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STRONG CONVERGENCE THEOREMS FOR FIXED POINT PROBLEMS OF ASYMPTOTICALLY QUASI-ϕ-NONEXPANSIVE MAPPINGS IN THE INTERMEDIATE SENSE

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Abstract. In this paper, we introduce a general iterative algorithm for asymptotically quasi-ϕ-nonexpansive mappings in the intermediate sense to have the strong convergence in the framework of Banach spaces. The results presented in the paper improve and extend the corresponding results announced by many authors.

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1. Introduction

Let E be a real Banach space with the dual space E^*. Let C be a nonempty closed convex subset of E. Let $T : C \to C$ be a nonlinear mapping. We denote by $F(T)$ the set of fixed points of T.

A mapping $T : C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

Three classical iteration processes are often used to approximate a fixed point of nonexpansive mapping. The first one is introduced by Halpern [3] and is defined as follows: Take an initial point $x_0 \in C$ arbitrarily and define $\{x_n\}$ recursively by

$$x_{n+1} = t_n x_0 + (1 - t_n) T x_n, \quad n \in \mathbb{N}, \tag{1.1}$$

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where $\{t_n\}_{n=1}^{\infty}$ is a sequence in the interval $[0,1]$. The second iteration process is now known as Mann's iteration process [6] which is defined as

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \quad n \in \mathbb{N}, \tag{1.2}$$

where the initial point x_1 is taken in C arbitrarily and the sequence $\{\alpha_n\}_{n=1}^{\infty}$ is in the interval $[0,1]$. The third iteration process is referred to as Ishikawa's iteration process [5] which is defined recursively by

$$\begin{cases} y_n = \beta_n x_n + (1-\beta_n)Tx_n, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)Ty_n, \quad n \in \mathbb{N}, \end{cases} \tag{1.3}$$

where the initial point x_1 is taken in C arbitrarily, $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are sequences in the interval $[0,1]$.

In general not much is known regarding the convergence of the iteration processes (1.1)-(1.3) unless the underlying space E has elegant properties which we briefly mention here.

Recently, Matsushita and Takahashi [7] proved strong convergence theorems for approximation of fixed points of relatively nonexpansive mappings in a uniformly convex and uniformly smooth Banach space. More precisely, they proved the following theorem.

Theorem 1.1. *Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E, let T be a relatively nonexpansive mapping from C into itself and let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \le \alpha_n < 1$ and $\limsup_{n\to\infty} \alpha_n < 1$. Suppose that $\{x_n\}$ is given by*

$$\begin{cases} x_0 = x \in C, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \\ H_n = \{z \in C : \phi(z, y_n) \le \phi(z, x_n)\} \\ W_n = \{z \in C : \langle x_n - z, Jx - Jx_n\rangle \ge 0\}, \\ x_{n+1} = \Pi_{H_n \cap W_n} x_0, \quad n = 0, 1, 2, \cdots, \end{cases} \tag{1.4}$$

where J is the duality mapping on E. If $F(T)$ is nonempty, then $\{x_n\}$ converges strongly to $\Pi_{F(T)}x_0$, where $\Pi_{F(T)}$ is the generalized projection from C onto $F(T)$.

In [4], Hao introduced the following iterative scheme for approximating a fixed point of asymptotically quasi-ϕ-nonexpansive mappings in the intermediate sense in a reflexive, strictly convex and smooth Banach space: $x_0 \in E$, $C_1 = C$, $x_1 = \Pi_{C_1}x_0$,

$$\begin{cases} y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JT^n x_n), \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \le \phi(z, x_n) + \xi_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \ge 1, \end{cases} \tag{1.5}$$

where $\xi_n = \max\{0, \sup_{p\in F(T), x\in C}(\phi(p, T^n x) - \phi(p, x))\}$.

Motivated by the fact above, the purpose of this paper is to prove a strong convergence theorem for finding a fixed point of asymptotically quasi-ϕ-nonexpansive mappings in the intermediate sense in a reflexive, strictly convex and smooth Banach space, which has the Kadec-Klee property.

