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GLOBAL CONVERGENCE METHODS FOR NONSMOOTH EQUATIONS WITH FINITELY MANY MAXIMUM FUNCTIONS AND THEIR APPLICATIONS†

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Abstract. Nonsmooth equations with finitely many maximum functions is often used in the study of complementarity problems, variational inequalities and many problems in engineering and mechanics. In this paper, we consider the global convergence methods for nonsmooth equations with finitely many maximum functions. The steepest decent method and the smoothing gradient method are used to solve the nonsmooth equations with finitely many maximum functions. In addition, the convergence analysis and the applications are also given. The numerical results for the smoothing gradient method indicate that the method works quite well in practice.

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1. Introduction

By the widely used in the problems of image restoration, variable selection, stochastic equilibrium and optimal control, nonsmooth equations and their related problems have been widely studied by many authors(see[1-16]). In this paper, we consider the nonsmooth equations with finitely many maximum functions

$$\begin{aligned} \max_{j\in J_1} f_{1j}(x) &= 0 \\ &\vdots \\ \max_{j\in J_n} f_{nj}(x) &= 0 \end{aligned} \tag{1}$$

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where $x \in R^n, f_{ij} : R^n \to R$ are continuously differentiable functions, $j \in J_i, i = 1, \ldots, n, J_i, i = 1, \ldots, n$ are finite index sets. This system of nonsmooth equations with finitely many maximum functions has specific application background, for instance, complementarity problems, variational inequality problems and many problems in national defense, economic, financial, engineering and management lead to this system of equations.(see for instance [9-10]). Obviously, (1) is a system of semismooth equations. For simplicity, we denote

$$f_i(x) = \max_{j \in J_i} f_{ij}(x), x \in R^n, i = 1, \ldots, n \tag{2}$$

$$F(x) = (f_1(x), \ldots, f_n(x))^T, x \in R^n \tag{3}$$

$$J_i(x) = \{j \in J_i | f_{ij}(x) = f_{ji}(x)\}, x \in R^n, i = 1, \ldots, n \tag{4}$$

Thus, the equations (1) can be briefly written as

$$F(x) = 0, x \in R^n. \tag{5}$$

The value function of $F(x)$ is defined as

$$f(x) = \frac{1}{2}\|F(x)\|^2.$$

Then, (5) can be solved by solving the following problem

$$\min_{x \in R^n} f(x) \tag{6}$$

We consider using the iterative method for solving (6)

$$x_{k+1} = x_k + \alpha_k d_k, k = 0, 1, \ldots, .$$

where $\alpha_k > 0$ is stepsize, d_k is a search direction.

This paper is organized as follows. In Section 2, when f is smooth function, we present the steepest method for solving it and give its global convergence result. When f is a nonsmooth function, we call it a nondifferentiable problem. There are many papers (see for instance [4,7,8,12,13,14,15,16]) deal with this problem. we give the smoothing gradient method for solving it and give the convergence analysis. In Section 3, we discuss the applications of the methods, this further illustrated the system of nonsmooth equations with finitely many maximum functions is related to solve the optimization in theory. In the last section, we discuss the application of the method for the related minimax optimization. The numerical results are also given.

Notation. Throughout the paper, $\|.\|$ denotes the l_2 norm, $R_+ = \{x | x \geq 0, x \in R\}$, g_k denote the gradient of f at x_k.

2. The methods and their convergence analysis

Case(I). Firstly, when f is smooth function, we give the steepest method for solving it. The steepest method is one of the most used method for solving unconstrained optimization (One can see for [11]).

Method 2.1

Step 1. Choose $\sigma_1 \in (0, 0.5), \sigma_2 \in (\sigma_1, 1)$. Give initial point $x_0 \in R^n$, Let

$k := 0$.

Step 2. Compute $g_k = \nabla f(x_k)$, let $d_k = -g_k$, determine α_k by Wolfe line search, where $\alpha_k = \max\{\rho^0, \rho^1 \ldots\}$ and ρ^i satisfying

$$f(x_k + \alpha_k d_k) \le f(x_k) + \sigma_1 \rho^i g_k^T d_k \tag{7}$$

and

$$g(x_k + \alpha_k d_k)^T d_k \ge \sigma_2 g_k^T d_k \tag{8}$$

Set $x_{k+1} = x_k + \alpha_k d_k$.

Step 3. Let $k := k + 1$, go to step 2.

The global convergence of the Method 2.1 is given by the following theorem.

