (24) becomes

$$L.H.S(24) = \frac{(4n^8 - 8n^7 - 61n^6 + 114n^5 + 247n^4 - 424n^3 + 32n^2)(b-a)^{2n+1}}{36[(n+1)!]^2(2n-3)(4n^2-1)2^{2n}} \tag{25}$$

and the right-hand side of inequality (24) becomes

$$R.H.S(24) = \Big(\frac{4n^8 - 8n^7 - 61n^6 + 114n^5 + 247n^4 - 424n^3 + 32n^2}{36[(n+1)!]^2(2n-3)(4n^2-1)2^{2n}}\Big)^{\frac{1}{2}} C(b-a)^{2n+1}. \tag{26}$$

It follows from (24), (25) and (26) that

$$C \geq \frac{1}{6(n+1)!2^n}\sqrt{\frac{4n^8 - 8n^7 - 61n^6 + 114n^5 + 247n^4 - 424n^3 + 32n^2}{(2n-3)(4n^2-1)}},$$

proving that the constant $\frac{1}{6(n+1)!2^n}\sqrt{\frac{4n^8-8n^7-61n^6+114n^5+247n^4-424n^3+32n^2}{(2n-3)(4n^2-1)}}$ is the best possible in (23). □

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References

1. P. Cerone and S.S. Dragomir, Trapezoidal type rules from an inequalities point of view, *Handbook of Analytic-Computational Methods in Applied Mathematics,* CRC Press N. Y.(2000), 65-134.
2. P. Cerone and S.S. Dragomir, Midpoint type rules from an inequalities point of view, *Handbook of Analytic-Computational Methods in Applied Mathematics,* CRC Press N. Y.(2000), 135-200.
3. S.S. Dragomir, R.P. Agarwal and P. Cerone, On Simpson's inequality and applications. *J. Inequal. Appl.* 5(2000), 533-579.
4. W.J. Liu, Y. Jiang and A. Tuna, A unified generalization of some quadrature rules and error bounds, *Appl. Math. Comput.,* 219(9), 2013, 4765-4774.
5. Z. Liu, Some inequalities of perturbed trapezoid type, *J. Inequal. in Pure and Appl. Math.,* 7(2), Article 47, 2006.
6. Z. Liu, A note on perturbed midpoint inequalities, *Soochow J. Math.,* 33(1), 2007, 101-109.
7. Z. Liu, More on the averaged midpoint-trapezoid type rules, *Appl. Math. Comput.,* 218(4), 2011, 1389-1398.
8. Z. Liu, On generalizations of some classical integral inequalities, *J. Math. Inequal.,* 7(2), 2013, 255-269.
9. C.E.M. Pearce, J. Pečarić, N. Ujević, S. Varošanec, Generalizations of some inequalities of Ostrowski-Gruss type, *Math. Inequal. Appl.,* 3(1), 2000, 25-34.
10. N. Ujević, A generalization of Ostrowski's inequality and applications in numerical integration, *Appl. Math. Lett.,* 17(2), 2004, 133-137.
11. N. Ujević, Error inequalities for a generalized trapezoid rule, *Appl. Math. Lett.,* 19(1), 2006, 32-37.

12. N. Ujević, G. Erceg, A generalization of the corrected midpoint-trapezoid rule and error bounds, *Appl. Math. Comput.*, 184(2), 2007, 216-222.

Feixiang Chen received his MS degree from Xi'an Jiaotong University. Since 2009 he has been teaching at Chongqing Three Gorges University. His research interests include integral inequalities on several kinds of convex functions and applications.

School of Mathematics and Statistics, Chongqing Three Gorges University, Wanzhou, 404000, P.R. China.
e-mail: cfx2002@126.com