2. Preliminaries

Let E be a real Banach space and let E^* be the dual space of E. The duality mapping $J : E \to 2^{E^*}$ is defined by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}.$$

By Hahn-Banach theorem, $J(x)$ is nonempty.

The modulus of convexity of E is the function $\delta_E : (0, 2] \to [0, 1]$ defined by

$$\delta_E(\varepsilon) = \inf\{1 - \|\frac{x + y}{2}\| : \|x\| = \|y\| = 1, \varepsilon = \|x - y\|\}.$$

E is said to be uniformly convex if $\forall\varepsilon \in (0, 2]$, there exists a $\delta = \delta(\varepsilon) > 0$ such that for $x, y \in E$ with $\|x\| \le 1$, $\|y\| \le 1$ and $\|x - y\| \ge \varepsilon$, then $\|\frac{x+y}{2}\| \le 1 - \delta$. Equivalently, E is uniformly convex if and only if $\delta_E(\varepsilon) > 0$, $\forall\varepsilon \in (0, 2]$. E is strictly convex if for all $x, y \in E$, $x \ne y$, $\|x\| = \|y\| = 1$, we have $\|\lambda x+(1-\lambda)y\| < 1$, $\forall\lambda \in (0, 1)$. The space E is said to be smooth if the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for all $x, y \in S(E) = \{z \in E : \|z\| = 1\}$. It is also said to be uniformly smooth if the limit exists uniformly in $x, y \in S(E)$.

It is well known that if E is uniformly smooth, then J is norm-to-norm uniformly continuous on each bounded subset of E. If E is smooth, then J is single-valued.

Recall that a Banach space E has the Kadec-Klee property if for any sequence $\{x_n\} \subset E$ and $x \in E$ with $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$, then $\|x_n - x\| \to 0$ as $n \to \infty$. It is well known that if E is a uniformly convex Banach space, then E has the Kadec-Klee property.

In what follows, we always use $\phi : E \times E \to \mathbb{R}$ to denote the Lyapunov functional defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \forall x, y \in E.$$

It follows from the definition of ϕ that

$$(\|x\| - \|y\|)^2 \le \phi(x, y) \le (\|x\| + \|y\|)^2, \quad \forall x, y \in E, \tag{2.1}$$

and

$$\phi(x, J^{-1}(\lambda Jy + (1 - \lambda)Jz)) \le \lambda\phi(x, y) + (1 - \lambda)\phi(x, z), \quad \forall x, y, z \in E. \tag{2.2}$$

Following Alber [1], the generalized projection $\Pi_C : E \to C$ is defined by

$$\Pi_C(x) = \{u \in C : \phi(u, x) = \min_{y\in C} \phi(y, x)\}, \quad \forall x \in E.$$

The existence and uniqueness of the operator Π_C follows from the properties of the function $\phi(x,y)$ and strict monotonicity of mapping J (see [1,2,10]).

Lemma 2.1 ([1]). *Let E be a reflexive, strictly convex and smooth Banach space and C be a nonempty closed convex subset of E. Then the following conclusions hold:*

(a) $\phi(x,\Pi_C y)+\phi(\Pi_C y,y)\le\phi(x,y),\quad \forall x\in C, y\in E$;

(b) If $x\in E$ and $z\in C$, then $z=\Pi_C x \Leftrightarrow \langle z-y, Jx-Jz\rangle\ge 0,\quad \forall y\in C$;

(c) For $x,y\in E$, $\phi(x,y)=0$ if and only if $x=y$.

Remark 2.1. If E is a real Hilbert space, then $\phi(x,y)=\|x-y\|^2$ and Π_C is the metric projection P_C of E onto C.

Definition 2.2. Let C be a nonempty closed convex subset of E and let T be a mapping from C into itself. A point $p\in C$ is said to be an asymptotic fixed point of T if C contains a sequence $\{x_n\}$, which converges weakly to p and $\lim_{n\to\infty}\|x_n-Tx_n\|=0$.

The set of asymptotic fixed points of T is denoted by $\tilde{F}(T)$.