Theorem 2.1. *Let $\{x_k\}$ generated by the Method 2.1. $f(x)$ is lower bounded. For any $x_0 \in R^n$, $\bigtriangledown f(x)$ is existence and uniformly continuous on the level set*

$$L(x_0) = \{x \in R^n | f(x) \le f(x_0)\}.$$

Then we have

$$\lim_{k\to\infty} \|\nabla f(x_k)\| = 0.$$

Proof. Suppose that the theorem is not true, then there exist a subsequence (we still denote the index by k)such that

$$\|g_k\| \ge \varepsilon > 0.$$

By d_k is a descent direction and (7) , we can see that $\{f(x_k)\}$ is monotonically decreasing sequence. Since $f(x_k)$ is lower bounded. So the limitation of $f(x_k)$ is existence. Thus, we have

$$f(x_k) - f(x_{k+1}) \to 0 (k \to \infty).$$

Set $s_k = \alpha_k d_k$. From (7), we know that

$$0 \le -g_k^T(\alpha_k d_k) = -g_k^T s_k \le \frac{1}{\sigma_1}(f(x_k) - f(x_{k+1})) \to 0.$$

Due to the angle between d_k and $-g_k$ is $\theta_k = 0$. Then

$$0 \le \|g_k\|\|s_k\| \cos\theta_k = -g_k^T s_k \to 0.$$

Note that $\|g_k\| \ge \varepsilon > 0$, hence we must have $\|s_k\| \to 0$.
And because $\nabla f(x)$ is uniformly continuous on the level set, we have

$$\nabla f(x_{k+1})^T s_k = g_k^T s_k + o(\|s_k\|).$$

That is

$$\lim_{k\to\infty} \frac{\nabla f(x_{k+1})^T s_k}{g_k^T s_k} = 1.$$

This contradiction with (8) and $\sigma_2 < 1$. So we have

$$\|g_k\| \to 0 (k \to \infty).$$

That is

$$\lim_{k\to\infty} \|\nabla f(x_k)\| = 0.$$

□

Case (II). When f is locally Lipschitz continuous but not necessarily differentiable function. The generalized gradient of f at x is defined by

$$\partial f(x) = \texttt{conv}\{\lim_{x_i \to x, x_i \in D_f} \nabla f(x_i)\},$$

where "conv" denotes the convex hull of set. D_f is the set of points at which f is differentiable.

Firstly, we introduce the definition of smoothing function.

Definition 2.2 ([3])**.** Let $f : R^n \to R$ be continuous function. We call $\widetilde{f} : R^n \times R_+ \to R$ a smooth function of f, if $\widetilde{f}(\cdot, \mu)$ is continuously differentiable in R^n for any fixed $\mu > 0$ and

$$\lim_{z \to x, \mu \downarrow 0} \widetilde{f}(z, \mu) = f(x), \tag{9}$$

for any $x \in R^n$.

In the following, we present a smoothing gradient algorithm for (6).

Method 2.2

Step 1. Choose $\sigma_1 \in (0, 0.5)$, $\sigma_2 \in (\sigma_1, 1)$ $\gamma > 0$ $\gamma_1 \in (0, 1)$, give a initial point $x_0 \in R^n$, Let $k := 0$.

Step 2. Compute $g_k = \nabla \widetilde{f}(x_k, \mu_k)$, let $d_k = -g_k$, determine α_k by the Wolfe line search, where $\alpha_k = \max\{\rho^0, \rho^1 \ldots\}$ and ρ^i satisfying

$$\widetilde{f}(x_k + \alpha_k d_k, \mu_k) \le \widetilde{f}(x_k, \mu_k) + \sigma_1 \rho^i g_k^T d_k$$

and

$$\nabla \widetilde{f}(x_k + \alpha_k d_k, \mu_k)^T d_k \ge \sigma_2 g_k^T d_k.$$

Set $x_{k+1} = x_k + \alpha_k d_k$.

Step 3. if $\|\nabla \widetilde{f}(x_{k+1}, \mu_k)\| \ge \gamma \mu_k$, then set $\mu_{k+1} = \mu_k$; otherwise, $\mu_{k+1} = \gamma_1 \mu_k$.

Step 4. Let $k := k + 1$, go to Step 2.

Then, we give the convergence result of Method 2.2.