Definition 2.3. A mapping $T:C\to C$ is said to be

(1) relatively nonexpansive if $\tilde{F}(T)=F(T)\ne\phi$ and

$$\phi(p,Tx)\le\phi(p,x)$$

for all $x\in C$ and $p\in F(T)$;

(2) quasi-ϕ-nonexpansive if $F(T)\ne\phi$ and

$$\phi(p,Tx)\le\phi(p,x)$$

for all $x\in C$ and $p\in F(T)$;

(3) asymptotically quasi-ϕ-nonexpansive if $F(T)\ne\phi$ and there exists a sequence $\{k_n\}\subset[0,\infty)$ with $k_n\to 1$ as $n\to\infty$ such that

$$\phi(p,T^n x)\le k_n\phi(p,x)$$

for all $x\in C$, $p\in F(T)$ and $n\ge 1$;

(4) asymptotically quasi-ϕ-nonexpansive in the intermediate sense if $F(T)\ne\phi$ and

$$\limsup_{n\to\infty}\sup_{p\in F(T),x\in C}(\phi(p,T^n x)-\phi(p,x))\le 0.$$

Put

$$\xi_n=\max\{0,\sup_{p\in F(T),x\in C}(\phi(p,T^n x)-\phi(p,x))\}.$$

Remark 2.2. From the definition, it is obvious that $\xi_n\to 0$ as $n\to\infty$ and

$$\phi(p,T^n x)\le\phi(p,x)+\xi_n,\quad \forall p\in F(T), x\in C. \tag{2.3}$$

Remark 2.3. (1) It is easy to see that the class of quasi-ϕ-nonexpansive mappings contains the class of relatively nonexpansive mappings.

(2) The class of asymptotically quasi-ϕ-nonexpansive mappings is more general than the class of relatively asymptotically nonexpansive mappings.

(3) The class of asymptotically quasi-ϕ-nonexpansive mappings in the intermediate sense is a generalization of the class of asymptotically quasi-nonexpansive mappings in the intermediate sense in the framework.

Recall that T is said to be asymptotically regular on C if for any bounded subset K of C,

$$\lim_{n\to\infty} \sup_{x\in K} \|T^{n+1}x - T^n x\| = 0.$$

Definition 2.4. A mapping $T : C \to C$ is said to be closed if for any sequence $\{x_n\} \subset C$ with $x_n \to x$ and $Tx_n \to y$, $Tx = y$.

Lemma 2.5 ([4]). *Let E be a reflexive, strictly convex and smooth Banach space such that both E and E^* have the Kadec-Klee property. Let C be a nonempty closed and convex subset of E. Let $T : C \to C$ be a closed and asymptotically quasi-ϕ-nonexpansive mapping in the intermediate sense. Then $F(T)$ is a closed convex subset of C.*

3. Main results

Theorem 3.1. *Let E be a reflexive, strictly convex and smooth Banach space such that both E and E^* have the Kadec-Klee property. Let C be a nonempty closed convex subset of E. Let $T : C \to C$ be a closed, asymptotically regular and asymptotically quasi-ϕ-nonexpansive mapping in the intermediate sense. Let $\{\alpha_n\}$ be a sequence in $[0,1]$ and $\{\beta_n\}$ be a sequence in $(0,1)$ satisfying the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$.

Let $\{x_n\}$ be a sequence generated by

$$\begin{cases} x_1 \in E \quad \textit{chosen arbitrarily}, \\ C_1 = C, \\ y_n = J^{-1}[\alpha_n J x_1 + (1-\alpha_n)(\beta_n J x_n + (1-\beta_n) J T^n x_n)], \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \le \alpha_n \phi(z, x_1) + (1-\alpha_n)\phi(z, x_n) + \xi_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \ge 1, \end{cases} \tag{3.1}$$

where $\xi_n = \max\{0, \sup_{p\in F(T), x\in C}(\phi(p, T^n x) - \phi(p, x))\}$, $\Pi_{C_{n+1}}$ is the generalized projection of E onto C_{n+1}. If $F(T)$ is bounded in C, then $\{x_n\}$ converges strongly to $\Pi_{F(T)} x_1$.

Proof. It follows from Lemma 2.2 that $F(T)$ is a closed convex subset of C, so that $\Pi_{F(T)}x$ is well defined for any $x \in C$.

We split the proof into six steps.

Step 1. We first show that C_n, $n \geq 1$, is nonempty, closed and convex.

It is obvious that $C_1 = C$ is closed and convex. Suppose that C_n is closed and convex for some $n \geq 2$. For $z_1, z_2 \in C_{n+1}$, we see that $z_1, z_2 \in C_n$. It follows that $z = tz_1 + (1-t)z_2 \in C_n$, where $t \in (0,1)$. Notice that

$$\phi(z_1, y_n) \leq \alpha_n \phi(z_1, x_1) + (1-\alpha_n)\phi(z_1, x_n) + \xi_n,$$

and

$$\phi(z_2, y_n) \leq \alpha_n \phi(z_2, x_1) + (1-\alpha_n)\phi(z_2, x_n) + \xi_n.$$

These are equivalent to

$$\begin{aligned} &2\alpha_n\langle z_1, Jx_1\rangle + 2(1-\alpha_n)\langle z_1, Jx_n\rangle - 2\langle z_1, Jy_n\rangle \\ &\quad \leq \alpha_n\|x_1\|^2 + (1-\alpha_n)\|x_n\|^2 - \|y_n\|^2 + \xi_n, \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} &2\alpha_n\langle z_2, Jx_1\rangle + 2(1-\alpha_n)\langle z_2, Jx_n\rangle - 2\langle z_2, Jy_n\rangle \\ &\quad \leq \alpha_n\|x_1\|^2 + (1-\alpha_n)\|x_n\|^2 - \|y_n\|^2 + \xi_n. \end{aligned} \tag{3.3}$$

Multiplying t and $1-t$ on both sides of (3.2) and (3.3), respectively, we obtain that

$$\begin{aligned} &2\alpha_n\langle z, Jx_1\rangle + 2(1-\alpha_n)\langle z, Jx_n\rangle - 2\langle z, Jy_n\rangle \\ &\quad \leq \alpha_n\|x_1\|^2 + (1-\alpha_n)\|x_n\|^2 - \|y_n\|^2 + \xi_n. \end{aligned}$$

That is,

$$\begin{aligned} \|z\|^2 - 2\langle z, Jy_n\rangle + \|y_n\|^2 &\leq \alpha_n(\|z\|^2 - 2\langle z, Jx_1\rangle + \|x_1\|^2) \\ &\quad + (1-\alpha_n)(\|z\|^2 - 2\langle z, Jx_n\rangle + \|x_n\|^2) + \xi_n. \end{aligned}$$

Therefore, we have

$$\phi(z, y_n) \leq \alpha_n \phi(z, x_1) + (1-\alpha_n)\phi(z, x_n) + \xi_n.$$

This implies that C_{n+1} is closed and convex for all $n \geq 1$.

Step 2. We show that $F(T) \subset C_n$, $\forall n \geq 1$.

For $n = 1$, we have $F(T) \subset C_1 = C$. Now, assume that $F(T) \subset C_n$ for some $n \geq 2$. Put $w_n = J^{-1}(\beta_n Jx_n + (1-\beta_n)JT^n x_n)$. For each $x^* \in F(T)$, we obtain from (2.2) and (2.3) that

$$\begin{aligned} \phi(x^*, y_n) &= \phi(x^*, J^{-1}(\alpha_n Jx_1 + (1-\alpha_n)Jw_n)) \\ &\leq \alpha_n \phi(x^*, x_1) + (1-\alpha_n)\phi(x^*, w_n) \end{aligned}$$

and

$$\begin{aligned} \phi(x^*, w_n) &= \phi(x^*, J^{-1}(\beta_n Jx_n + (1-\beta_n)JT^n x_n)) \\ &\leq \beta_n \phi(x^*, x_n) + (1-\beta_n)\phi(x^*, T^n x_n) \end{aligned}$$

$$\begin{aligned}&\le \beta_n\phi(x^*,x_n)+(1-\beta_n)(\phi(x^*,x_n)+\xi_n)\\&\le \phi(x^*,x_n)+\xi_n.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\phi(x^*,y_n)&\le \alpha_n\phi(x^*,x_1)+(1-\alpha_n)(\phi(x^*,x_n)+\xi_n)\\&\le \alpha_n\phi(x^*,x_1)+(1-\alpha_n)\phi(x^*,x_n)+\xi_n.\end{aligned}$$

So, $x^*\in C_{n+1}$. It implies that $F(T)\subset C_{n+1}$.