Theorem 2.3. *Suppose that $\widetilde{f}(\cdot, \mu)$ is a smoothing function of f. If for any fixed $\mu > 0$, $\widetilde{f}(\cdot, \mu)$ satisfies the conditions as in Theorem 2.1, then $\{x_k\}$ generated by Method 2.2 satisfies*

$$\lim_{k \to \infty} \mu_k = 0$$

and

$$\lim_{k \to \infty} \inf \|\nabla \widetilde{f}(x_k, \mu_{k-1})\| = 0.$$

Proof. Define $K = \{k|\mu_{k+1} = \gamma_1\mu_k\}$. If K is finite set, then there exists an interger $\bar{k}$ such that for all $k > \bar{k}$

$$\|\nabla\widetilde{f}(x_k, \mu_{k-1})\| \geq \gamma\mu_{k-1} \tag{10}$$

Then $\mu_k = \mu_{\bar{k}} = \bar{\mu}$ in step 3 of Method 2.2.

Since $\widetilde{f}(\cdot, \bar{\mu})$ is a smoothing function, Method 2.2 reduces to solve

$$\min_{x\in R^n} \widetilde{f}(x, \bar{\mu}).$$

Hence, from the above Theorem 2.1, we can deduce that

$$\lim_{k\to\infty} \inf \|\nabla\widetilde{f}(x_k, \bar{\mu})\| = 0,$$

which contradicts with (10). This show that K must be infinite. And we know

$$\lim_{k\to\infty} \mu_k = 0.$$

Since K is infinite, we can assume that $K = \{k_0, k_1, \ldots\}$, where $k_0 < k_1 < \ldots$ Then we have

$$\lim_{i\to\infty} \|\nabla\widetilde{f}(x_{k_i+1}, \mu_{k_i}\| \leq \gamma \lim_{i\to\infty} \mu_{k_i} = 0.$$

We get the theorem. □

From above Theorem 2.3 and the gradient consistency discussion in [3,6], we can get the following result.

Theorem 2.4. *Any accumulation point x^* generated by Method 2.2 is a clarkr stationary point. This is*

$$0 \in \partial f(x^*).$$

3. The applications of the methods

3.1. Application in solving generalized complementarity problem. Consider the generalized complementarity problem (GCP) as in [5], Find a $x \in R^n$ such that

$$F(x) \geq 0, G(x) \geq 0, F(x)^T G(x) = 0. \tag{11}$$

where $F = (F_1, F_2, \ldots, F_n)^T, G = (G_1, G_2, \ldots, G_n)^T, F_i : R^n \to R (i = 1, \ldots, n)$ and $G_i : R^n \to R (i = 1, \ldots, n)$ are continuously differentiable functions.

To solve (11) is equivalent to solve the following equations

$$\min\{F_i(x), G_i(x)\} = 0 \tag{12}$$

By $\min(x, y) = x - (x - y)_+$, we know that (12) is equivalent to

$$F_i(x) - (F_i(x) - G_i(x))_+ = 0 \tag{13}$$

Let $\rho : R \to R_+$ be a piecewise continuous density function satisfying

$$\rho(s) = \rho(-s)(s \in R) \text{ and } \int_R \rho(s)ds = 1. \tag{14}$$

Let $\kappa := \int_{-\infty}^{\infty} |s|\rho(s)ds < \infty$, then for any fixed $\mu > 0$, there is a continuous function

$$\phi(t,\mu) := \int_{-\infty}^{\infty} (t-\mu s)_+ \rho(s)ds$$

satisfying

$$0 \le \phi(t,\mu) - (t)_+ \le \kappa\mu.$$

From the definition of smoothing function, we know that $\phi(\cdot,\mu)$ is a smoothing function of $(t)_+$.

Choose

$$\rho(s) = \begin{cases} 0, & \text{if } |s| \ge \frac{1}{2}; \\ 1, & \text{if } |s| < \frac{1}{2}. \end{cases}$$

Then

$$\phi(t,\mu) = \int_{-\infty}^{\infty} (t-\mu s)_+ \rho(s)ds = \begin{cases} (t)_+, & \text{if } |t| \ge \frac{\mu}{2}; \\ \frac{t^2}{2\mu} + \frac{t}{2} + \frac{\mu}{8}, & \text{if } |t| < \frac{\mu}{2}. \end{cases}$$

is a smoothing function of $(t)_+$. Then, let $t = F_i(x) - G_i(x), i = 1, \ldots, n$, we have

$$\phi((F_i(x) - G_i(x)),\mu) := \int_{-\infty}^{\infty} ((F_i(x) - G_i(x)) - \mu s)_+ \rho(s)ds$$

$$= \begin{cases} (F_i(x) - G_i)_+, & \text{if } |F_i(x) - G_i| \ge \frac{\mu}{2}; \\ \frac{(F_i(x)-G_i)^2}{2\mu} + \frac{F_i(x)-G_i}{2} + \frac{\mu}{8}, & \text{if } |F_i(x) - G_i| < \frac{\mu}{2}. \end{cases}$$