Step 3. We prove that $\{x_n\}$ is bounded and $\lim_{n\to\infty}\phi(x_n,x_1)$ exists.

Since $x_n=\Pi_{C_n}x_1$, we have from Lemma 2.1 that

$$\langle x_n-y, Jx_1-Jx_n\rangle\ge 0,\quad \forall y\in C_n.$$

Again, since $F(T)\subset C_n$, we have

$$\langle x_n-x^*, Jx_1-Jx_n\rangle\ge 0,\quad \forall x^*\in F(T).$$

It follows from Lemma 2.1 that for each $u\in F(T)$ and for each $n\ge 1$,

$$\begin{aligned}\phi(x_n,x_1)&=\phi(\Pi_{C_n}x_1,x_1)\\&\le \phi(u,x_1)-\phi(u,x_n)\\&\le \phi(u,x_1).\end{aligned}$$

Therefore, $\{\phi(x_n,x_1)\}$ is bounded. By virtue of (2.1), $\{x_n\}$ is also bounded. Again, since $x_n=\Pi_{C_n}x_1$, $x_{n+1}=\Pi_{C_{n+1}}x_1$ and $x_{n+1}\in C_{n+1}\subset C_n$ for all $n\ge 1$, we have

$$\phi(x_n,x_1)\le \phi(x_{n+1},x_1),\quad \forall n\ge 1.$$

This implies that $\{\phi(x_n,x_1)\}$ is nondecreasing and bounded. Hence, $\lim_{n\to\infty}\phi(x_n,x_1)$ exists.

Step 4. Next, we prove that $x_n\to\overline{x}$, where $\overline{x}$ is some point in C.

Now, since $\{x_n\}$ is bounded and the space E is reflexive, we may assume that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i}\rightharpoonup\overline{x}$. Since C_n is closed and convex, it is easy to see that $\overline{x}\in C_n$ for each $n\ge 1$. This implies that

$$\phi(x_{n_i},x_1)\le\phi(\overline{x},x_1),\quad \forall n_i.$$

On the other hand, it follows from the weak lower semicontinuity of the norm that

$$\begin{aligned}\phi(\overline{x},x_1)&=\|\overline{x}\|^2-2\langle\overline{x},Jx_1\rangle+\|x_1\|^2\\&\le\liminf_{n\to\infty}(\|x_{n_i}\|^2-2\langle x_{n_i},Jx_1\rangle+\|x_1\|^2)\\&=\liminf_{n_i\to\infty}\phi(x_{n_i},x_1)\\&\le\phi(\overline{x},x_1),\end{aligned}$$

which implies that $\phi(x_{n_i},x_1)\to\phi(\overline{x},x_1)$ as $n_i\to\infty$. Hence, $\|x_{n_i}\|\to\|\overline{x}\|$ as $n_i\to\infty$. In view of the Kadec Klee property of E, we see that $x_{n_i}\to\overline{x}$ as $n_i\to$

∞. If there exists another subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to x^* \in C$, we have

$$\begin{aligned}\phi(\overline{x}, x^*) &= \lim_{n_i, n_j \to \infty} \phi(x_{n_i}, x_{n_j}) \\ &= \lim_{n_i, n_j \to \infty} \phi(x_{n_i}, \Pi_{C_{n_j}} x_1) \\ &\leq \lim_{n_i, n_j \to \infty} [\phi(x_{n_i}, x_1) - \phi(\Pi_{C_{n_j}} x_1, x_1)] \\ &= \lim_{n_i, n_j \to \infty} [\phi(x_{n_i}, x_1) - \phi(x_{n_j}, x_1)] \\ &= 0,\end{aligned}$$

which implies $\overline{x} = x^*$. This shows that $x_n \to \overline{x}$.

Step 5. Now we prove that $\overline{x} \in F(T)$.

Since $x_n = \Pi_{C_n} x_1$, $x_{n+1} = \Pi_{C_{n+1}} x_1 \in C_{n+1} \subset C_n$ and $\lim_{n\to\infty} \phi(x_n, x_1)$ exists, we see that

$$\begin{aligned}\phi(x_{n+1}, x_n) &= \phi(x_{n+1}, \Pi_{C_n} x_1) \\ &\leq \phi(x_{n+1}, x_1) - \phi(\Pi_{C_n} x_1, x_1) \\ &= \phi(x_{n+1}, x_1) - \phi(x_n, x_1).\end{aligned}$$

Hence, we have

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0.$$

Since $x_{n+1} \in C_{n+1}$, $x_n \to \overline{x}$ and $\alpha_n \to 0$, it follows from (3.1) and Remark 2.2 that

$$\phi(x_{n+1}, y_n) \leq \alpha_n \phi(x_{n+1}, x_1) + (1 - \alpha_n)\phi(x_{n+1}, x_n) + \xi_n \to 0 \tag{3.4}$$

as $n \to \infty$. This implies that

$$\lim_{n\to\infty} (\|x_{n+1}\| - \|y_n\|)^2 = 0.$$

Therefore we obtain

$$\lim_{n\to\infty} \|y_n\| = \|\overline{x}\|. \tag{3.5}$$

and so

$$\lim_{n\to\infty} \|Jy_n\| = \|J\overline{x}\|. \tag{3.6}$$

This shows that $\{Jy_n\}$ is bounded. Since E is reflexive, E^* is reflexive. Without loss of generality, we can assume that $J(y_n) \rightharpoonup \overline{y} \in E^*$. In view of reflexivity of E, we see that $J(E) = E^*$. Hence, there exists $y \in E$ such that $Jy = \overline{y}$. This implies that $J(y_n) \rightharpoonup Jy$. And

$$\begin{aligned}\phi(x_{n+1}, y_n) &= \|x_{n+1}\|^2 - 2\langle x_{n+1}, Jy_n\rangle + \|y_n\|^2 \\ &= \|x_{n+1}\|^2 - 2\langle x_{n+1}, Jy_n\rangle + \|Jy_n\|^2.\end{aligned} \tag{3.7}$$

Taking $\liminf_{n\to\infty}$ for both sides of (3.7), we have from (3.4) that

$$\begin{aligned}0 &\geq \|\overline{x}\|^2 - 2\langle \overline{x}, Jy\rangle + \|Jy\|^2 \\ &= \|\overline{x}\|^2 - 2\langle \overline{x}, Jy\rangle + \|y\|^2 = \phi(\overline{x}, y),\end{aligned}$$

which shows that $\overline{x} = y$ and so

$$J(y_n) \rightharpoonup J\overline{x}.$$

It follows from (3.6) and the Kadec-Klee property of E^* that $J(y_n) \to J\overline{x}$. Since J^{-1} is norm-weak-continuous, we have

$$y_n \rightharpoonup \overline{x}. \tag{3.8}$$

It follows from (3.5),(3.8) and the Kadec-Klee property of E that we have

$$y_n \to \overline{x}. \tag{3.9}$$

On the other hand, since $\{x_n\}$ is bounded and T is asymptotically quasi-ϕ-nonexapnsive in the intermediate sense, for any given $p \in F(T)$, we have from (2.3) that

$$\phi(p, T^n x_n) \leq \phi(p, x_n) + \xi_n, \quad n \geq 1.$$

This implies that $\{T^n x_n\}$ is bounded. Since

$$\begin{aligned}\|w_n\| &= \|J^{-1}(\beta_n J x_n + (1-\beta_n) J T^n x_n)\| \\ &\leq \beta_n \|x_n\| + (1-\beta_n)\|T^n x_n\| \\ &\leq \max\{\|x_n\|, \|T^n x_n\|\},\end{aligned}$$

it implies that $\{w_n\}$ is also bounded. From (3.1), we have

$$\begin{aligned}\lim_{n\to\infty} \|Jy_n - Jw_n\| &= \lim_{n\to\infty} \alpha_n \|Jx_1 - (\beta_n J x_n + (1-\beta_n) J T^n x_n)\| \\ &= \lim_{n\to\infty} \alpha_n \|Jx_1 - Jw_n\| \\ &= 0.\end{aligned}$$