We know that the smoothing function of $F_i(x) - (F_i(x) - G_i(x))_+$ is

$$F_i(x) - \phi((F_i(x) - G_i(x)),\mu) \quad i = 1, \ldots, n. \tag{15}$$

So, we can transform (13) into

$$F_i(x) - \phi((F_i(x) - G_i(x)),\mu) = 0 \quad i = 1, \ldots, n. \tag{16}$$

Then, we can use the Method 2.2 to solve (16).

3.2. Application in solving linear maximum equations. Here, we consider the equations of maximum functions in [16]. Let $F : R \to R$ is a finite equations of maximum functions,

$$F(t) = \max_{i=1,\ldots,m} \{f_i(t)\},$$

where $f_i : R \to R$ is a affine linear,

$$f_i(t) = p_i(t) + q_i,$$

where $p_i, q_i \in R(i = 1, \ldots, m, m \in N)$ are scalars. Follow the affine structure of F, we know that F is Lipschitz and convex. Generally assumption

$$p_1 < p_2 < \ldots < p_{m-1} < p_m, m \in N. \tag{17}$$

And there exists $-\infty = t_1 < t_2 < \ldots < t_m < t_{m+1} = \infty$, such that

$$p_i t_{i+1} + q_i = p_{i+1} t_{i+1} + q_{i+1}, \quad \forall \quad i = 1, \ldots, m-1. \tag{18}$$

And

$$F(t)=\begin{cases} p_1 t+q_1, & \text{if } t\le t_2;\\ p_i t+q_i, & \text{if } t\in[t_i,t_{i+1}];\\ p_m t+q_m, & \text{if } t\ge t_r \qquad i\in\{2,\ldots,m-1\}.\end{cases} \tag{19}$$

For the above linear affine equations of maximum functions, the smoothing function for the linear equations of maximum functions can be defined as follows. Let $\rho : R\to R$ is a piecewise continuous density function such that

$$\rho\ge 0$$

and

$$\int_R |t|\rho(t)dt<+\infty.$$

We define a distribution function that goes with ρ by F ,i.e.,

$$F : R\to[0,1],\quad F(x)=\int_{-\infty}^{x}\rho(t)dt$$

Similar to [2], we can find the smoothing function $F(t)$ of this special equations of maximum functions by convolution

$$\widetilde{f}(t,\mu):=\int_R F(t-\mu s)\rho(s)ds \tag{20}$$

For this linear affine finite equations of maximum functions

$$F(t)=\max_{i=1,\ldots,m}\{f_i(t)\}=0.$$

Using the above convolution, we can transform it into

$$\widetilde{f}(t,\mu)=\max_{i=1,\ldots,m}\{\widetilde{f}_i(t,\mu)\}=0,$$

and we can use the Method 2.2 to solve it.

4. Application in related minimax optimization problem

In this section, we consider the minimax optimization problem(see in [15])

$$\min f(x),$$

where $f(x)=\max_{i=1,\ldots,m} f_i(x)$. $f_1(x),\ldots,f_m(x) : R^n\to R$ are twice continuous differentiable functions. Minimax problems are widely used in engineering design, optimal control, circuit design and computer-aided-design. Usually, minimax problems can be approached by reformulating them into smooth problems with constraints or by dealing with the nonsmooth objective directly.

In this paper, we also use the smoothing function (see for instance [15])

$$\widetilde{f}(x,\mu)=\mu ln\sum_{i=1}^{m}\texttt{exp}\ (\frac{f_i(x)}{\mu}),$$

to approximate the function $f(x)$. In the following, we can see that using the Method 2.2 to solve the minimax optimization problem works quite well from the numerical result. We use the examples in [4]. All codes are finished in MATLAB 8.0. Throughout our computational experiments, the parameters used in the Method 2.2 are chosen as

$$\delta_1 = 0.25, \delta_2 = 0.75, \gamma = \gamma_1 = 0.5,$$

In our implementation, we use $\|\Delta x\| \leq 10^{-5}$ as the stopping rule. x_0 is the initial point, x^* is the optimal value point, $f(x^*)$ is optimal value, k is the iterations.