It follows from (3.9) that $Jw_n \to J\overline{x}$ as $n \to \infty$. Since J^{-1} is norm-weakly-continuous, this implies that

$$w_n \rightharpoonup \overline{x} \tag{3.10}$$

as $n \to \infty$. Note that

$$\begin{aligned}\lim_{n\to\infty} |\|w_n\| - \|\overline{x}\|| &= \lim_{n\to\infty} |\|Jw_n\| - \|J\overline{x}\|| \\ &\leq \lim_{n\to\infty} \|Jw_n - J\overline{x}\| \\ &= 0.\end{aligned}$$

This together with (3.10) shows that

$$w_n \to \overline{x}$$

as $n \to \infty$. Since $x_n \to \overline{x}$, we have $Jx_n \to J\overline{x}$. Since

$$Jw_n - J\overline{x} = \beta_n J x_n + (1-\beta_n) J T^n x_n - J\overline{x}$$

$$= \beta_n(Jx_n - J\overline{x}) + (1-\beta_n)(JT^n x_n - J\overline{x}),$$

$$\begin{aligned}\lim_{n\to\infty}(1-\beta_n)\|JT^n x - J\overline{x}\| &\le \lim_{n\to\infty}\|Jw_n - J\overline{x}\| + \lim_{n\to\infty}\beta_n\|Jx_n - J\overline{x}\| \\ &= 0. \end{aligned} \tag{3.11}$$

By condition (ii) and (3.11), we have that

$$\lim_{n\to\infty}\|JT^n x_n - J\overline{x}\| = 0. \tag{3.12}$$

Since J^{-1} is norm-weakly-continuous, this implies that

$$T^n x_n \rightharpoonup \overline{x}. \tag{3.13}$$

It follows from (3.12) that

$$\begin{aligned}\lim_{n\to\infty}|\|T^n x_n\| - \|\overline{x}\|| &= \lim_{n\to\infty}|\|JT^n x_n\| - \|J\overline{x}\|| \\ &\le \lim_{n\to\infty}\|JT^n x_n - J\overline{x}\| \\ &= 0.\end{aligned}$$

This together with (3.13) and the Kadec-Klee property of E shows that

$$T^n x_n \to \overline{x}$$

as $n \to \infty$. Again, by the asymptotic regularity of T, we have

$$\begin{aligned}\|T^{n+1}x_n - \overline{x}\| &\le \|T^{n+1}x_n - T^n x_n\| + \|T^n x_n - \overline{x}\| \\ &\to 0\end{aligned}$$

as $n \to \infty$. That is, $\lim_{n\to\infty} TT^n x_n = \overline{x}$. It follows from the closedness of T that $T\overline{x} = \overline{x}$, i.e., $\overline{x} \in F(T)$.

Step 6. Finally, we prove that $x_n \to \overline{x} = \Pi_{F(T)}x_1$.
Let $w = \Pi_{F(T)}x_1$. Since $w \in F(T) \subset C_n$ and $x_n = \Pi_{C_n}x_1$, we have

$$\phi(x_n, x_1) \le \phi(w, x_1), \quad \forall n \ge 1.$$

This implies that

$$\begin{aligned}\phi(\overline{x}, x_1) &= \lim_{n\to\infty}\phi(x_n, x_1) \\ &\le \phi(w, x_1).\end{aligned} \tag{3.14}$$

From the definition of $\Pi_{F(T)}x_1$, $\overline{x} \in F(T)$ and (3.14), we see that $\overline{x} = w$. This completes the proof. □

Remark 3.1. If we take $\alpha_n = 0$ for all $n \in \mathbb{N}$, then the iterative scheme (3.1) reduces to following scheme:

$$\begin{cases} x_0 \in E \quad \text{chosen arbitrarily,} \\ C_1 = C, \\ y_n = J^{-1}(\beta_n Jx_n + (1-\beta_n)JT^n x_n) \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \le \phi(z, x_n) + \xi_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \ge 1, \end{cases}$$

where

$$\xi_n = \max\{0, \sup_{p \in F(T), x \in C} (\phi(p, T^n x) - \phi(p, x))\},$$

which is (1.2) and an improvement to (1.1).

In the framework of Hilbert spaces, Theorem 3.1 is reduced to the following.

Corollary 3.2. *Let E be a Hilbert space. Let C be a nonempty closed convex subset of H. Let $T : C \to C$ be a closed, asymptotically regular and asymptotically quasi-ϕ-nonexpansive mapping in the intermediate sense. Let $\{\alpha_n\}$ be a sequence in $[0,1]$ and $\{\beta_n\}$ be a sequence in $(0,1)$ satisfying the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0$*;*
(ii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$.
Let $\{x_n\}$ be a sequence generated by

$$\begin{cases} x_1 \in E \quad \textit{chosen arbitrarily,} \\ C_1 = C, \\ y_n = \alpha_n x_1 + (1-\alpha_n)(\beta_n x_n + (1-\beta_n)T^n x_n), \\ C_{n+1} = \{z \in C_n : \|z - y_n\|^2 \le \alpha_n \|z - x_1\|^2 + (1-\alpha_n)\|z - x_n\|^2 + \xi_n\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \ge 1, \end{cases}$$

where

$$\xi_n = \max\{0, \sup_{p \in F(T), x \in C} (\|p - T^n x\|^2 - \|p - x\|^2)\},$$

$P_{C_{n+1}}$ is the metric projection from E onto C_{n+1}. If $F(T)$ is bounded in C, then $\{x_n\}$ converges strongly to $P_{F(T)} x_1$.

Proof. . If E is a Hilbert space, then $J = I$ (the identity mapping) and $\phi(x, y) = \|x - y\|^2$. We can obtain the desired conclusion easily from Theorem 3.1. This completes the proof.

If T is quasi-ϕ- nonexpansive, then Theorem 3.1 is reduced to the following without involving boundedness of $F(T)$ and asymptotically regularity on C. □

Corollary 3.3. *Let E be a reflexive, strictly convex and smooth Banach space such that both E and E^* have the Kadec-Klee property. Let C be a nonempty closed convex subset of E. Let $T : C \to C$ be a closed, quasi-ϕ-nonexpansive mapping with $F(T) \neq \phi$. Let $\{\alpha_n\}$ be a sequence in $[0, 1]$ and $\{\beta_n\}$ be a sequence in $(0, 1)$ satisfying the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$.
Let $\{x_n\}$ be a sequence generated by

$$\begin{cases} x_1 \in E \quad \textit{chosen arbitrarily}, \\ C_1 = C. \\ y_n = J^{-1}(\alpha_n Jx_1 + (1-\alpha_n)(\beta_n Jx_n + (1-\beta_n)JTx_n)), \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \le \alpha_n \phi(z, x_1) + (1-\alpha_n)\phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \ge 1, \end{cases}$$

where $\Pi_{C_{n+1}}$ is the generalized projection of E onto C_{n+1}. Then $\{x_n\}$ converges strongly to $\Pi_{F(T)}x_1$.

Remark 3.2. (1) By Remark 3.1, Theorem 3.1 extends Theorem 2.1 of Hao [4].

(2) Theorem 3.1 generelized Theorem 3.1 of Matsushita and Takahashi [7] in the following respects:

(i) from the relatively nonexpansive mapping to the asymptotically quasi-ϕ-nonexpansive mapping in the intermediate sense;

(ii) from a uniformly convex and uniformly smooth Banach space to a reflexive, strictly convex and smooth Banach space.

(3) Corollary 3.1 generalized and improves Corollary 2.5 of Hao [4], Theorem 3.4 of Nakajo and Takahashi [8] and Theorem 2.1 of Su and Qin [9] in the following aspects:

(i) Algorithm of Corollary 3.1 is different from algorithms in [4,8,9].

(ii) Corollary 3.1 includes Corollary 2.5 of Hao [4] as a special case.

(iii) The set Q_n in [8,9] have been relaxed.

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