Example 4.1 ([4]).

$$\min \quad \max_{i=1,2,3} f_i(x),$$

where

$$f_1(x) = x_1^2 + x_2^4,$$
$$f_2(x) = (2 - x_1)^2 + (2 - x_2)^2,$$
$$f_3(x) = 2exp(-x_1 + x_2).$$

Table 4.1 Numerical results for Example 4.1.

x_0	x^*	$f(x^*)$	k
(3.7,-4)	(1.139924928941916, 0.898573376287522)	1.955901416401007	63
(1.9,-0.4)	(1.139902887689143, 0.898586313882269)	1.955901308067330	60
(2.9,-0.4)	(1.139961860026445 , 0.898544495795374)	1.955901637122538	58
(2.7,-1)	(1.139960526989222, 0.898545538496570)	1.955901629030341	56
(1.5,-0.7)	(1.139897807096254, 0.898598213333535)	1.955901259220984	50
(1,-1)	(1.139864150412696, 0.898624534951500)	1.955901069399404	49

Example 4.2 ([4]).

$$\min \quad \max_{i=1,2,3} f_i(x),$$

where

$$f_1(x) = x_1^4 + x_2^2,$$
$$f_2(x) = (2 - x_1)^2 + (2 - x_2)^2,$$
$$f_3(x) = 2exp(-x_1 + x_2).$$

Consider the following nonlinear programming problem as in [4].

$$\min \quad F(x)$$
$$\texttt{Subject to} \quad g_i(x) \geq 0 \quad i = 2, 3, \cdots, m.$$

Table 4.2 Numerical results for Example 4.2.

x_0	x^*	$f(x^*)$	k
(4,-3.9)	(1.000889228841187, 0.997874108053042)	2.007908913059054	79
(3,-2.5)	(1.000889228842974, 0.997874108051301)	2.007908913059060	77
(3,-1.8)	(1.000889228835420, 0.997874108058792)	2.007908913059036	75
(2.7,3)	(1.000889229157946, 0.997874107690151)	2.007908913060094	74
(1.6,-0.8)	(1.000889229150102, 0.997874107698151)	2.007908913060069	69
(1,-1)	(1.000889228839357, 0.997874108054933)	2.007908913059048	72

Bandler and Charalambous (see [1]) proved that for sufficiently large α_i, the optimum of the nonlinear programming problem coincides with the following minimax function:

$$f(x) = \max_{1\leq i\leq m} f_i(x),$$

where

$$f_1(x) = F(x), \quad f_i(x) = F(x) - \alpha_i g_i(x), \quad 2 \leq i \leq m, \quad \alpha_i > 0,$$

Example 4.3 (Rosen-Suzuki Problem).

$$\begin{aligned} F(x) &= x_1^2 + x_2^2 + 2x_3^2 + x_4^2 - 5x_1 - 5 * x_2 - 21x_3 + 7x_4, \\ g_2(x) &= -x_1^2 - x_2^2 - x_3^2 - x_4^2 - x_1 + x_2 - x_3 + x_4 + 8, \\ g_3(x) &= -x_1^2 - 2x_2^2 - x_3^2 - 2x_4^2 + x_1 + x_4 + 10, \\ g_4(x) &= -x_1^2 - x_2^2 - x_3^2 - 2x_1 + x_2 + x_4 + 5. \end{aligned}$$

Here, we use

$$\alpha_2 = \alpha_3 = \alpha_4 = 10.$$

The numerical results for Example 4.3 are listed in Table 4.3.

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Table 4.3 Numerical results for Example 4.3.

x_0	x^*	$f(x^*)$	k
(0.2,1.4,2.5,-0.3)	(-0.012563047488575, 0.999654230708825, 2.005391998500046, -0.985708619962418)	-43.883347012428914	142
(0.12,1.3,2.4,-0.2)	(-0.013347871663692, 0.999803796437638, 2.005910162837592, -0.985035832435970)	-43.883196842326250	147
(0.13,1.3,2.2,-0.1)	(-0.013652924639572, 0.999714592460183, 2.006153660052175, -0.984752320557744)	-43.883131676657008	189
(0.17,1.2,2.4,-0.2)	(-0.012342323315547, 0.999736026294434, 2.005210732080403, -0.985916181709301)	-43.883390746471470	179
(0.2,1.1,2.3,-0.2)	(-0.013239463173810, 0.999733070420246, 2.005852934619233, -0.985121416579123)	-43.883216781622913	170
(0.15,1.4,2.6,-0.16)	(-0.013761884871110, 0.999903150166336, 2.006177533247161, -0.984683649647571)	-43.883114292014980	169

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